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Hardy type unique continuation properties for abstract Schrödinger equations and applications

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Abstract. In this paper, Hardy's uncertainty principle and unique continuation properties of Schrödinger equations with operator potentials in Hilbert space-valued L^2 classes are obtained. Since the Hilbert space H and linear operators are arbitrary, by choosing the appropriate spaces and operators we obtain numerous classes of Schrödinger type equations and its finite and infinite many systems which occur in a wide variety of physical systems.

Keywords: Schrödinger equations, positive operators, groups of operators, unique continuation, Hardy's uncertainty principle

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1 Introduction

Here, the unique continuation properties of the following abstract Schrödinger equation

$$i\partial_t u + \Delta u + A(x)\, u + V(x,t)\, u = 0, \qquad x \in \mathbb{R}^n,\ t \in [0,T], \tag{1.1}$$

are studied, where $A = A(x)$ is a linear and $V(x,t)$ is a given potential operator functions in a Hilbert space H; Δ denotes the Laplace operator in $\mathbb{R}^n$ and $u = u(x,t)$ is the H-valued unknown function. This linear result was then applied to show that two regular solutions u_1, u_2 of non-linear abstract Schrödinger equations

$$i\partial_t u + \Delta u + A((x))\, u = F(u,\bar{u}), \qquad x \in \mathbb{R}^n,\ t \in [0,T] \tag{1.2}$$

for general non-linearities F must agree in $\mathbb{R}^n \times [0,T]$, when $u_1 - u_2$ and its gradient decay faster than any quadratic exponential at times 0 and T.

Hardy's uncertainty principle and unique continuation properties for Schrödinger equations studied e.g in [4–7] and the references therein. Abstract differential equations studied e.g. in [2, 12–15, 17–19, 23, 25]. However, there seems to be no such abstract setting for nonlinear Schrödinger equations except the local existence of weak solution (cf. [15]). In contrast to

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these results we will study the unique continuation properties of abstract Schrödinger equations with the operator potentials. Since the Hilbert space H is arbitrary and A is a possible linear operator, by choosing H and A we can obtain numerous classes of Schrödinger type equations and its systems which occur in the different processes. Our main goal is to obtain sufficient conditions on a solution u, the operator A, potential V and the behavior of the solution at two different times t_0 and t_1 which guarantee that $u(x,t) \equiv 0$ for $x \in \mathbb{R}^n$, $t \in [0,T]$. If we choose H to be a concrete Hilbert space, for example $H = L^2(\Omega)$, $A = L$, where Ω is a domain in $\mathbb{R}^m$ with sufficiently smooth boundary and L is a regular elliptic operator then, we obtain the unique continuation properties of the anisotropic Schrödinger equation

$$\partial_t u = i(\Delta u + Lu) + V(x,t)u, \qquad x \in \mathbb{R}^n,\ y \in \Omega,\ t \in [0,T]. \tag{1.3}$$

Moreover, let we choose $H = L^2(0,1)$ and A to be differential operator with Wentzell–Robin boundary condition defined by

$$D(A) = \left\{u \in W^{2,2}(0,1),\ Au(j) = 0,\ j = 0,1\right\}, \tag{1.4}$$
$$A(x)u = a(x,y)u^{(2)} + b(x,y)u^{(1)},$$

where a, b are sufficiently smooth functions on $\mathbb{R}^n \times (0,1)$ and $V(x,t)$ is a integral operator so that

$$V(x,t)u = \int_0^1 K(x,y,t)u(x,y,t)\,dy,$$

where, $K = K(x,\tau,t)$ is a complex valued bounded function. From our general results we obtain the unique continuation properties of the Wentzell–Robin type boundary value problem (BVP) for the following Schrödinger equation

$$\begin{gathered}\partial_t u = i\left(\Delta u + a\frac{\partial^2 u}{\partial y^2} + b\frac{\partial u}{\partial y}\right) + \int_0^1 K(x,y,t)u(x,y,t)\,dy, \\ x \in \mathbb{R}^n,\ y \in (0,1),\ t \in [0,T],\end{gathered} \tag{1.5}$$

$$a\partial_y^2 u(x,j,t) + b\partial_y u(x,j,t) = 0, \qquad j = 0,1. \tag{1.6}$$

Note that, the regularity properties of Wentzell–Robin type BVP for elliptic equations were studied e.g. in [10, 11] and the references therein. Moreover, if put $H = l_2$ and choose A to be a infinite matrix $\left[a_{mj}\right]$, $m,j = 1,2,\ldots,\infty$, then we derive the unique continuation properties of the following system of Schrödinger equation

$$\partial_t u_m = i\left[\Delta u_m + \sum_{j=1}^{\infty}\left(a_{mj}(x) + b_{mj}(x,t)\right)u_j\right], \qquad x \in \mathbb{R}^n,\ t \in (0,T), \tag{1.7}$$

where a_{mj} are continuous and b_{mj} are bounded functions.

Let E be a Banach space. $L^p(\Omega;E)$ denotes the space of strongly measurable E-valued functions that are defined on the measurable subset $\Omega \subset \mathbb{R}^n$ with the norm

$$\|f\|_{L^p} = \|f\|_{L^p(\Omega;E)} = \left(\int_\Omega \|f(x)\|_E^p\,dx\right)^{\frac{1}{p}}, \qquad 1 \le p < \infty.$$

Let H be a Hilbert space and

$$\|u\| = \|u\|_H = (u,u)_H^{\frac{1}{2}} = (u,u)^{\frac{1}{2}} \quad \text{for } u \in H.$$

For $p = 2$ and $E = H$, $L^p(\Omega; E)$ becomes a H-valued function space with inner product:

$$(f, g)_{L^2(\Omega;H)} = \int_\Omega (f(x), g(x))_H \, dx, \qquad f, g \in L^2(\Omega; H).$$

Here, $W^{s,2}(\mathbb{R}^n; H)$, $-\infty < s < \infty$ denotes the H-valued Sobolev space of order s which is defined as:

$$W^{s,2} = W^{s,2}(\mathbb{R}^n; H) = (I - \Delta)^{-\frac{s}{2}} L^2(\mathbb{R}^n; H)$$

with the norm

$$\|u\|_{W^{s,2}} = \left\| (I - \Delta)^{\frac{s}{2}} u \right\|_{L^2(\mathbb{R}^n;H)} < \infty.$$

It clear that $W^{0,2}(\mathbb{R}^n; E) = L^2(\mathbb{R}^n; H)$. Let H_0 and H be two Hilbert spaces and H_0 is continuously and densely embedded into H. Let $W^{s,2}(\mathbb{R}^n; H_0, H)$ denote the Sobolev–Lions type space, i.e.,

$$W^{s,2}(\mathbb{R}^n; H_0, H) = \Big\{ u \in W^{s,2}(\mathbb{R}^n; H) \cap L^2(\mathbb{R}^n; H_0), \\ \|u\|_{W^{s,2}(\mathbb{R}^n;H_0,H)} = \|u\|_{L^2(\mathbb{R}^n;H_0)} + \|u\|_{W^{s,2}(\mathbb{R}^n;H)} < \infty \Big\}.$$

Let $C(\Omega; E)$ denote the space of E-valued uniformly bounded continuous functions on Ω with norm

$$\|u\|_{C(\Omega;E)} = \sup_{x \in \Omega} \|u(x)\|_E.$$

$C^m(\Omega; E)$ will denote the spaces of E-valued uniformly bounded strongly continuous and m-times continuously differentiable functions on Ω with norm

$$\|u\|_{C^m(\Omega;E)} = \max_{0 \le |\alpha| \le m} \sup_{x \in \Omega} \|D^\alpha u(x)\|_E.$$

Here, $O_r = \{x \in \mathbb{R}^n, \ |x| < r\}$ for $r > 0$. Let $\mathbb{N}$ denote the set of all natural numbers, $\mathbb{C}$ denote the set of all complex numbers. Let E_1 and E_2 be two Banach spaces. $B(E_1, E_2)$ will denote the space of all bounded linear operators from E_1 to E_2. For $E_1 = E_2 = E$ it will be denoted by $B(E)$. By $(E_1, E_2)_{\theta,p}$, $0 < \theta < 1, 1 \le p \le \infty$ we will denote the interpolation spaces obtained from $\{E_1, E_2\}$ by the K-method [24, §1.3.2]. Here, $S = S(\mathbb{R}^n; E)$ denotes the E-valued Schwartz class, i.e. the space of E-valued rapidly decreasing smooth functions on $\mathbb{R}^n$, equipped with its usual topology generated by seminorms. $S(\mathbb{R}^n; \mathbb{C})$ will be denoted by just S. Let $S'(\mathbb{R}^n; E)$ denote the space of all continuous linear operators, $L : S \to E$, equipped with topology of bounded convergence.

Let $A = A(x)$, $x \in \mathbb{R}^n$ be closed linear operator in E with independent on $x \in \mathbb{R}^n$ domain $D(A)$ that is dense on E. The Fourier transformation of $A(x)$, i.e. $\hat{A} = FA = \hat{A}(\xi)$ is a linear operator defined as

$$\hat{A}(\xi) u(\varphi) = A(x) u(\hat{\varphi}) \quad \text{for } u \in S'(\mathbb{R}^n; E), \ \varphi \in S(\mathbb{R}^n).$$

(For details see e.g. [1, Section 3]).

For linear operators A and B, $[A, B]$-denotes a commutator operator, i.e.

$$[A, B] = AB - BA.$$

These kind of operators are of fundamental importance in real analysis, potential theory and in the study of elliptic and parabolic differential equations (see e.g. [9, 16, 22] and references therein).

Sometimes we use one and the same symbol C without distinction in order to denote positive constants which may differ from each other even in a single context. When we want to specify the dependence of such a constant on a parameter, say α, we write C_α.

2 Main results

Let $A = A(x)$, $x \in \mathbb{R}^n$ be closed linear operator in a Hilbert space H with independent on $x \in \mathbb{R}^n$ domain $D(A)$ that is dense on H. Let

$$H(A) = \left\{ u \in D(A),\ \|u\|_{H(A)} = \|Au\|_H + \|u\|_H < \infty \right\},$$

$$X = L^2(\mathbb{R}^n; H), \quad X(A) = L^2(\mathbb{R}^n; H(A)), \quad X(A) = L^2(\mathbb{R}^n; H(A))$$

$$X^\infty(A) = L^\infty(\mathbb{R}^n; H(A)), \quad Y^s = W^{s,2}(\mathbb{R}^n; H), \quad Y^s(A) = W^{s,2}(\mathbb{R}^n; H(A)),$$

$$B = L^\infty(\mathbb{R}^n; B(H)) \quad \text{and} \quad \mu(t) = \alpha t + \beta(1-t).$$

Definition 2.1. A function $u \in L^\infty(0, T; X(A))$ is called a local weak solution to (1.1) on $(0, T)$ if u satisfies (1.1). In particular, if $(0, T)$ coincides with $\mathbb{R}$, then u is called a global weak solution to (1.1). If the solution of (1.1) belongs to $C\left([0, T]; X(A) \cap Y^2\right)$, then it is called a strong solution.

Our main result in this paper is the following.

Theorem 2.2. *Assume that the following condition are satisfied:*

(1) $A = A(x)$ *and* $\frac{\partial A}{\partial x_k}$ *are symmetric operators in a Hilbert space* H *with independent on* $x \in \mathbb{R}^n$ *domain* $D\left(\frac{\partial A}{\partial x_k}\right) = D(A)$ *that is dense on* H*. Moreover,* $(A(x)u, u) \geq 0$ *and* $(A(x)u, u) \in L^2(\mathbb{R}^n)$ *for* $u \in D(A)$*;*

(2)

$$\sum_{k=1}^{n} \left(x_k \left[A \frac{\partial f}{\partial x_k} - \frac{\partial A}{\partial x_k} f \right], f \right)_X \geq 0 \quad \text{for } f \in L^\infty\left(0, T; Y^1(A)\right);$$

(3) $A(x) A^{-1}(x_0) \in L^1(\mathbb{R}^n; B(H))$ *for some* $x_0 \in \mathbb{R}^n$ *and* $V(x, t) \in B(H)$ *for* $(x, t) \in \mathbb{R}^n \times [0, 1]$*;*

(4) either, $V(x, t) = V_1(x) + V_2(x, t)$*, where* $V_1(x) \in B(H)$ *for* $x \in \mathbb{R}^n$ *and*

$$M_1 = \sup_{x \in \mathbb{R}^n} \|V_1(x)\|_{B(H)} < \infty, \qquad \sup_{t \in [0,1]} \left\| e^{|x|^2 \mu^{-2}(t)} V_2(\cdot, t) \right\|_B < \infty$$

or

$$\lim_{r \to \infty} \|V\|_{L^1(0,1; L^\infty(\mathbb{R}^n / O_r); B(H))} = 0;$$

(5) $u \in C([0, 1]; X(A))$ *is a solution of the equation* (1.1) *and*

$$\left\| e^{\beta^{-2}|x|^2} u(\cdot, 0) \right\|_X < \infty, \qquad \left\| e^{\alpha^{-2}|x|^2} u(\cdot, 1) \right\|_X < \infty.$$

where

$$\alpha,\ \beta > 0, \qquad \alpha\beta < 2.$$

Then $u(x, t) \equiv 0$.

As a result of Theorem 2.2 we get the following Hardy's uncertainty principle result for the non linear equation (1.2).

Theorem 2.3. *Suppose that the assumptions (1)–(2) of Theorem 2.2 are satisfied. Let* $u_1, u_2 \in C\left([0,1]; Y^k(A)\right)$, $k \in \mathbb{Z}^+$ *be strong solutions of the equation* (1.2) *with* $k > \frac{n}{2}$. *Moreover, assume:*

(1) $F \in C^k\left(\mathbb{C}^2, \mathbb{C}\right)$ *and* $F(0) = \partial_u F(0) = \partial_{\bar{u}} F(0) = 0$. *There are* α, $\beta > 0$ *with* $\alpha\beta < 2$ *such that*

$$e^{-\beta^{-2}|x|^2}\left(u_1(\cdot,0) - u_2(\cdot,0)\right) \in X, \qquad e^{-\alpha^{-2}|x|^2}\left(u_1(\cdot,1) - u_2(\cdot,1)\right) \in X$$

(2) there exists a constant $B_0 > 0$ *such that*

$$\|F(u,\bar{u})\|_H \leq B_0 \|u\|_{(H(A),H)_{\frac{1}{p},p}}$$

for all $u \in (H(A), H)_{\frac{1}{p},p}$.

Then $u_1 \equiv u_2$.

One of the results we get is the following one.

Theorem 2.4. *Assume that the all conditions of Theorem 2.2 are satisfied. Suppose* $\Delta + A + V_1$ *generates a bounded continuous group. Let* $u \in C([0,1]; X(A))$ *be a solution of* (1.1). *Then* $\left\|e^{|x|^2\mu^{-2}(t)}u(\cdot,t)\right\|_X^{\frac{1}{\mu(t)}}$ *is logarithmically convex in* $[0,1]$ *and there is* $N = N(\alpha,\beta)$ *such that*

$$\left\|e^{|x|^2\mu^{-2}(t)}u(\cdot,t)\right\|_X^{\frac{1}{\mu(t)}} \leq e^{N\left(M_1+M_2+M_1^2+M_2^2\right)}\left\|e^{\beta^{-2}|x|^2}u(\cdot,0)\right\|_X^{\beta(1-t)\mu(t)}\left\|e^{\alpha^{-2}|x|^2}u(\cdot,1)\right\|_X^{\alpha t\mu(t)},$$

when

$$M_2 = e^{2B(V_2)}\sup_{t\in[0,1]}\left\|e^{|x|^2\mu^{-2}(t)}V_2(\cdot,t)\right\|_B, \qquad B(V_2) = \sup_{t\in[0,1]}\|\operatorname{Re} V_2(\cdot,t)\|_B.$$

Moreover,

$$\sqrt{t(1-t)}\left\|e^{|x|^2\mu^{-2}(t)}\nabla u\right\|_{L^2(\mathbb{R}^n\times[0,1];H)} \leq e^{N\left(M_1+M_2+M_1^2+M_2^2\right)}\left[\left\|e^{\beta^{-2}|x|^2}u(\cdot,0)\right\|_X + \left\|e^{\alpha^{-2}|x|^2}u(\cdot,1)\right\|_X\right].$$

Consider the Cauchy problem for abstract parabolic equations with variable operator coefficients

$$\partial_t u = \Delta u + A(x)u + V(x,t)u, \tag{2.1}$$

$$u(x,0) = f(x),\ x \in \mathbb{R}^n,\ t \in [0,1],$$

where $A(x)$ is a linear and $V(x,t)$ is the given potential operator functions in H. By employing Theorem 2.2 we obtain the following result for the abstract parabolic equation (2.1):

Theorem 2.5. *Assume the assumptions (1)–(3) of Theorem 2.2 are satisfied. Suppose that* $u \in L^\infty(0,1; X(A)) \cap L^2\left(0,1; Y^1\right)$ *is a solution of* (2.1) *and*

$$\|f\|_X < \infty, \left\|e^{\delta^{-2}|x|^2}u(\cdot,1)\right\|_X < \infty$$

for some $\delta < 1$. *Then,* $f(x) \equiv 0$ *for* $x \in \mathbb{R}^n$.

First of all, we generalize the result of G. H. Hardy (see e.g. [20, p. 131]) about uncertainty principle for Fourier transform:

Lemma 2.6. *Let $f(x)$ be H-valued function for $x \in \mathbb{R}^n$ and*

$$\|f(x)\| = O\left(e^{-\frac{|x|^2}{\beta^2}}\right), \qquad \left\|\hat{f}(\xi)\right\| = O\left(e^{-\frac{4|\xi|^2}{\alpha^2}}\right), \qquad x, \xi \in \mathbb{R}^n \quad \text{for } \alpha\beta < 4.$$

Then $f(x) \equiv 0$. Also, if $\alpha\beta = 4$, then $\|f(x)\|$ is a constant multiple of $e^{-\frac{|x|^2}{\beta^2}}$.

Proof. Indeed, by employing Phragmén–Lindelöf theorem to the classes of Hilbert-valued analytic functions and by reasoning as in [8] we obtain the assertion.

Consider the Cauchy problem for free abstract Schrödinger equation

$$i\partial_t u + \Delta u + Au = 0, \qquad x \in \mathbb{R}^n, \ t \in [0,1], \tag{2.2}$$

$$u(x,0) = f(x),$$

where $A = A(x)$ is a linear operator in a Hilbert space H with independent on x domain $D(A)$.

The above result can be rewritten for solution of the (2.2) on $\mathbb{R}^n \times (0,\infty)$. Indeed, assume

$$\|u(x,0)\| = O\left(e^{-\frac{|x|^2}{\beta^2}}\right), \qquad \|u(x,T)\| = O\left(e^{-\frac{|x|^2}{\alpha^2}}\right) \quad \text{for } \alpha\beta < 4T.$$

Then $u(x,t) \equiv 0$. Also, if $\alpha\beta = 4T$, then u has as a initial data a constant multiple of $e^{-\left(\frac{1}{\beta^2}+\frac{i}{4T}\right)|x|^2}$. □

Lemma 2.7. *Assume that A is a symmetric operator in H with independent on $x \in \mathbb{R}^n$ domain $D(A)$ that is dense on H. Moreover, $A(x)A^{-1}(x_0) \in L^1(\mathbb{R}^n; B(H))$ for some $x_0 \in \mathbb{R}^n$. Then for $f \in W^{s,2}(\mathbb{R}^n; H)$, $s \geq 0$ there is a generalized solution of (2.2) expressing as*

$$u(x,t) = F^{-1}\left[e^{i\hat{A}_\xi t}\hat{f}(\xi)\right], \qquad \hat{A}_\xi = \hat{A}(\xi) - |\xi|^2, \tag{2.3}$$

where F^{-1} is the inverse Fourier transform and $\hat{A}(\xi)$ denotes the Fourier transform of $A(x)$.

Proof. By applying the Fourier transform to the problem (2.2) we get

$$i\partial_t \hat{u}(\xi,t) + \hat{A}_\xi \hat{u}(\xi,t) = 0, \qquad x \in \mathbb{R}^n, \ t \in [0,1], \tag{2.4}$$

$$\hat{u}(\xi,0) = \hat{f}(\xi), \qquad \xi \in \mathbb{R}^n.$$

It is clear to see that the solution of (2.4) can be exspressed as

$$\hat{u}(\xi,t) = e^{i\hat{A}_\xi t}\hat{f}(\xi).$$

Hence, we obtain (2.3) □

3 Estimates for solutions

We need the following lemmas for proving the main results. Consider the abstract Schrödinger equation

$$\partial_t u = (a+ib)\left[\Delta u + Au + Vu + F(x,t)\right], \qquad x \in \mathbb{R}^n,\ t \in [0,1], \tag{3.1}$$

where a, b are real numbers, $A = A(x)$ is a linear operator, $V = V(x,t)$ is a given potential operator function in H and $F(x,t)$ is a given H-valued function.

Condition 3.1. Assume that:

(1) $A = A(x)$ is a symmetric operator in Hilbert space H with independent on $x \in \mathbb{R}^n$ domain $D(A)$ that is dense on H; moreover, $(A(x)u,u) \geq 0$ and $(A(x)u,u) \in L^2(\mathbb{R}^n)$ for $u \in D(A)$;

(2) $\frac{\partial A}{\partial x_k}$ are symmetric operators in H with independent on $x \in \mathbb{R}^n$ domain $D\left(\frac{\partial A}{\partial x_k}\right) = D(A)$. Moreover,

$$\sum_{k=1}^{n}\left(x_k\left[A\frac{\partial f}{\partial x_k} - \frac{\partial A}{\partial x_k}f\right], f\right)_X \geq 0, \quad \text{for } f \in L^\infty\left(0,T;Y^1(A)\right); \tag{3.2}$$

(3) $a > 0$, $b \in \mathbb{R}$; $V = V(x,t) \in B(H)$.

Let

$$|\nabla v|_H^2 = \sum_{k=1}^{n}\left\|\frac{\partial v}{\partial x_k}\right\|^2 \quad \text{for } v \in W^{1,2}(\mathbb{R}^n;H).$$

Lemma 3.2. *Assume that the Condition 3.1 holds. Then the solution u of (3.1) belonging to $L^\infty(0,1;X(A)) \cap L^2\left(0,1;Y^1\right)$ satisfies the following estimate*

$$e^{M_T}\left\|e^{\phi(\cdot,T)}u(\cdot,T)\right\|_X \leq \left\|e^{\gamma|x|^2}u(\cdot,0)\right\|_X + \varkappa\left\|e^{\phi(t)}F\right\|_{L^1(0,T;X)} + a\left\|(Au,u)\right\|_X, \tag{3.3}$$

where

$$\phi(x,t) = \frac{\gamma a|x|^2}{a + 4\gamma\left(a^2+b^2\right)t}, \quad M_T = \|a \operatorname{Re} V - b \operatorname{Im} V\|_B, \quad \varkappa = \sqrt{a^2+b^2}, \quad \gamma \geq 0.$$

Proof. Let $v = e^{\varphi}u$, where φ is a real-valued function to be chosen later. The function v verifies

$$\partial_t v = Sv + Kv + (a+ib)e^{\varphi}F, \ (x,t) \in \mathbb{R}^n \times [0,1],$$

where S, K are symmetric and skew-symmetric operators respectively given by

$$S = aA_1 - ibB_1 + \varphi_t + a\operatorname{Re} V - b\operatorname{Im} V, \ K = ibA_1 - aB_1 + i\left(b\operatorname{Re} v + a\operatorname{Im} v\right),$$

here

$$A_1 = \Delta + A(x) + |\nabla\varphi|^2, \qquad B_1 = 2\nabla\varphi.\nabla + \Delta\varphi.$$

By differentiating the inner product in X, we get

$$\begin{aligned}\partial_t \|v\|_X^2 &= 2\operatorname{Re}(Sv,v)_X + 2\operatorname{Re}(Kv,v)_X \\ &\quad + 2\operatorname{Re}\left((a+ib)e^{v}F,v\right)_X + 2\operatorname{Re}(a+ib)(Vv,v)_X, \qquad t \geq 0.\end{aligned} \tag{3.4}$$

A formal integration by parts gives that

$$\operatorname{Re}(Sv,v)_X = -a\int_{\mathbb{R}^n} |\nabla v|_H^2\,dx + \int_{\mathbb{R}^n}\left(a\,|\nabla\varphi|^2+\varphi_t\right)\|v\|^2\,dx + a\int_{\mathbb{R}^n}(Av,v)\,dx,$$

$$\operatorname{Re}(Kv,v)_X = -a\gamma\int\limits_{\mathbb{R}^n}\left[(2\nabla\varphi.\nabla v,v)+\Delta\varphi\,\|v\|^2\right]dx. \tag{3.5}$$

It is clear that

$$\operatorname{Re}(a+ib)(Vv,v)_X = a\int\limits_{\mathbb{R}^n}(\operatorname{Re}Vv,v)\,dx - b\int\limits_{\mathbb{R}^n}(\operatorname{Im}Vv,v)\,dx,$$

$$\operatorname{Re}\left((a+ib)\,e^{\varphi}F,v\right)_X = a\operatorname{Re}\int\limits_{\mathbb{R}^n}(e^{\varphi}F,v)\,dx - b\operatorname{Im}\int\limits_{\mathbb{R}^n}(e^{\varphi}F,v)\,dx$$
$$= ae^{\varphi}\operatorname{Re}(F,v)_X - be^{\varphi}\operatorname{Im}(F,v)_X.$$

Then by using the Cauchy–Schwarz inequality, by assumptions (2), (3), in view of (3.4) and (3.5) we obtain

$$\partial_t\|v\|_X^2 \leq 2\,\|a\operatorname{Re}V - b\operatorname{Im}V\|_B\,\|v\|_X^2 + 2\varkappa\,\|e^{\varphi}F(t,\cdot)\|_X\,\|v\|_X + a\,\|(Av,v)\|_X,$$

where a, b and φ are such that

$$\left(a+\frac{b^2}{a}\right)|\nabla\varphi|^2+\varphi_t \leq 0 \quad \text{in } \mathbb{R}_+^{n+1}. \tag{3.6}$$

The remaining part of the proof is obtained by reasoning as in [7, Lemma 1] .

When $\varphi(x,t) = q(t)\,\psi(x)$, it suffices that

$$\left(a+\frac{b^2}{a}\right)q^2(t)\,|\nabla\psi|^2 + q'(t)\,\psi(x) \leq 0. \tag{3.7}$$

If we put $\psi(x) = |x|^2$ then (3.6) holds, when

$$q'(t) = -4\left(a+\frac{b^2}{a}\right)q^2(t), \qquad q(0)=\gamma,\ \gamma\geq 0. \tag{3.8}$$

Let

$$\psi_r(x) = \begin{cases} |x|^2, & |x|<r, \\ \infty, & |x|>r. \end{cases}$$

Regularize ψ_r with a radial mollifier θ_ρ and set

$$\varphi_{\rho,r}(x,t) = q(t)\,\theta_\rho * \psi_r(x), \qquad v_{\rho,r}(x,t) = e^{\varphi_{\rho,r}}u,$$

where $q(t) = \gamma a\left[a+4\gamma\left(a^2+b^2\right)t\right]^{-1}$ is the solution to (3.7). Because the right hand side of (3.5) only involves the first derivatives of φ, ψ_r is Lipschitz and bounded at infinity,

$$\theta_\rho * \psi_r(x) \leq \theta_\rho * |x|^2 = C(n)\,\rho^2$$

and (3.6) holds uniformly in ρ and r, when φ is replaced by $\varphi_{\rho,r}$. Hence, it follows that the estimate

$$e^{M_T}\left\|e^{\phi(T)}u(T)\right\|_X \leq M_T\left\|e^{\gamma|x|^2}u(0)\right\|_X + \sqrt{a^2+b^2}\,\|e^{\varphi_{\rho,r}}F\|_{L^1(0,T;X)}$$

holds uniformly in ρ and r. The assertion is obtained after letting ρ tend to zero and r to infinity. □

Remark 3.3. It should be noted that if $H=\mathbb{C}$, $A=0$ and $V(x,t)$ is a complex valued function, then the abstract condition (3.3) can be replaced by

$$M_T=\|a\operatorname{Re}V-b\operatorname{Im}V\|_{L^1(0,T;L^\infty(\mathbb{R}^n))}<\infty.$$

Let

$$Q(t)=(f,f)_X,\quad D(t)=(Sf,f)_X,\quad N(t)=D(t)Q^{-1}(t),\quad \partial_t S=S_t.$$

In a similar way as in [7, Lemma 2] we have the following result.

Lemma 3.4. *Assume that* $S=S(t)$ *is a symmetric,* $K=K(t)$ *is a skew-symmetric operators in* H, $G(x,t)$ *is a positive and* $f(x,t)$ *is an* X*-valued reasonable function. Then,*

$$\begin{aligned} Q''(t)&=2\partial_t\operatorname{Re}(\partial_t f-Sf-Kf,f)_X+2(S_tf+[S,K]f,f)_X\\ &+\|\partial_t f-Sf+Kf\|_X^2-\|\partial_t f-Sf-Kf\|_X^2 \end{aligned} \tag{3.9}$$

and

$$\partial_t N(t)\geq Q^{-1}(t)\left[(S_tf+[S,K]f,f)_X-\frac{1}{2}\|\partial_t f-Sf-Kf\|_X^2\right].$$

Moreover, if

$$\|\partial_t f-Sf-Kf\|_H\leq M_1\|f\|_H+G(x,t),\qquad S_t+[S,K]\geq -M_0$$

for $x\in\mathbb{R}^n$, $t\in[0,1]$ *and*

$$M_2=\sup_{t\in[0,1]}\|G(.,t)\|_{L^2(\mathbb{R}^n)}\left(\|f(\cdot,t)\|_X\right)^{-1}<\infty.$$

Then $Q(t)$ *is logarithmically convex in* $[0,1]$ *and there is a constant* M *such that*

$$Q(t)\leq e^{M\left(M_0+M_1+M_2+M_1^2+M_2^2\right)}Q^{1-t}(0)\,Q^t(1),\qquad 0\leq t\leq 1.$$

Lemma 3.5. *Assume that the Condition 3.1 holds. Moreover, suppose*

$$\sup_{t\in[0,1]}\|V(\cdot,t)\|_B\leq M_1,\qquad \left\|e^{\gamma|x|^2}u(\cdot,0)\right\|_X<\infty,$$

$$\left\|e^{\gamma|x|^2}u(\cdot,1)\right\|_X<\infty,\qquad M_2=\sup_{t\in[0,1]}\frac{\left\|e^{\gamma|x|^2}F(\cdot,t)\right\|_X}{\|u\|_X}<\infty.$$

Then, for solution $u\in L^\infty(0,1;X(A))\cap L^2\left(0,1;Y^1\right)$ *of* (3.1), $e^{\gamma|x|^2}u(\cdot,t)$ *is logarithmically convex in* $[0,1]$ *and there is a constant* N *such that*

$$\left\|e^{\gamma|x|^2}u(\cdot,t)\right\|_X\leq e^{NM(a,b)}\left\|e^{\gamma|x|^2}u(\cdot,0)\right\|_X^{1-t}\left\|e^{\gamma|x|^2}u(\cdot,1)\right\|_X^{t}, \tag{3.10}$$

where

$$M(a,b)=\varkappa^2\left(\gamma M_1^2+M_2^2\right)+\varkappa\left(M_1+M_2\right)$$

when $0\leq t\leq 1$.

Proof. Let $f = e^{\gamma\varphi}u$, where φ is a real-valued function to be chosen. The function $f(x)$ verifies

$$\partial_t f = Sf + Kf + (a+ib)\left(Vf + e^{\gamma\varphi}F\right) \quad \text{in } \mathbb{R}^n \times [0,1], \tag{3.11}$$

where S, K are symmetric and skew-symmetric operator, respectively given by

$$S = aA_1 - ib\gamma B_1 + \varphi_t + a\operatorname{Re}V - b\operatorname{Im}V,$$
$$K = ibA_1 - a\gamma B_1 + i\left(b\operatorname{Re}v + a\operatorname{Im}v\right),$$

where

$$A_1 = \Delta + A(x) + \gamma^2 |\nabla\varphi|^2, \qquad B_1 = 2\nabla\varphi.\nabla + \Delta\varphi.$$

A calculation shows that,

$$\begin{aligned} S_t + [S,K] = {} & \gamma\partial_t^2\varphi + 2\gamma^2 a\nabla\varphi.\nabla\varphi_t - 2ib\gamma\left(2\nabla\varphi_t.\nabla + \Delta\varphi_t\right) \\ & - \gamma\varkappa^2\left[4\nabla.\left(D^2\varphi\nabla\right) - 4\gamma^2 D^2\varphi\nabla\varphi + \Delta^2\varphi\right] + 2\left[A(x)\nabla\varphi.\nabla - \nabla\varphi.\nabla A\right]. \end{aligned} \tag{3.12}$$

If we put $\varphi = |x|^2$, then (3.12) reduces to the following

$$S_t + [S,K] = -\gamma\varkappa^2\left[8\Delta - 32\gamma^2|x|^2\right] + \frac{d}{dt}A + 2\left[A(x)\nabla\varphi.\nabla - \nabla\varphi.\nabla A\right].$$

Moreover by assumption (2),

$$\begin{aligned} (S_t f + [S,K]f, f) = {} & \gamma\varkappa\int_{\mathbb{R}^n}\left(8|\nabla f|_H^2 + 32\gamma^2|x|^2\|f\|^2\right)dx \\ & + 2\int_{\mathbb{R}^n}\left(\left[A(x)\nabla\varphi.\nabla f - \nabla\varphi.\nabla Af\right], f\right)dx \geq 0. \end{aligned} \tag{3.13}$$

This identity, the condition on V and (3.13) imply that

$$\|\partial_t f - Sf - Kf\|_X \leq \varkappa^2\left(M_1\|f\|_X + e^{\gamma\varphi}\|F\|_X\right). \tag{3.14}$$

If we knew that the quantities and calculations involved in the proof of Lemma 3.4 (similar as in [7, Lemma 2]) were finite and correct, when $f = e^{\gamma|x|^2}u$ we would have the logarithmic convexity of $Q(t) = \left\|e^{\gamma|x|^2}u(\cdot,t)\right\|_X$ and the estimate (3.10) from Lemma 3.4. But this fact is verifying by reasoning as in [7, Lemma 3]. □

Let

$$\sigma = \sqrt{t(1-t)}e^{\gamma|x|^2}, \qquad Y = L^2\left([0,1]\times\mathbb{R}^n; H\right).$$

Lemma 3.6. *Assume that a, b, u, A and V are as in Lemma 3.5 and $\gamma > 0$. Then,*

$$\|\sigma\nabla u\|_Y + \|\sigma|x|u\|_Y \leq N\left[(1+M_1)\right]\left[\sup_{t\in[0,1]}\left\|e^{\gamma|x|^2}u(.,t)\right\|_X + \sup_{t\in[0,1]}\left\|e^{\gamma|x|^2}F(.,t)\right\|_Y\right],$$

where N is bounded number, when γ and $\varkappa$ are bounded below.

Proof. The integration by parts shows that

$$\int_{\mathbb{R}^n}\left(|\nabla f|_H^2+4\gamma^2|x|^2\|f\|\right)dx=\int_{\mathbb{R}^n}\left[e^{2\gamma|x|^2}\left(|\nabla u|_H^2-2n\gamma\right)\|u\|^2\right]dx,$$

when $f=e^{\gamma|x|^2}u$, while integration by parts, the Cauchy–Schwarz inequality and the identity, $n=\nabla\cdot x$, give that

$$\int_{\mathbb{R}^n}\left(|\nabla f|_H^2+4\gamma^2|x|^2\|f\|\right)dx\geq 2\gamma n\|f\|_X^2.$$

The sum of the last two formulae gives the inequality

$$2\int_{\mathbb{R}^n}\left(|\nabla f|_H^2+4\gamma^2|x|^2\|f\|\right)dx\geq\int_{\mathbb{R}^n}e^{\gamma|x|^2}|\nabla f|_H^2\,dx. \tag{3.15}$$

Integration over $[0,1]$ of $t(1-t)$ times the formula (3.6) for $Q''(t)$ and integration by parts, shows that

$$\begin{aligned}&2\int_0^1 t(1-t)\left(S_tf+[S,K]f,f\right)_X dt+\int_0^1 Q(t)\,dt\\&\leq Q(1)+Q(0)+2\int_0^1(1-2t)\operatorname{Re}\left(\partial_tf-Sf-Kf,f\right)_X dt\\&\quad+\int_0^1 t(1-t)\|\partial_tf-Sf-Kf\|_X^2\,dt.\end{aligned} \tag{3.16}$$

Assuming again that the last two calculations are justified for $f=e^{\gamma|x|^2}$. Then (3.13)–(3.16) imply the assertion. □

4 Appell transformation in abstract function spaces

Let

$$\rho(t)=\alpha(1-t)+\beta t,\qquad\varphi(x,t)=\frac{(\alpha-\beta)\left|x^2\right|}{4(a+ib)\rho(t)},$$
$$\nu(s)=\left[\gamma\alpha\beta\rho^2(s)+\frac{(\alpha-\beta)a}{4(a^2+b^2)}\rho(s)\right].$$

Lemma 4.1. *Assume A and V are as in Lemma 3.5 and $u=u(x,s)$ is a solution of the equation*

$$\partial_su=(a+ib)\left[\Delta u+Au+V(y,s)u+F(y,s)\right],\qquad y\in\mathbb{R}^n,\ s\in[0,1].$$

Let $a+ib\neq 0$, $\gamma\in\mathbb{R}$ and α, $\beta\in\mathbb{R}_+$. Set

$$\tilde{u}(x,t)=\left(\sqrt{\alpha\beta}\rho^{-1}(t)\right)^{\frac{n}{2}}u\left(\sqrt{\alpha\beta}x\rho^{-1}(t),\beta t\rho^{-1}(t)\right)e^{\varphi}. \tag{4.1}$$

Then, $\tilde{u}(x,t)$ verifies the equation

$$\partial_t\tilde{u}=(a+ib)\left[\Delta\tilde{u}+A\tilde{u}+\tilde{V}(x,t)u+\tilde{F}(x,t)\right],\qquad x\in\mathbb{R}^n,\ t\in[0,1]$$

with

$$\tilde{V}(x,t)=\alpha\beta\rho^{-2}(t)V\left(\sqrt{\alpha\beta}x\rho^{-1}(t),\beta t\rho^{-1}(t)\right),$$

$$\tilde{F}(x,t) = \left(\sqrt{\alpha\beta}\rho^{-1}(t)\right)^{\frac{n}{2}+2}\left(\sqrt{\alpha\beta}x\rho^{-1}(t),\beta t\rho^{-1}(t)\right).$$

Moreover,

$$\left\|e^{\gamma|x|^2}\tilde{F}(\cdot,t)\right\|_X = \alpha\beta\rho^{-2}(t)\,e^{\nu|y|^2}\|F(s)\|_X\,,$$
$$\left\|e^{\gamma|x|^2}\tilde{u}(\cdot,t)\right\|_X = e^{\nu|y|^2}\|u(s)\|_X\,,$$

when $s = \mu(t)$ *and* $\gamma \in \mathbb{R}$.

Proof. If u is a solution of the equation

$$\partial_s u = (a+ib)\left[\Delta u + Au + Q(y,s)\right], \qquad y \in \mathbb{R}^n,\ s \in [0,1] \tag{4.2}$$

then, the function $u_1(x,t) = u\left(\sqrt{r}x, rt+\tau\right)$ verifies

$$\partial_t u_1 = (a+ib)\left[\Delta u_1 + Au_1 + rQ\left(\sqrt{r}x, rt+\tau\right)\right], \qquad y \in \mathbb{R}^n,\ s \in [0,1]$$

and $u_2(x,t) = t^{-\frac{n}{2}}u\left(\frac{x}{t},\frac{1}{t}\right)e^{\frac{|x|^2}{4(a+ib)t}}$ is a solution to

$$\partial_t u_2 = -(a+ib)\left[\Delta u_2 + Au + t^{-\left(2+\frac{n}{2}\right)}Q\left(\frac{x}{t},\frac{1}{t}\right)e^{\frac{|x|^2}{4(a+ib)t}}\right], \qquad y \in \mathbb{R}^n,\ s \in [0,1].$$

These two facts and the sequel of changes of variables below verify the lemma, when $\alpha > \beta$, i.e.

$$u\left(\sqrt{\frac{\alpha\beta}{\alpha-\beta}}x, \frac{\alpha\beta}{\alpha-\beta}t - \frac{\beta}{\alpha-\beta}\right)$$

is a solution to the same non-homogeneous equation but with right-hand side

$$\frac{\alpha\beta}{\alpha-\beta}Q\left(\sqrt{\frac{\alpha\beta}{\alpha-\beta}}x, \frac{\alpha\beta}{\alpha-\beta}t - \frac{\beta}{\alpha-\beta}\right).$$

The function,

$$\frac{1}{(\alpha-t)^{\frac{n}{2}}}u\left(\frac{\sqrt{\alpha\beta}x}{\sqrt{\alpha-\beta}(\alpha-t)}, \frac{\alpha\beta}{(\alpha-\beta)(\alpha-t)} - \frac{\beta}{\alpha-\beta}\right)e^{\frac{|x|^2}{4(a+ib)(\alpha-t)}}$$

verifies (4.2) with right-hand side

$$\frac{\alpha\beta}{(\alpha-\beta)(\alpha-t)^{\frac{n}{2}+2}}Q\left(\frac{\sqrt{\alpha\beta}x}{\sqrt{\alpha-\beta}(\alpha-t)}, \frac{\alpha\beta}{(\alpha-\beta)(\alpha-t)} - \frac{\beta}{\alpha-\beta}\right)e^{\frac{|x|^2}{4(a+ib)(\alpha-t)}}.$$

Replacing (x,t) by $\left(\sqrt{\alpha-\beta}x, (\alpha-\beta)t\right)$ we get that

$$\rho^{-\frac{n}{2}}(t)\,u\left(\sqrt{\alpha\beta}\rho^{-1}(t)\,x, \frac{\alpha\beta\rho^{-1}(t)-\beta}{\alpha-\beta}\right)e^{(\alpha-\beta)\frac{|x|^2\rho(t)}{4(a+ib)}} \tag{4.3}$$

is a solution of (4.2) but with right-hand

$$\rho^{-\left(\frac{n}{2}+2\right)}(t)\,Q\left(\sqrt{\alpha\beta}\rho^{-1}(t)\,x, \frac{\alpha\beta\rho^{-1}(t)-\beta}{\alpha-\beta}\right)e^{(\alpha-\beta)\frac{|x|^2\rho(t)}{4(a+ib)}}. \tag{4.4}$$

Finally, observe that

$$s = \beta t\rho(t) = \frac{\alpha\beta\rho^{-1}(t)-\beta}{\alpha-\beta}$$

and multiply (4.3) and (4.4) we obtain the assertion for $\alpha > \beta$. The case $\beta > \alpha$ follows by reversing by changes of variables, $s' = 1-s$ and $t' = 1-t$. □

5 Variable coefficients. Proof of Theorem 2.4

We are ready to prove Theorem 2.4. Let

$$B = L^1\left(0,1;L^\infty\left(\mathbb{R}^n;B\left(H\right)\right)\right).$$

Proof of Theorem 2.4. We may assume that $\alpha \neq \beta$. The case $\alpha = \beta$ follows from the latter by replacing β by $\beta+\delta$, $\delta > 0$, and letting δ tend to zero. We may also assume that $\alpha < \beta$. Otherwise, replace u by $\bar{u}(1-t)$. Assume $a > 0$. Set $W = \Delta + A + V_1$. By Lemma 2.7 the problem

$$\partial_t u = \left(a+ib\right)\left[\Delta u + A\left(x\right)u + V_1\left(x\right)u\right], \qquad x \in \mathbb{R}^n,\ t \in [0,1], \tag{5.1}$$
$$u\left(x,0\right) = u_0\left(x\right).$$

has a solution $u = U\left(t\right)u_0 = e^{t(a+ib)W}u_0 \in C\left([0,1];X\left(A\right)\right)$, where

$$U\left(t\right) = F^{-1}\left[e^{-Q(\xi)}\right], \qquad Q\left(\xi\right) = \left(a+ib\right)\left[-\left|\xi\right|^2 + \hat{A}\left(\xi\right) + \hat{V}_1\left(\xi\right)\right],$$

here, F^{-1} is the inverse Fourier transform, $\hat{A}\left(\xi\right)$, $\hat{V}_1\left(\xi\right)$ respectively denote the Fourier transforms of $A\left(x\right)$, $V_1\left(x\right)$. By reasoning as the Duhamel principle we get that the problem

$$\partial_t u = \left(a+ib\right)\left[\Delta u + A\left(x\right)u + V\left(x,t\right)\right]u, \qquad x \in \mathbb{R}^n,\ t \in [0,1],$$

$$u\left(x,0\right) = u_0\left(x\right)$$

has a solution expressing as

$$u\left(x,t\right) = H\left(t\right)u_0 + i\int_0^t H\left(t-s\right)V_2\left(x,s\right)u\left(x,s\right)ds \quad \text{for } x \in \mathbb{R}^n,\ s \in [0,1], \tag{5.2}$$

where

$$e^{itW} = H\left(t\right) = H\left(t,x\right) = F^{-1}\left[e^{iQ(\xi)}\right].$$

For $0 \leq \varepsilon \leq 1$ set

$$F_\varepsilon\left(x,t\right) = \frac{i}{\varepsilon+i}e^{\varepsilon tW}V_2\left(x,t\right)u\left(x,t\right) \tag{5.3}$$

and

$$u_\varepsilon\left(x,t\right) = e^{(\varepsilon+i)tW}u_0 + \left(\varepsilon+i\right)\int_0^t e^{(\varepsilon+i)(t-s)W}F_\varepsilon\left(x,s\right)u\left(x,s\right)ds. \tag{5.4}$$

Then, $u_\varepsilon\left(x,t\right) \in L^\infty\left(0,1;X\left(A\right)\right) \cap L^2\left(\mathbb{R}^n;Y^1\right)$ and satisfies

$$\partial_t u_\varepsilon = \left(\varepsilon+i\right)\left(Wu + F_\varepsilon\right) \quad \text{in } \mathbb{R}^n \times [0,1],$$

$$u_\varepsilon\left(\cdot,0\right) = u_0\left(\cdot\right).$$

The identities

$$e^{(z_1+z_2)W} = e^{z_1W}e^{z_2W}, \quad \text{when } \operatorname{Re} z_1,\ \operatorname{Re} z_2 \geq 0,$$

and (5.2)–(5.4) show that

$$u_\varepsilon\left(x,t\right) = e^{\varepsilon tW}u\left(x,t\right), \quad \text{for } t \in [0,1]. \tag{5.5}$$

In particular, the equality $u_\varepsilon(x,1)=e^{\varepsilon W}u(x,1)$, Lemma 3.2 with $a+ib=\varepsilon$, $\gamma=\frac{1}{\beta}$, $F\equiv 0$ and the fact that $u_\varepsilon(0)=u(0)$ imply that

$$\left\|e^{\frac{|x|^2}{\beta^2+4\varepsilon}}u_\varepsilon(\cdot,1)\right\|_X\leq e^{\varepsilon\|V_1\|_B}\left\|e^{\frac{|x|^2}{\beta^2}}u(\cdot,1)\right\|_X,$$
$$\left\|e^{\frac{|x|^2}{\alpha^2}}u_\varepsilon(\cdot,0)\right\|_X=\left\|e^{\frac{|x|^2}{\alpha^2}}u(\cdot,0)\right\|_X.$$

A second application of Lemma 3.2 with $a+ib=\varepsilon$, $F\equiv 0$, the value of $\gamma=\mu^{-2}(t)$ and (5.2) show that
$$\left\|\varepsilon^{\frac{|x|^2}{\mu^2(t)+4\varepsilon t}}F_\varepsilon(\cdot,t)\right\|_X\leq e^{\varepsilon\|V_1\|_B}\left\|\varepsilon^{\frac{|x|^2}{\mu^2(t)}}V_2(\cdot,t)\right\|_B\|u(\cdot,t)\|_X$$

for $t\in[0,1]$. Setting, $\alpha_\varepsilon=\alpha+2\varepsilon$ and $\beta_\varepsilon=\beta+2\varepsilon$, the last three inequalities give that

$$\left\|e^{\frac{|x|^2}{\beta_\varepsilon^2}}u_\varepsilon(\cdot,1)\right\|_X\leq e^{\varepsilon\|V_1\|_B}\left\|e^{\frac{|x|^2}{\beta^2}}u(\cdot,1)\right\|_X, \tag{5.6}$$

$$\left\|e^{\frac{|x|^2}{\alpha_\varepsilon^2}}u_\varepsilon(\cdot,0)\right\|_X\leq e^{\varepsilon\|V_1\|_B}\left\|e^{\frac{|x|^2}{\alpha^2}}u(\cdot,0)\right\|_X,$$
$$\left\|\varepsilon^{|x|^2\mu^{-2}(t)}F_\varepsilon(\cdot,t)\right\|_X\leq e^{\varepsilon\|V_1\|_B}\left\|\varepsilon^{|x|^2\mu^{-2}(t)}V_2(\cdot,t)\right\|_B\|u(\cdot,t)\|_X,\qquad t\in[0,1]. \tag{5.7}$$

A third application of Lemma 3.2 with $a+ib=b$, $F\equiv 0$, $\gamma=0$, and (5.2), (5.5) implies that

$$\|F_\varepsilon(\cdot,t)\|_X\leq e^{\varepsilon\|V_1\|_B}\|V_2(\cdot,t)\|_B\|u(\cdot,t)\|_X, \tag{5.8}$$
$$\|u_\varepsilon(\cdot,t)\|_X\leq e^{\varepsilon\|V_1\|_B}\|u(\cdot,t)\|_X,\qquad t\in[0,1].$$

Set $\gamma_\varepsilon=\frac{1}{\alpha_\varepsilon\beta_\varepsilon}$ and let

$$\tilde{u}_\varepsilon(x,t)=\left(\sqrt{\alpha_\varepsilon\beta_\varepsilon}\rho_\varepsilon^{-1}(t)\right)^{\frac{n}{2}}u\left(\sqrt{\alpha_\varepsilon\beta_\varepsilon}x\rho_\varepsilon^{-1}(t),\beta_\varepsilon t\rho_\varepsilon^{-1}(t)\right)e^{\varphi_\varepsilon}$$

be the function associated to u_ε in Lemma 4.1, where $a+ib=\varepsilon+i$ and α, β are replaced respectively by α_ε, β_ε when

$$\rho_\varepsilon(t)=\alpha_\varepsilon(1-t)+\beta_\varepsilon t,\qquad \varphi_\varepsilon=\varphi_\varepsilon(x,t)=\frac{(\alpha_\varepsilon-\beta_\varepsilon)|x|^2}{4(a+ib)\rho_\varepsilon(t)}.$$

Because $\alpha<\beta$, $\tilde{u}_\varepsilon\in L^\infty(0,1;X)\cap L^2\left(0,1;Y^1\right)$ and satisfies the equation

$$\partial_t\tilde{u}_\varepsilon=(\varepsilon+i)\left(\Delta\tilde{u}_\varepsilon+A(x)\tilde{u}_\varepsilon+\tilde{V}_1^\varepsilon(x,t)\tilde{u}_\varepsilon+\tilde{F}_\varepsilon(x,t)\right)\quad\text{in }\mathbb{R}^n\times[0,1],$$

where

$$\tilde{V}_1^\varepsilon(x,t)=\alpha_\varepsilon\beta_\varepsilon\rho_\varepsilon^{-2}(t)V_1\left(\sqrt{\alpha_\varepsilon\beta_\varepsilon}\rho_\varepsilon^{-1}(t)x\right),\qquad\sup_{t\in[0,1]}\left\|\tilde{V}_1^\varepsilon(\cdot,t)\right\|_B\leq\frac{\beta}{\alpha}M_1,$$

$$\tilde{F}_\varepsilon(x,t)=\left[\sqrt{\alpha_\varepsilon\beta_\varepsilon}\rho_\varepsilon^{-1}(t)\right]^{\frac{n}{2}+2}F_\varepsilon\left(\sqrt{\alpha_\varepsilon\beta_\varepsilon}\rho_\varepsilon^{-1}(t)x,\beta_\varepsilon t\rho_\varepsilon^{-1}(t)\right)e^{\varphi_\varepsilon}, \tag{5.9}$$

$$\left\| e^{\gamma_\varepsilon |x|^2} \tilde{F}_\varepsilon (\cdot, t) \right\|_X \leq \frac{\beta}{\alpha} \left\| e^{\rho_\varepsilon^{-2} |x|^2} F_\varepsilon (\cdot, t) \right\|_X, \quad (5.10)$$

$$\left\| \tilde{F}_\varepsilon (., t) \right\|_X \leq \frac{\beta}{\alpha} \left\| F_\varepsilon (., t) \right\|_X$$

and

$$\left\| e^{\gamma_\varepsilon |x|^2} \tilde{u}_\varepsilon (\cdot, t) \right\|_X = \left\| e^{\left[\rho_\varepsilon^{-2}(s) + \varphi_\varepsilon(s,t)\right] |y|^2} u_\varepsilon (\cdot, s) \right\|_X, \quad (5.11)$$

$$\left\| \tilde{u}_\varepsilon (\cdot, t) \right\|_X \leq \left\| u (\cdot, s) \right\|_X,$$

when $s = \beta_\varepsilon \rho_\varepsilon^{-1} (t)$. The above identity when t is zero or one and (5.6) shows that

$$\left\| e^{\gamma_\varepsilon |x|^2} \tilde{u}_\varepsilon (\cdot, 0) \right\|_X \leq \left\| e^{\frac{|x|^2}{\beta^2}} u (\cdot, 0) \right\|_X,$$

$$\left\| e^{\gamma_\varepsilon |x|^2} \tilde{u}_\varepsilon (\cdot, 1) \right\|_X \leq e^{\varepsilon \|V\|_B} \left\| V_2 (\cdot, t) \right\|_B \left\| e^{\frac{|x|^2}{\beta^2}} u (\cdot, 1) \right\|_X. \quad (5.12)$$

On the other hand,

$$N_1^{-1} \left\| u (\cdot, 0) \right\|_X \leq \left\| u (\cdot, t) \right\|_X \leq N_1 \left\| u (\cdot., 0) \right\|_X, \qquad t \in [0, 1], \quad (5.13)$$

where

$$N_1 = e^{B(V_2)}, \qquad B (V_2) = \sup_{t \in [0,1]} \left\| \operatorname{Re} V_2 (\cdot, t) \right\|_B.$$

The energy method imply that

$$\partial_t \left\| \tilde{u}_\varepsilon (\cdot, t) \right\|_X^2 \leq 2\varepsilon \left\| \tilde{V}_1^\varepsilon (x, t) \right\|_B \left\| \tilde{u}_\varepsilon (\cdot, t) \right\|_X^2 + 2 \left\| \tilde{F}_\varepsilon (x, t) \right\|_X \left\| \tilde{u}_\varepsilon (\cdot, t) \right\|_X. \quad (5.14)$$

Let $0 = t_0 < t_1 < \cdots < t_m = 1$ be a uniformly distributed partition of $[0, 1]$, where m will be chosen later. The inequalities (5.8)–(5.10), (5.13) and (5.14) imply that there is N_2, which depends on $\frac{\beta}{\alpha}$, $\|V_1\|_B$ and $B(V_2)$ such that

$$\left\| \tilde{u}_\varepsilon (\cdot, t_i) \right\|_X \leq e^{\frac{\varepsilon\beta}{\alpha} \|V_1\|_B} \left\| \tilde{u}_\varepsilon (\cdot, t) \right\|_X + N_2 \sqrt{t_i - t_{i-1}} \left\| u (\cdot, 0) \right\|_X \quad (5.15)$$

for $t \in [t_{i-1}, t_i]$ and $i = 1, 2, \ldots, m$. Choose now m so that

$$N_2 \max_i \sqrt{t_i - t_{i-1}} \leq \frac{1}{4N_1}. \quad (5.16)$$

Because $\lim_{\varepsilon \to 0} \left\| \tilde{u}_\varepsilon (\cdot, t) \right\|_X = \left\| u (\cdot, s) \right\|_X$ when $s = \beta t \rho (t)$ and (5.13), there is ε_0 such that

$$\left\| \tilde{u}_\varepsilon (\cdot, t_i) \right\|_X \geq \frac{1}{4N_1} \left\| u (\cdot, 0) \right\|_X, \quad \text{when } 0 < \varepsilon \leq \varepsilon_0, \ i = 1, 2, \ldots, m \quad (5.17)$$

and now, (5.15)–(5.17) show that

$$\left\| \tilde{u}_\varepsilon (\cdot, t) \right\|_X \geq \frac{1}{4N_1} \left\| u (\cdot, 0) \right\|_X, \quad \text{when } 0 < \varepsilon \leq \varepsilon_0, \ t \in [0, 1]. \quad (5.18)$$

It is now simple to verify that (5.18), the first inequality in (5.7), (5.10) and (5.13) imply that

$$\sup_{t \in [0,1]} \frac{\left\| e^{\gamma_\varepsilon |x|^2} \tilde{F}_\varepsilon (\cdot, t) \right\|_X}{\left\| \tilde{u}_\varepsilon (\cdot, t) \right\|_X} \leq \frac{4\beta}{\alpha} M_2 (\varepsilon), \quad \text{when } 0 < \varepsilon \leq \varepsilon_0, \quad (5.19)$$

where

$$M_2(\varepsilon) = e^{\sup\limits_{t\in[0,1]} \|\operatorname{Re} V_2(\cdot,t)\|_B + \varepsilon \|V_1\|_B} \sup_{t\in[0,1]} \left\| e^{|x|^2 \mu^{-2}(t)} V_2(\cdot,t) \right\|_B.$$

By using Lemma 3.5, (5.12), (5.9) and (5.19) to show that $\left\|e^{\gamma_\varepsilon |x|^2} \tilde{u}_\varepsilon(\cdot,t)\right\|_X$ is logarithmically convex in $[0,1]$ and that

$$\left\| e^{\gamma |x|^2} \tilde{u}_\varepsilon(\cdot,t) \right\|_X \leq e^{NM(a,b)} \left\| e^{\gamma |x|^2} \tilde{u}_\varepsilon(0) \right\|_X^{1-t} \left\| e^{\gamma |x|^2} \tilde{u}_\varepsilon(1) \right\|_X^{t}, \tag{5.20}$$

when $0 < \varepsilon \leq \varepsilon_0$, $t \in [0,1]$ and $N = N(\alpha,\beta)$. Then, Lemma 3.6 gives that

$$\begin{aligned}
&\|\eta \nabla \tilde{u}_\varepsilon\|_Z + \|\eta |x| \tilde{u}_\varepsilon\|_Z \\
&\leq N(1+M_1)\left[\sup_{t\in[0,1]} \left\| e^{\gamma|x|^2} \tilde{u}_\varepsilon(\cdot,t) \right\|_X + \sup_{t\in[0,1]} \left\| e^{\gamma|x|^2} \tilde{F}_\varepsilon(\cdot,t) \right\|_Y \right] \\
&\leq N e^{N\left(M_0+M_1+M_2(\varepsilon)+M_1^2+M_2^2(\varepsilon)\right)} \left[\left\| e^{\frac{|x|^2}{\beta^2}} u(\cdot,0) \right\|_X + \left\| e^{\frac{|x|^2}{\alpha^2}} u(\cdot,1) \right\|_X \right],
\end{aligned}$$

when $0 < \varepsilon \leq \varepsilon_0$, the logarithmic convexity and regularity of u follow from the limit of the identity in (5.11), the final limit relation between the variables s and t, $s = \beta t \rho(t)$ and letting ε tend to zero in (5.20) and the above inequality. □

By reasoning as in [4, Lemma 6] we obtain the following lemma.

Lemma 5.1. *Let the assumption (1) of the Condition 3.1 holds and $\|V\|_B \leq \varepsilon_0$ for a $\varepsilon_0 > 0$. Let $u \in C([0,1];X(A))$ be a solution of the equation*

$$\partial_t u = i\left[\Delta u + Au + V(x,t)u + F(x,t)\right], \qquad x \in \mathbb{R}^n,\ t \in [0,1].$$

Then,

$$\sup_{t\in[0,1]} \left\| e^{\lambda\cdot x} u(\cdot,t) \right\|_X \leq N\left[\left\| e^{\lambda\cdot x} u(\cdot,0) \right\|_X + \left\| e^{\lambda\cdot x} u(\cdot,1) \right\|_X + \left\| e^{\lambda\cdot x} F(\cdot,t) \right\|_{L^1(0,1;X)} \right],$$

where $\lambda \in \mathbb{R}^n$ and $N > 0$ is a constant.

Theorem 5.2. *Let the assumption (1) of the Condition 3.1 hold and*

$$V \in B \quad \text{and} \quad \lim_{r\to\infty} \|V\|_{O(r)} = 0.$$

Suppose that α, β are positive numbers and

$$\left\| e^{\frac{|x|^2}{\beta^2}} u(\cdot,0) \right\|_X < \infty, \qquad \left\| e^{\frac{|x|^2}{\alpha^2}} u(\cdot,1) \right\|_X < \infty.$$

Let $u \in C([0,1];X(A))$ be a solution of the equation

$$\partial_t u = i\left[\Delta u + A(x)u + V(x,t)u\right], \qquad x \in \mathbb{R}^n,\ t \in [0,1].$$

Then, there is a $N = N(\alpha, \beta)$ *such that*

$$\sup_{t\in[0,1]} \left\| e^{|x|^2\mu^{-2}(t)} u(\cdot,t) \right\|_X + \left\| \sqrt{t(1-t)} e^{|x|^2\mu^{-2}(t)} \nabla u \right\|_{L^2(\mathbb{R}^n\times[0,1];H)}$$
$$\leq N e^{B(V)} \left[\left\| e^{\frac{|x|^2}{\beta^2}} u(\cdot,0) \right\|_X + \left\| e^{\frac{|x|^2}{\alpha^2}} u(\cdot,1) \right\|_X + \sup_{t\in[0,1]} \|u(\cdot,t)\|_X \right],$$

where

$$B(V) = \sup_{t\in[0,1]} \|V\|_B .$$

Proof. Assume that $u(y,s)$ verifies the equation

$$\partial_s u = i\left[\Delta u + A(y)u + V(y,s)u + F(y,s)\right], \qquad y \in \mathbb{R}^n,\ s \in [0,1].$$

Set $\gamma = (\alpha\beta)^{-1}$ and let

$$\tilde{u}(x,t) = \left(\sqrt{\alpha\beta}\rho^{-1}(t)\right)^{\frac{n}{2}} u\left(\sqrt{\alpha\beta}x\rho^{-1}(t), \beta t\rho^{-1}(t)\right) e^{\varphi}. \tag{5.21}$$

The function (5.21) is a solution of

$$\partial_t u = i\left[\Delta u + A(x)u + V(x,t)u\right], \qquad x \in \mathbb{R}^n,\ t \in [0,1]$$

with

$$\tilde{V}(x,t) = \alpha\beta\delta^{-2}(t)\, V\left(\sqrt{\alpha\beta}x\rho^{-1}(t), \beta t\rho^{-1}(t)\right),$$

$$\sup_{t\in[0,1]} \left\|\tilde{V}(\cdot,t)\right\|_B \leq \max\left(\frac{\alpha}{\beta}, \frac{\beta}{\alpha}\right) \sup_{t\in[0,1]} \|V(\cdot,t)\|_B, \qquad \lim_{r\to\infty} \left\|\tilde{V}(\cdot,t)\right\|_{O(r)} = 0$$

and

$$\left\| e^{\gamma|x|^2} \tilde{u}(\cdot,t) \right\|_X = \left\| e^{\mu^2(t)|x|^2} u(\cdot,s) \right\|_X, \tag{5.22}$$
$$\|\tilde{u}(\cdot,t)\|_X = \|u(\cdot,s)\|_X \quad \text{when } s = \beta t\mu(t).$$

Choose $r > 0$ such that $\left\|\tilde{V}(\cdot,t)\right\|_{O(r)} \leq \varepsilon_0$ we get

$$\partial_t \tilde{u} = i\left[\Delta\tilde{u} + A\tilde{u} + \tilde{V}_r(x,t)u + \tilde{F}_r(x,t)\right], \qquad x \in \mathbb{R}^n,\ t \in [0,1],$$

with

$$\tilde{V}_r(x,t) = \chi_{\mathbb{R}^n/O_r} \tilde{V}(x,t), \qquad \tilde{F}_r(x,t) = \chi_{O_r} \tilde{V}(x,t)\,\tilde{u}.$$

Then using the Lemma 5.1 we obtain

$$\sup_{t\in[0,1]} \left\| e^{\lambda\cdot x}\tilde{u}(\cdot,t) \right\|_X \leq N\left[\left\| e^{\lambda\cdot x}\tilde{u}(\cdot,0) \right\|_X + \left\| e^{\lambda\cdot x}\tilde{u}(\cdot,1) \right\|_X + e^{|\lambda|r} \left\|\tilde{V}(\cdot,t)\right\|_B \sup_{t\in[0,1]} \|u(\cdot,t)\|_X \right].$$

Replace λ by $\lambda\sqrt{\gamma}$ in the above inequality, square both sides, multiply all by $e^{-\frac{|\lambda^2|}{2}}$ and integrate both sides with respect to λ in $\mathbb{R}^n$. This and the identity

$$\int_{\mathbb{R}^n} e^{2\sqrt{\gamma}\lambda . x - \frac{|\lambda|^2}{2}} d\lambda = (2\pi)^{\frac{n}{2}} e^{2\gamma|x|^2}$$

imply the inequality

$$\sup_{t\in[0,1]}\|\tilde{u}(\cdot,t)\|_X$$
$$\leq N\left[\left\|e^{2\gamma|x|^2}\tilde{u}(\cdot,0)\right\|_X+\left\|e^{2\gamma|x|^2}\tilde{u}(\cdot,1)\right\|_X+\left\|e^{2\gamma r^2}\tilde{V}(\cdot,t)\right\|_B\sup_{t\in[0,1]}\|\tilde{u}(\cdot,t)\|_X\right]. \quad (5.23)$$

This inequality and (5.22) imply that

$$\sup_{t\in[0,1]}\|\tilde{u}(\cdot,t)\|_X\leq N\left[\left\|e^{\frac{|x|^2}{\beta^2}}\tilde{u}(\cdot,0)\right\|_X+\left\|e^{\frac{|x|^2}{\beta^2}}\tilde{u}(\cdot,1)\right\|_X+\sup_{t\in[0,1]}\|V(\cdot,t)\|_B\sup_{t\in[0,1]}\|u(\cdot,t)\|_X\right]$$

for some new constant N.

To prove the regularity of u we proceed as in (5.2)–(5.4). The Duhamel formula shows that

$$u_\varepsilon(x,t)=e^{itW}u_0+i\int_0^t e^{i(t-s)W}V_2(x,s)\,u(x,s)\,ds,\qquad x\in\mathbb{R}^n,\ t\in[0,1]. \quad (5.24)$$

For $0\leq\varepsilon\leq 1$, set

$$\tilde{F}_\varepsilon(x,t)=\frac{i}{\varepsilon+i}e^{\varepsilon t(\Delta+A)}\tilde{V}(x,t)\,\tilde{u}(x,t), \quad (5.25)$$

and

$$\tilde{u}_\varepsilon(x,t)=e^{(\varepsilon+i)t(\Delta+A)}u_0+(\varepsilon+i)\int_0^t e^{(\varepsilon+i)(t-s)(\Delta+A)}\tilde{F}(x,s)\,u(x,s)\,ds,$$
$$x\in\mathbb{R}^n,\ t\in[0,1]. \quad (5.26)$$

The identities

$$e^{(z_1+z_2)(\Delta+A)}=e^{z_1(\Delta+A)}.e^{z_2(\Delta+A)}\quad\text{for }\operatorname{Re}z_1,\ \operatorname{Re}z_2\geq 0$$

and (5.24)–(5.26) show that

$$\tilde{u}_\varepsilon(x,t)=e^{\varepsilon t(\Delta+A)}\tilde{u}(x,t)\quad\text{for }t\in[0,1]. \quad (5.27)$$

From Lemma 3.2 with $a+ib=\varepsilon$, (5.27) and (5.25) we get that

$$\begin{aligned}\sup_{t\in[0,1]}\left\|e^{\gamma_\varepsilon|x|^2}\tilde{u}_\varepsilon(\cdot,t)\right\|_X&\leq\sup_{t\in[0,1]}\left\|e^{\gamma|x|^2}\tilde{u}(\cdot,t)\right\|_X,\\ \sup_{t\in[0,1]}\left\|e^{\gamma_\varepsilon|x|^2}\tilde{F}_\varepsilon(\cdot,t)\right\|_X&\leq e^{\tilde{V}_0}\sup_{t\in[0,1]}\left\|e^{\gamma|x|^2}\tilde{F}(\cdot,t)\right\|_X,\end{aligned} \quad (5.28)$$

where

$$\gamma_\varepsilon=\frac{\gamma}{1+4\gamma\varepsilon},\qquad\tilde{V}_0=\sup_{t\in[0,1]}\|\tilde{V}\|_B.$$

Then, Lemma 3.6, (5.28) and (5.23) show that

$$\left\|e^{\gamma_\varepsilon|x|^2}u(\cdot,t)\right\|_{L^2(\mathbb{R}^n\times[0,1];H)}+\left\|\sqrt{t(1-t)}e^{\gamma_\varepsilon|x|^2}\nabla u\right\|_{L^2(\mathbb{R}^n\times[0,1];H)}$$
$$\leq Ne^{NV_0}\left[\left\|e^{\frac{|x|^2}{\beta^2}}u(\cdot,0)\right\|_X+\left\|e^{\frac{|x|^2}{\alpha^2}}u(\cdot,1)\right\|_X+\sup_{t\in[0,1]}\|u(\cdot,t)\|_X\right],$$

where

$$V_0=\sup_{t\in[0,1]}\|V(x,t)\|_B.$$

The Theorem 5.2 follows from this inequality, from (5.21)–(5.23) and letting ε tend to zero. □

6 A Hardy type abstract uncertainty principle. Proof of Theorem 2.2

The assertion about the Carleman inequality in Lemma 6.1 below is the following monotonicity or frequency function argument related to Lemma 3.4. When $u \in C([0,1];X)$ is a solution to the free abstract Schrödinger equation

$$\partial_t u - i\left(\Delta u + A\left(x\right)u\right) = 0, \qquad x \in \mathbb{R}^n,\ t \in [0,1],$$

satisfies

$$\left\|e^{\gamma|x|^2} u\left(\cdot,0\right)\right\|_X < \infty \qquad \left\|e^{\gamma|x|^2} u\left(\cdot,1\right)\right\|_X < \infty$$

and

$$f = e^{\varkappa} u, \qquad Q\left(t\right) = \left(f\left(\cdot,t\right), f\left(\cdot,t\right)\right)_X,$$

where

$$\varkappa\left(x,t\right) = \mu|x + rt(1-t)|^2 - \frac{r^2 t(1-t)}{8\mu}, \qquad \sigma\left(\varepsilon,t\right) = \frac{\left(1+\varepsilon\right)t(1-t)}{16\mu}.$$

Then, $\log Q\left(t\right)$ is logarithmically convex in $[0,1]$, when $0 < \mu < \gamma$.

The formal application of the above argument to a $C([0,1];X)$ solution of the equation

$$\partial_t u - i\left[\Delta u + Au + V\left(x,t\right)u\right] = 0, \qquad x \in \mathbb{R}^n,\ t \in [0,1], \tag{6.1}$$

implies a similar result, when V is a bounded potential, though the justification of the correctness of the assertions involved in the corresponding formal application of Lemma 3.4 were formal. In fact, we can only justify these assertions, when the potential V verifies the first condition in Theorem 2.2 or when we can obtain the additional regularity of the gradient of u in the strip, as in Theorem 5.2. Here, we choose to prove Theorem 2.2 using the Carleman inequality in Lemma 6.1 in place of the above convexity argument. The reason for our choice is that it is simpler to justify the correctness of the application of the Carleman inequality to a $C([0,1];X)$ solution to (6.1) than the corresponding monotonicity or logarithmic convexity of the solution.

Lemma 6.1. *Let the assumptions (1)–(2) of Conditon 3.1 hold. Moreover,*

$$V \in B \quad \text{and} \quad \lim_{r\to\infty} \|V\|_{O(r)} = 0.$$

Then the estimate

$$r\sqrt{\frac{\varepsilon}{8\mu}}\left\|e^{\varkappa-\sigma} v\right\|_{L^2(\mathbb{R}^{n+1};H)} \leq \left\|e^{\varkappa-\sigma}\left[\partial_t u - i\left(\Delta u + Au\right)\right]v\right\|_{L^2(\mathbb{R}^{n+1};H)}$$

holds, when $\varepsilon > 0$, $\mu > 0$, $r > 0$ and $v \in C_0^\infty\left(\mathbb{R}^{n+1}; H\left(A\right)\right)$.

Proof. Let $f = e^{\varkappa-\sigma} v$. Then,

$$e^{\varkappa-\sigma}\left[\partial_t u - i\left(\Delta u + Au\right)\right]v = \partial_t f + Sf - Kf.$$

From (3.8)–(3.10) with $\gamma = 1$, $a + ib = i$ and $\varphi\left(x,t\right) = \varkappa\left(x,t\right) - \sigma\left(\varepsilon,t\right)$ we have

$$S = -4\mu i(x + rt(1-t)e_1)\cdot\nabla - 2\mu n i + 2\mu r(1-2t)(x_1 + rt(1-t)) - \sigma,$$

$$K = i\left(\triangle + A\right) + 4^2 i|x + rt(1-t)e_1|^2$$

$$\begin{aligned} S_t + [S,K] = & -8^3|x + rt(1-t)e_1|^2 - 4r(x_1 + rt(1-t)) \\ & + 2\mu r^2(1-2t)^2 + \frac{(1+\varepsilon)\, r^2}{8\mu} + -4i\mu r(1-2t)\partial_{x_1} \end{aligned}$$

and

$$\begin{aligned} (S_t f + [S,K]f, f)_X = & \, 32^3 \int_{\mathbb{R}^n} \left| x + rt(1-t)e_1 - \frac{r}{16\mu^2} e_1 \right|^2 \|f\|^2 \, dx + \frac{\varepsilon r^2}{8\mu} \int_{\mathbb{R}^n} \|f\|^2 \, dx \\ & + 8\mu \int_{\mathbb{R}^n} \|\nabla_{x'} f\|_H^2 \, dx + 8\mu \int_{\mathbb{R}^n} \left\| i\partial_{x_1} f - r\left(\frac{1}{2} - t\right) f \right\|^2 dx \\ \geq & \, \frac{\varepsilon r^2}{8\mu} \int_{\mathbb{R}^n} \|f\|^2 \, dx. \end{aligned} \tag{6.2}$$

Following the standard method to handle L_2-Carleman inequalities, the symmetric and skew-symmetric parts of $\partial_t - S - K$, as a space-time operator, are respectively $-S$ and $\partial_t - K$, and $[-S, \partial_t - K] = S_t + [S,K]$. Thus,

$$\begin{aligned} & \|\partial_t f - Sf - Kf\|^2_{L^2(\mathbb{R}^{n+1};H)} \\ & = \|\partial_t f - Kf\|^2_{L^2(\mathbb{R}^{n+1};H)} + \|Sf\|^2_{L^2(\mathbb{R}^{n+1};H)} - 2\,\mathrm{Re} \int_{\mathbb{R}^n} \int_{-\infty}^{\infty} (Sf, \partial_t f - Kf) \, dxdt \\ & \geq \int_{\mathbb{R}^n} \int_{-\infty}^{\infty} ([-S, \partial_t - K]f, f) \, dxdt \\ & = \int_{-\infty}^{\infty} (S_t f + [S,K]\, f, f)_H \, dt, \end{aligned} \tag{6.3}$$

and the Lemma 6.1 follows from (5.2) and (5.3). □

Proof of Theorem 2.2. Let u be as in Theorem 2.2, $\tilde{u}$ and $\tilde{V}$ be corresponding functions defined in Lemma 4.1, when $a + ib = i$. Then, $\tilde{u} \in C([0,1]; X\left(A\right))$ is a solution of the equation

$$\partial_t u - i\left[\Delta u + Au + \tilde{V}u\right] = 0, \; x \in \mathbb{R}^n, \; t \in [0,1]$$

and

$$\left\| e^{\gamma|x|^2} \tilde{u}\left(\cdot, 0\right) \right\|_X < \infty, \qquad \left\| e^{\gamma|x|^2} \tilde{u}\left(\cdot, 1\right) \right\|_X < \infty \quad \text{for } \gamma = \frac{1}{\alpha\beta}, \; \gamma > \frac{1}{2}.$$

The proof of Theorem 2.4 shows that in either case

$$N_\gamma = \sup_{t\in[0,1]} \left[\left\| e^{\gamma_\varepsilon |x|^2} \tilde{u}\left(\cdot, t\right) \right\|_{L^2(\mathbb{R}^n\times[0,1];H)} + \left\| \sqrt{t\left(1-t\right)} e^{\gamma_\varepsilon |x|^2} \nabla \tilde{u} \right\|_{L^2(\mathbb{R}^n\times[0,1];H)} \right] < \infty. \tag{6.4}$$

For given $r > 0$, choose μ and ε such that

$$\frac{(1+\varepsilon)^{\frac{3}{2}}}{2\left(1-\varepsilon\right)^3} < \mu \leq \frac{\gamma}{1+\varepsilon} \tag{6.5}$$

and let η_M and θ_r be smooth functions verifying, $\theta_M\left(x\right) = 1$, when $|x| \leq M$, $\theta_M\left(x\right) = 0$, when $|x| > 2M$, $M \geq 2r$, $\eta_r \in C_0^\infty(0,1)$, $0 \leq \eta_r\left(t\right) \leq 1$, $\eta_r\left(t\right) = 1$ for $t \in \left[\frac{1}{r}, 1 - \frac{1}{r}\right]$ and $\eta_r\left(t\right) = 0$

for $t \in \left[0, \frac{1}{2r}\right] \cup \left[1-\frac{1}{2r}, 1\right]$. Then, $v(x,t) = \eta_r(t)\,\theta_M(x)\,\tilde{u}(x,t)$ is compactly supported in $\mathbb{R}^n \times (0,1)$ and

$$\partial_t v - i\left[\Delta v + Av + \tilde{V}v\right] = \eta_r'(t)\,\theta_M(x)\,\tilde{u}(x,t) - \left(2\nabla\theta_M.\nabla\tilde{u} + \tilde{u}\Delta\theta_M\right)\eta_r. \qquad (6.6)$$

The terms on the right-hand side of (6.6) are supported, where

$$\mu|x + rt(1-t)|^2 \leq \gamma\,|x|^2 + \frac{\gamma}{\varepsilon},$$

$$\mu|x + rt(1-t)e_1|^2 \leq \gamma\,|x|^2 + \frac{\gamma}{\varepsilon}r^2.$$

Apply now Lemma 6.1 to v with the values of μ and ε chosen in (6.5). This, the bounds for $\mu|x+rt(1-t)e_1|^2$ in each of the parts of the support of

$$\partial_t v - i\left[\Delta v + Av + \tilde{V}v\right]$$

and the natural bounds for $\nabla\theta_M$, $\triangle\theta_M$ and η_r' show that there is a constant N_ε such that

$$r\left\|e^{\varkappa-\sigma}v\right\|_{L^\infty(\mathbb{R}^n\times[0,1];H)} \leq N_\varepsilon\left\|\tilde{V}\right\|_B\left\|e^{\varkappa-\sigma}v\right\|_{L^2(\mathbb{R}^n\times[0,1];H)} + N_\varepsilon r e^{\frac{\gamma}{\varepsilon}}\sup_{t\in[0,1]}\left\|e^{\gamma|x|^2}\tilde{u}(\cdot,t)\right\|_X$$
$$+ N_\varepsilon M^{-1}e^{\frac{\gamma}{\varepsilon}r^2}\left\|e^{\gamma|x|^2}\left(\|\tilde{u}\| + \|\nabla\tilde{u}\|_H\right)\right\|_{L^2(\mathbb{R}^n\times\sigma_r)}, \qquad (6.7)$$

where

$$\sigma_r = \left[\frac{1}{2r}, 1-\frac{1}{2r}\right].$$

The first term on the right hand side of (6.7) can be hidden in the left hand side, when $r \geq 2N_\varepsilon\left\|\tilde{V}\right\|_B$, while the last tends to zero, when M tends to infinity by (6.4). This and the fact that $v = \tilde{u}$ in $O_{r_\varepsilon} \times \left[\frac{1-\varepsilon}{2}, \frac{1+\varepsilon}{2}\right]$, where

$$\varkappa - \sigma \geq \frac{r^2}{16\mu}\left(4\mu^2(1-\varepsilon)^6 - (1+\varepsilon)^3\right), \qquad r_\varepsilon = \frac{\varepsilon\left(1-\varepsilon^2\right)^2 r}{4};$$

and (6.5) show that

$$e^{C(\gamma,\varepsilon)}\|\tilde{u}\|_{L^2(O(r,\varepsilon);H)} \leq N_{\gamma,\varepsilon}, \qquad O(r,\varepsilon) = O_{\frac{r}{8}} \times \left[\frac{1-\varepsilon}{2}, \frac{1+\varepsilon}{2}\right] \qquad (6.8)$$

when $r \geq 2N_\varepsilon\left\|\tilde{V}\right\|_B$. At the same time

$$\left(B\left(\tilde{V}\right)\right)^{-1}\|\tilde{u}(\cdot,0)\|_X \leq \|\tilde{u}(\cdot,t)\|_X \leq B\left(\tilde{V}\right)\|\tilde{u}(\cdot,1)\|_X \qquad (6.9)$$

for $0 \leq t \leq 1$ and $B\left(\tilde{V}\right) = \sup_{t\in[0,1]}\left\|\tilde{V}\right\|_B$. Moreover, from (6.4) we get

$$\|\tilde{u}(\cdot,t)\|_X \leq \|\tilde{u}(\cdot,t)\|_{L^2\left(O_{\frac{r}{8}};H\right)} + e^{\frac{-\gamma r^2}{64}}N_\gamma \quad \text{when } 0 \leq t \leq 1. \qquad (6.10)$$

Then, (6.8)–(6.10) show that there is a constant $N_{\gamma,\varepsilon,V}$, which such that

$$e^{C(\gamma,\varepsilon)r^2}\|\tilde{u}(\cdot,0)\|_X \leq N_{\gamma,\varepsilon,V}.$$

For $r \to \infty$ we obtain $u \equiv 0$. □

Proof of Theorem 2.5.

First of all, we show the following.

Lemma 6.2. *Let the assumptions (1)–(2) of Condition 3.1 hold. Moreover, let*

$$V \in B, \qquad \lim_{r\to\infty} \|V\|_{O(r)} = 0.$$

Then the estimate

$$r\sqrt{\frac{\varepsilon}{8\mu}} \left\|e^{\varkappa-\sigma+\chi} v\right\|_{L^2(\mathbb{R}^{n+1};H)} \leq \left\|e^{\varkappa-\sigma+\chi}\left[\partial_t u - \Delta u - Au\right] v\right\|_{L^2(\mathbb{R}^{n+1};H)} \tag{6.11}$$

holds, when $\varepsilon > 0$, $\mu > 0$, $r > 0$ and $v \in C_0^\infty\left(\mathbb{R}^{n+1}; H(A)\right)$, where

$$\chi(t) = \frac{r^2 t(1-t)(1-2t)}{6}.$$

Proof. Let $f = e^{\varkappa+\chi-\sigma} v$. Then,

$$e^{\varkappa+\chi-\sigma}\left[\partial_t u - (\Delta u + Au)\right] v = \partial_t f - Sf - Kf.$$

From (3.8)–(3.10) with $\gamma = 1$, $a + ib = 1$ and $\varphi(x,t) = \varkappa(x,t) + \chi(t) - \sigma(\varepsilon, t)$ we have

$$\begin{aligned} S &= \Delta + A + 4\mu^2 i |x + rt(1-t)e_1|^2 + 2\mu n i \\ &\quad + 2\mu r(1-2t)(x_1 + rt(1-t)) - \sigma + \left(t^2 - t + \frac{1}{6}\right) r^2, \\ K &= -4\mu(x + rt(1-t)e_1)\cdot\nabla - 2\mu n, \\ S_t + [S,K] &= -8\mu\triangle + 32\mu^3 |x + rt(1-t)e_1|^2 \\ &\quad + 4\mu r(4\mu(1-2t-1)((x_1 + rt(1-t)) + (2t-1) r^2 + \frac{(1+\varepsilon) r^2}{8\mu} \end{aligned}$$

and

$$\begin{aligned} (S_t f + [S,K]f, f)_X &= 32\mu^3 \int_{\mathbb{R}^n} \left| x + rt(1-t)e_1 + \frac{(4\mu(1-2t-1) r}{16\mu^2} e_1 \right|^2 \|f\|^2 dx \\ &\quad + 8\mu \int_{\mathbb{R}^n} |\nabla f|_H^2 dx + \frac{\varepsilon r^2}{8\mu} \int_{\mathbb{R}^n} \|f\|^2 dx \geq \frac{\varepsilon r^2}{8\mu} \int_{\mathbb{R}^n} \|f\|^2 dx. \end{aligned} \tag{6.12}$$

Then from (6.12) a similar way as Lemma 6.1 we obtain the estimate (6.11). □

Proof of Theorem 2.5. Assume that u verifies the conditions in Theorem 2.5 and let $\tilde{u}$ be the Appel transformation of u defined in Lemma 4.1 with $a + ib = 1$, $\alpha = 1$ and $\beta = 1 + \frac{2}{\beta}$. $\tilde{u} \in L^\infty(0,1; X(A)) \cap L^2\left(0,1; Y^1\right)$ is a solution of the equation

$$\partial_t u = \Delta u + Au + \tilde{V} u, \qquad x \in \mathbb{R}^n,\ t \in [0,1]$$

with $\tilde{V}$ a bounded potential in $\mathbb{R}^n \times [0,1]$ and $\gamma = \frac{1}{2\delta}$. Then, we have

$$\left\|e^{\gamma|x|^2} \tilde{u}(\cdot,0)\right\|_X = \|\tilde{u}(\cdot,0)\|_X, \qquad \left\|e^{\gamma|x|^2} \tilde{u}(\cdot,1)\right\|_X = \|\tilde{u}(\cdot,1)\|_X.$$

From Lemma 3.5 and Lemma 3.6 with $a+ib=1$, we have

$$\sup_{t\in[0,1]}\left\|e^{\gamma|x|^2}\tilde{u}(\cdot,t)\right\|_X+\left\|\sqrt{t(1-t)}e^{\gamma|x|^2}\nabla\tilde{u}\right\|_{L^2(\mathbb{R}^n\times[0,1];H)}$$
$$\leq e^{\left(M_1+M_1^2\right)}\left[\left\|e^{\gamma|x|^2}\tilde{u}(\cdot,0)\right\|_X+\left\|e^{\gamma|x|^2}\tilde{u}(\cdot,1)\right\|_X\right],$$

where

$$M_1=\left\|\tilde{V}\right\|_B.$$

The proof is finished by setting $v(x,t)=\theta_M(x)\eta_r(t)\tilde{u}(x,t)$, by using Carleman inequality (6.11) and a similar argument that we used to prove Theorem 2.2. □

7 Unique continuation properties for the system of Schrödinger equations

Consider the system of Schrödinger equation

$$\partial_t u_m=i\left[\Delta u_m+\sum_{j=1}^N a_{mj}u_j+\sum_{j=1}^N b_{mj}u_j\right],\qquad x\in\mathbb{R}^n,\ t\in(0,T),\tag{7.1}$$

where $u=(u_1,u_2,\dots,u_N)$, $u_j=u_j(x,t)$, a_{mj} and $b_{mj}=b_{mj}(x,t)$ are real-valued functions. Let $l_2=l_2(N)$ and $l_2^s=l_2^s(N)$ (see [24, § 1.18]). Let A be the operator in $l_2(N)$ defined by

$$D(A)=\left\{u=\{u_j\},\ \|u\|_{D(A)}=\left(\sum_{m,j=1}^N\left(a_{mj}u_j\right)^2\right)^{\frac{1}{2}}<\infty\right\},$$

$$A=\left[a_{mj}\right],\qquad m,j=1,2,\dots,N,\quad N\in\mathbb{N}\tag{7.2}$$

and

$$V(x,t)=\left[b_{mj}(x,t)\right],\qquad m,j=1,2,\dots,N,\quad N\in\mathbb{N}.$$

Let

$$X_2=L^2(\mathbb{R}^n;l_2),\qquad Y^{s,2}=H^{s,2}(\mathbb{R}^n;l_2).$$

From Theorem 2.2 we obtain the following result.

Theorem 7.1. *Assume:*

(1) $a_{mj}=a_{jm}\geq 0$, $a_{mj}\in C^{(1)}(\mathbb{R}^n)$, $\sum_{m,j=1}^N a_{mj}>0$, $(A(x)u,u)\in L^2(\mathbb{R}^n)$ *for* $u\in l_2^s$. *Moreover,*

$$\sum_{k=1}^n\left(x_k\left[A\frac{\partial f}{\partial x_k}-\frac{\partial A}{\partial x_k}f\right],f\right)_{X_2}\geq 0\quad\text{for } f\in C^1\left(\mathbb{R}^n;l_2^s\right);$$

(2)

$$\sup_{t\in[0,T],\,x\in\mathbb{R}^n}\left(\sum_{m,j=1}^N\left|b_{mj}(x,t)\right|^2\right)^{\frac{1}{2}}<\infty,\qquad\sup_{t\in[0,T],\,x\in\mathbb{R}^n}\left|e^{|x|^2\mu^{-2}(t)}\right|\left(\sum_{m,j=1}^N\left|b_{mj}(x,t)\right|^2\right)^{\frac{1}{2}}<\infty,$$

where

$$\mu(t)=\alpha t+\beta(1-t),\quad\alpha,\beta>0,\quad\alpha\beta<2;$$

(3) $u \in C([0,T]; l_2)$ *be a solution of the equation* (7.1) *and*

$$\left\| e^{\frac{|x|^2}{\beta^2}} u(\cdot,0) \right\|_{X_2} < \infty, \qquad \left\| e^{\frac{|x|^2}{\alpha^2}} u(\cdot,T) \right\|_{X_2} < \infty.$$

Then $u(x,t) \equiv 0$.

Proof. Consider the operators A and $V(x,t)$ in $H = l^2$ defined by (7.2). Then the problem (8.1)–(8.2) can be rewritten as the problem (1.1), where $u(x) = \{u_j(x)\}$, $f(x) = \{f_j(x)\}$, $j = 1,2,\ldots,N$, $x \in \mathbb{R}^n$ are the functions with values in $H = l^2(N)$. Since $a_{mj} = a_{jm}$ for $m, j = 1,2,\ldots,N$, $N \in \mathbb{N}$, it is easy to see that

$$(Ax,y)_{l(N)} = (x,Ay)_{l(N)} \quad \text{for all } x, y \in l(N),$$

i.e., A is a symmetric operator in l_2 and other conditions of Theorem 2.2 are satisfied. Hence, from Theorem 2.2 we obtain the conclusion. □

8 Unique continuation properties for anisotropic Schrödinger equation

Let us consider the following problem

$$\partial_t u = i\left[\Delta_x u + \sum_{|\alpha|=2m} a_\alpha D_y^\alpha u(x,y,t) + \int_G K(x,y,t)\, u(x,y,t)\, dy\right],$$
$$x \in \mathbb{R}^n,\ y \in G,\ t \in [0,T], \tag{8.1}$$

$$B_j u = \sum_{|\beta| \le m_j} b_{j\beta} D_y^\beta u(x,y,t) = 0, \qquad x \in \mathbb{R}^n,\ y \in \partial G, \quad j = 1,2,\ldots,m, \tag{8.2}$$

where G is a bounded domain in $\mathbb{R}^d$ for $d \ge 2$ with sufficiently smooth $(d-1)$-dimensional boundary ∂G, $a_\alpha = a_\alpha(x,y)$ are real valued function on $\Omega = \mathbb{R}^n \times G$, $b_{j\beta}$ are the real valued functions on ∂G, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_d)$, $\mu_i < 2m$, $K = K(x,y,t)$ is a complex valued bounded function in $\Omega \times [0,T]$ and

$$D_x^k = \frac{\partial^k}{\partial x^k}, \quad D_j = -i\frac{\partial}{\partial y_j}, \quad D_y = (D_1, \ldots, D_d), \quad y = (y_1, \ldots, y_d),$$

$$M_1 = \sup_{x \in \mathbb{R}^n} \|V_1(x)\|_{B(H)} < \infty, \qquad \sup_{t \in [0,1]} \left\| e^{|x|^2 \mu^{-2}(t)} V_2(\cdot,t) \right\|_B < \infty.$$

Theorem 8.1. *Let the following conditions be satisfied:*

(1) $\Omega \in C^2$, $a_\alpha(\cdot) \in C^{(1)}(\bar{\Omega})$ *and* $b_{j\beta} \in C(\partial\Omega)$ *such that* $A(x)$ *is symmetric in* $L^2(G)$, $(A(x)u,u) \in L^2(\mathbb{R}^n)$ *for* $u \in W^{2m}(G)$. *Moreover,*

$$\sum_{k=1}^n \left(x_k \left[A\frac{\partial f}{\partial x_k} - \frac{\partial A}{\partial x_k} f \right], f \right)_{L^2(\Omega)} \ge 0 \quad \textit{for } f \in C^1\left(\mathbb{R}^n; W^{2m,2}(G)\right);$$

(2)
$$\sup_{t\in[0,T],\,x\in\mathbb{R}^n} \|K(x,\cdot,t)\|_{L^2(G)} < \infty, \qquad \sup_{t\in[0,T],\,x\in\mathbb{R}^n} \left\|e^{|x|^2\mu^2(t)} K(x,\cdot,t)\right\|_{L^2(G)} < \infty,$$

where $\mu(t) = \alpha t + \beta(1-t)$, $\alpha, \beta > 0, \alpha\beta < 2$;

(3) $u \in C\left([0,T]; L^2(\Omega)\right)$ *be a solution of the equation* (8.1)–(8.2) *and*

$$\left\|e^{\frac{|x|^2}{\beta^2}} u(\cdot,\cdot,0)\right\|_{L^2(\Omega)} < \infty, \qquad \left\|e^{\frac{|x|^2}{\alpha^2}} u(\cdot,\cdot,T)\right\|_{L^2(\Omega)} < \infty.$$

Then $u(x,y,t) \equiv 0$.

Proof. Let us consider operators A and $V(x,t)$ in $H = L^2(G)$ that are defined by the equalities

$$D(A) = \left\{u \in W^{2m,2}(G),\, B_j u = 0,\ j = 1,2,\ldots,m\right\}, \qquad Au = \sum_{|\alpha|=2m} a_\alpha D_y^\alpha u(y),$$

$$V(x,t)\,u = \int_G K(x,y,t)\,u(x,y,t)\,dy.$$

Then the problem (8.1)–(8.2) can be rewritten as the problem (1.1), where $u(x) = u(x,\cdot)$, $f(x) = f(x,\cdot)$, $x \in \sigma$ are the functions with values in $H = L^2(G)$. In view of assumptions (1)–(4) all conditions of Theorem 2.2 are hold. Then Theorem 2.2 implies the assertion. □

9 Theorem 9.1.

Theorem 9.1. *Suppose the the following conditions are satisfied:*

(1) *let* $a(x,\cdot)$ *be positive,* $b(x,.)$ *be a real-valued function on* $(0,1)$, $a \in C^{(1)}([0,1]\times\mathbb{R}^n)$, *moreover,* A *is symmetric in* $L^2(0,1)$ *and* $(A(x)u,u) \in L^2(\mathbb{R}^n)$ *for* $u \in W^2(0,1)$

$$\sum_{k=1}^{n}\left(x_k\left[A\frac{\partial f}{\partial x_k} - \frac{\partial A}{\partial x_k}f\right], f\right)_{L^2(\sigma)} \geq 0 \quad \text{for } f \in C^1\left(\mathbb{R}^n; W^2(0,1)\right),$$

$b(\cdot,y) \in C(\mathbb{R}^n)$ *for a.e.* $y \in [0,1]$, $b(x,\cdot) \in L_\infty(0,1)$ *for a.e.* $x \in \mathbb{R}^n$ *and*

$$\exp\left(-\int_{\frac{1}{2}}^{x} b(\tau)\,a^{-1}(x,\tau)\,d\tau\right) \in L_1(0,1) \quad \text{for a.e.} x \in \mathbb{R}^n;$$

(2)
$$\sup_{t\in[0,T],\,x\in\mathbb{R}^n} \|K(x,\cdot,t)\|_{L^2(0,1)} < \infty, \qquad \sup_{t\in[0,T],\,x\in\mathbb{R}^n} \left\|e^{|x|^2\mu^2(t)} K(x,\cdot,t)\right\|_{L^2(0,1)} < \infty,$$

where $\mu(t) = \alpha t + \beta(1-t)$, $\alpha, \beta > 0$, $\alpha\beta < 2$;

(3) $u \in C\left([0,T]; L^2(\sigma)\right)$ *be a solution of the equation* (1.5)–(1.6) *and*

$$\left\|e^{\frac{|x|^2}{\beta^2}} u(.,0)\right\|_{L^2(\sigma)} < \infty, \qquad \left\|e^{\frac{|x|^2}{\alpha^2}} u(.,T)\right\|_{L^2(\sigma)} < \infty.$$

Then $u(x,y,t) \equiv 0$.

Proof. Let us consider the operator A in $H = L^2(0,1)$ defined by (1.4). Then (8.1)–(8.2) can be rewritten as the problem (1.1) where $u(x) = u(x,\cdot)$, $f(x) = f(x,\cdot)$, $x \in \sigma$ are the functions with values in $H = L^2(0,1)$. Hence, by virtue of assumptions (1)–(4), all conditions of Theorem 2.2 are satisfied. Then Theorem 2.2 implies the assertion. □

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