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Solitary wave of ground state type for a nonlinear Klein–Gordon equation coupled with Born–Infeld theory in $\mathbb{R}^2$

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Abstract. In this paper we prove the existence of nontrivial ground state solution for a nonlinear Klein–Gordon equation coupled with Born–Infeld theory in $\mathbb{R}^2$ involving unbounded or decaying radial potentials. The approach involves variational methods combined with a Trudinger–Moser type inequality and a symmetric criticality type result.

Keywords: Klein–Gordon equation, Born–Infeld theory, Trudinger–Moser inequality, unbounded or decaying radial potentials, critical exponential growth, Mountain-Pass Theorem.

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1 Introduction and main results

This paper was motivated by some works that had appeared in recent years concerning the following Klein–Gordon equation with Born–Infeld theory on $\mathbb{R}^3$:

$$\begin{cases} -\Delta u + [m^2 - (\omega+\phi)^2]u = |u|^{p-2}u, & x \in \mathbb{R}^3, \\ \Delta\phi + \beta\Delta_4\phi = 4\pi(\omega+\phi)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\Delta_4\phi = \operatorname{div}(|\nabla\phi|^2\nabla\phi)$. Such a system deduced by coupling the Klein–Gordon equation

$$\psi_{tt} - \Delta\psi + m^2\psi - |\psi|^{p-2}\psi = 0$$

with the Born–Infeld theory

$$\mathcal{L}_{\mathrm{BI}} = \frac{b^2}{4\pi}\left(1 - \sqrt{1 - \frac{1}{b^2}\left(|\mathbb{E}|^2 - |\mathbb{B}|^2\right)}\right),$$

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where $\psi = \psi(x,t) \in \mathbb{C}$ ($x \in \mathbb{R}^3$, $t \in \mathbb{R}$), m is a real constant and $2 < p < 6$, $\mathbb{E}$ is the electric field and $\mathbb{B}$ is the magnetic induction field. For more details on the physical aspects of the problem we refer the readers to see [13] and the references therein.

A few existence results for the system (1.1) have been proved via modern variational methods under various hypotheses on the nonlinear term. We recall some of them as follows. d'Avenia and Pisani [13] was pioneered work with this system. They found the existence of infinitely many radially symmetric solutions for system (1.1) by using $\mathbb{Z}_2$-Mountain Pass Theorem, when $4 < p < 6$ and $|\omega| < |m|$. Later, in [21] the range $p \in (2,4]$ was also covered provided $\sqrt{(\frac{p}{2}-1)}|m| > \omega > 0$. Replacing $|u|^{p-2}u$ by $|u|^{p-2}u + |u|^4 u$ in problem (1.1), Teng and Zhang in [26] get that problem

$$\begin{cases} -\Delta u + [m^2 - (\omega+\phi)^2]u = |u|^{p-2}u + |u|^4 u, & x \in \mathbb{R}^3, \\ \Delta\phi + \beta\Delta_4\phi = 4\pi(\omega+\phi)u^2, & x \in \mathbb{R}^3, \end{cases}$$

has at least a nontrivial solution by using Mountain Pass Theorem, when $4 < p < 6$ and $\omega < m$. Subsequently, replacing $|u|^{p-2}u$ by $|u|^{p-2}u + h(x)$ in problem (1.1), Chen and Li in [9] get the existence of two nontrivial solutions for nonhomogeneous problem

$$\begin{cases} -\Delta u + [m^2 - (\omega+\phi)^2]u = |u|^{p-2}u + h(x), & x \in \mathbb{R}^3, \\ \Delta\phi + \beta\Delta_4\phi = 4\pi(\omega+\phi)u^2, & x \in \mathbb{R}^3, \end{cases}$$

by using the Ekeland variational principle and the Mountain Pass Theorem, when $|m| > \omega > 0$ and $4 < p < 6$ or $\sqrt{(\frac{p}{2}-1)}|m| > \omega > 0$ and $2 < p \le 4$. Other related results about Klein–Gordon equation coupled with Born–Infeld theory on $\mathbb{R}^3$ can be found in [28] and [29]. By the way, we should point out that if $\beta = 0$ then problem (1.1) becomes

$$\begin{cases} -\Delta u + [m^2 - (\omega+\phi)^2]u = |u|^{p-2}u, & x \in \mathbb{R}^3, \\ \Delta\phi = 4\pi(\omega+\phi)u^2, & x \in \mathbb{R}^3, \end{cases}$$

for the well-known Klein–Gordon–Maxwell equations. Such problems have been intensively studied in recent years as for example in [6–8, 10–12, 14, 18, 19, 22].

In this paper we consider the following Klein–Gordon equation coupled with Born–Infeld theory:

$$\begin{cases} -\Delta u + \left[m^2 - (\omega+\phi)^2\right] V(|x|)u = K(|x|)f(u), & x \in \mathbb{R}^2, \\ \Delta\phi + \beta\Delta_4\phi = 4\pi(\omega+\phi)V(|x|)u^2, & x \in \mathbb{R}^2, \end{cases} \tag{1.2}$$

where ω is a positive frequency parameter, β depends on the so-called Born–Infeld parameter, m is a real constant, $\phi : \mathbb{R}^2 \to \mathbb{R}$ and V, $K : \mathbb{R}^2 \to \mathbb{R}$ are radial potentials which may be unbounded, singular at the origin or vanishing at infinity and the nonlinear term $f(s)$ is allowed to enjoy an critical exponential growth in the sense of the classical Trudinger–Moser inequality which will be stated later.

The bi-dimensional case is very special and quite delicate, because as we know for domains $\Omega \subset \mathbb{R}^2$ with finite volume, the Sobolev embedding theorem assures that $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ for any $q \in [1, +\infty)$, but, due to a function with a local singularity and this causes the failure of the embedding that $H_0^1(\Omega) \not\hookrightarrow L^\infty(\Omega)$. Therefore, and in order to overcome this trouble, the Trudinger–Moser inequality was established independently by Yudovič [17], Pohožaev [23] and Trudinger [27], came as a substitute of the Sobolev inequality. It asserts that the existence of a constant $\alpha > 0$ such that $H_0^1(\Omega) \hookrightarrow L_\phi(\Omega)$, where $L_\phi(\Omega)$ is the Orlicz space determined by the Young function $\phi(t) = e^{\alpha t^2} - 1$. Later, Moser in [20] sharpened this result by finding the best constant α in the embedding above. More precisely, he proved that for any $\alpha \leq 4\pi$, there exists a constant $c_0 > 0$ such that

$$\sup_{\|\nabla u\|_{L^2(\Omega)} \leq 1} \frac{1}{|\Omega|} \int_\Omega e^{\alpha u^2} \, dx \leq c_0. \tag{1.3}$$

Moreover, the constant 4π is sharp in the sense that if $\alpha > 4\pi$, then the supremum above will become infinity.

Throughout this work, the potentials $V,\ K : \mathbb{R}^2 \to \mathbb{R}$ are positive, radial and continuous functions assuming the following behaviors at the origin and infinity:

(V) There exist real numbers a_0 and a_∞ with $a_0, a_\infty > -2$ such that

$$\liminf_{r \to 0^+} \frac{V(r)}{r^{a_0}} > 0 \quad \text{and} \quad \liminf_{r \to +\infty} \frac{V(r)}{r^{a_\infty}} > 0;$$

(K) there exist real numbers b_0 and b_∞ with $b_\infty < a_\infty$, $b_0 > -2$ such that

$$\limsup_{r \to 0^+} \frac{K(r)}{r^{b_0}} < \infty \quad \text{and} \quad \limsup_{r \to +\infty} \frac{K(r)}{r^{b_\infty}} < \infty.$$

Hereafter, we say that $(V, K) \in \mathcal{K}$ if the assumptions (V) and (K) hold.

As we mentioned initially and motivated by the aforementioned works, we consider system (1.2) involving unbounded, singular at the origin or decaying to zero at infinity radial potentials. Recently, much attention has been paid to the Schrödinger equations with potentials with these kinds of behaviors. For example, we can cite [2, 24]. In [24], the authors studied the existence and multiplicity of solutions for the problem

$$\begin{cases} -\Delta u + V(|x|)u = K(|x|)f(u), & x \in \mathbb{R}^N \\ |u(x)| \to 0 \quad \text{as } |x| \to \infty, \end{cases}$$

where the nonlinearity considered was $f(s) = |s|^{p-2}s$, with $2 < p < 2^* = \frac{2N}{N-2}$ for $N \geq 3$ is the limiting exponent in the Sobolev embedding and $2 < p < \infty$ if $N = 2$. Succeeding this study, Albuquerque et al. in [2] studied the above problem in the critical case suggested by the so-entitled Trudinger–Moser inequality (1.3). To our best knowledge, there are no literature addressing the system (1.2) where the potentials V and K have these features and the nonlinearity f has exponential critical growth in two dimensions. Hence, our results are new and complement the above results to some extent.

In order to state our results, we need to introduce some notations. If $1 \leq p < \infty$ we define the weighted Lebesgue spaces

$$L^p(\mathbb{R}^2; K) := \left\{ u : \mathbb{R}^2 \to \mathbb{R} : u \text{ is measurable and } \int_{\mathbb{R}^2} K(|x|)|u|^p \, dx < \infty \right\},$$

equipped with the norm

$$\|u\|_{p;K} = \left(\int_{\mathbb{R}^2} K(|x|)|u|^p \, dx \right)^{\frac{1}{p}}.$$

Similarly, we can define $L^p(\mathbb{R}^2; V)$ with its correspondent norm

$$\|u\|_{p;V} = \left(\int_{\mathbb{R}^2} V(|x|)|u|^p \, \mathrm{d}x \right)^{\frac{1}{p}}.$$

We also define the Hilbert space

$$Y := \left\{ u \in L^2_{\mathrm{loc}}(\mathbb{R}^2) : |\nabla u| \in L^2(\mathbb{R}^2) \text{ and } \int_{\mathbb{R}^2} V(|x|)u^2 \, \mathrm{d}x < \infty \right\}$$

endowed with the norm $\|u\| := \sqrt{\langle u, u \rangle}$ induced by the scalar product

$$\langle u, v \rangle := \int_{\mathbb{R}^2} [\nabla u \nabla v + V(|x|)uv] \, \mathrm{d}x. \tag{1.4}$$

Let $C_0^\infty(\mathbb{R}^2)$ be the set of smooth functions with compact support. Equivalently, the functional space Y can be regarded as the completion of $C_0^\infty(\mathbb{R}^2)$ under the norm $\| \cdot \|$. Furthermore, the subspace

$$E := Y_{\mathrm{rad}} = \{u \in Y : u \text{ is radial}\},$$

which is closed in Y, and thus it is a Hilbert space itself. Also, denote by $\mathcal{D}$ the completion of $C_0^\infty(\mathbb{R}^2)$ with respect to the norm

$$\|\phi\|_{\mathcal{D}} := \left(\int_{\mathbb{R}^2} |\nabla \phi|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}^2} |\nabla \phi|^4 \, \mathrm{d}x \right)^{\frac{1}{4}}.$$

Remark 1.1. Under the behavior of V at infinity in the hypothesis (V) we can show that $\| \cdot \|$ defined above is a norm in Y. In fact, we only need to show that if $\|u\| = 0$, then $u \equiv 0$. If $\int_{\mathbb{R}^2} |\nabla u|^2 \, \mathrm{d}x = 0$, u is a constant, but since $\liminf_{|x| \to \infty} |x|^{-a_\infty} V(|x|) > 0$ we should have $u = 0$.

Here, we are interested in the case where the nonlinearity $f(s)$ has maximal growth on s which allows us to treat the problem (1.2) variationally. It is assumed that $f : \mathbb{R} \to \mathbb{R}$ is continuous, $f(0) = 0$ and f behaves like $e^{\alpha s^2}$ as $s \to \infty$.

In order to perform the minimax approach to the problem (1.2), we also need to make some suitable assumptions on the behavior of $f(s)$. More precisely, we shall assume the following growth conditions:

(f_0) (small order at the origin) $\lim_{s \to 0^+} \frac{f(s)}{s} = 0$;

(f_1) (critical exponential growth) there exists $\alpha_0 > 0$ such that

$$\lim_{s \to \infty} \frac{|f(s)|}{e^{\alpha s^2}} = 0, \quad \text{for any } \alpha > \alpha_0, \qquad \lim_{s \to \infty} \frac{|f(s)|}{e^{\alpha s^2}} = +\infty, \quad \text{for any } \alpha < \alpha_0;$$

(f_2) (Ambrosetti–Rabinowitz type condition) there exists $\theta > 2(\omega^2 + 1) > 2$ such that

$$0 \le \theta F(s) := \theta \int_0^s f(t) \, \mathrm{d}t \le s f(s), \qquad \forall s \in \mathbb{R};$$

(f_3) there exist $\vartheta > 2$ and $\mu > 0$ such that

$$F(s) \ge \frac{\mu}{\vartheta} |s|^\vartheta, \qquad \forall s \in \mathbb{R}.$$

In this work, we say that the pair (u, ϕ) is a weak solution of (1.2) if $(u, \phi) \in Y \times \mathcal{D}$ and it holds the equalities

$$\int_{\mathbb{R}^2} \left(\nabla u \cdot \nabla v + [m^2 - (\omega + \phi)^2] V(|x|) uv\right) \mathrm{d}x = \int_{\mathbb{R}^2} K(|x|) f(u) v \, \mathrm{d}x \tag{1.5}$$

and

$$-\int_{\mathbb{R}^2} \left(\frac{1}{4\pi} \left((1 + \beta |\nabla \phi|^2) \nabla \phi \cdot \nabla \eta\right) + V(|x|)(\phi + \omega) u^2 \eta \right) \mathrm{d}x = 0, \tag{1.6}$$

for all $v \in Y$ and $\eta \in \mathcal{D}$. We point out that from (f_0) the identically zero function is the trivial solution of (1.2). We say that a pair (u, ϕ) is called a *ground state solution* of system (1.2) if (u, ϕ) is a weak solution of (1.2) which has the least energy among all nontrivial weak solutions of system (1.2).

The main results we provide in this paper is announced below.

Theorem 1.2. *Suppose that $(V, K) \in \mathcal{K}$ and (f_0)–(f_3) are satisfied. If $|m| > \omega > 0$, then there exists $\mu_0 > 0$ such that system (1.2) has a nontrivial solution (u_0, ϕ), for all $\mu > \mu_0$, with u_0 nonnegative.*

Theorem 1.3. *Under the conditions of Theorem 1.2 and supposing that $s \mapsto \frac{f(s)}{s}$ is increasing for $s > 0$, then the solution obtained in Theorem 1.2 is a ground state.*

Remark 1.4. Our interest in ground states solutions is justified by the fact that they in general exhibit some type of stability and, from a physical point of view, the stability of a standing wave is a crucial point to establish the existence of stand waves solutions.

Remark 1.5. Our existence result complements the study [4, 10] in the sense that we study a class of systems with critical exponential growth and involving unbounded, singular or decaying radial potentials.

We observe that the hypotheses (f_0)–(f_3) have been used in many papers to find solutions using the classical Mountain-Pass Theorem introduced by Ambrosetti and Rabinowitz in the celebrated paper [5], see for instance [15, 16] and references therein. It is worth pointing out that when we deal with critical nonlinearities like the exponential at infinity and in the whole space, the problem becomes much more complicated due to the possible lack of compactness. There is other considerable difficulty in dealing with systems like (1.2), which we will treat throughout the text, due to a not very good variational structure since the indefiniteness of the action associated to this set of equations.

The rest of the paper is arranged as follows. In Section 2, we introduce some auxiliary embedding results. In Section 3, we establish a variational setting of our problem. Finally, Section 4 is devoted to the proof of the main results.

2 Some useful auxiliary embedding results

To prove Theorem 1.2 and for the reader's convenience, we need review some embedding lemmas and a Trudinger–Moser type inequality built in [3] (see also [2]) where one can refer to the proofs of these results and related comments.

In the following, B_r denotes the open ball in $\mathbb{R}^2$ centered at the origin with radius r and $B_R \setminus B_r$ denotes the annulus with interior radius r and exterior radius R. Throughout the paper, we use C or C_i $(i = 0, 1, 2, \dots)$ to denote (possibly different) positive constants.

Lemma 2.1 ([2, Lemma 2.1])**.** *Suppose that* (V) *holds. Then there exist* $C > 0$ *and* $R > 1$ *such that, for all* $u \in E$*, we have*

$$|u(x)| \leq C\|u\||x|^{-\frac{a_\infty+2}{4}}, \quad \text{for } |x| \geq R.$$

For any open set $A \subset \mathbb{R}^2$ we define $W^{1,2}_{\text{rad}}(A;V) = \{u_{|_A} : u \in E\}$.

Lemma 2.2 ([25, Lemma 3])**.** *Assume that* $(V, K) \in \mathcal{K}$*. For any fixed* $0 < r < R < \infty$*, the embeddings*

$$W^{1,2}_{\text{rad}}(B_R \setminus B_r; V) \hookrightarrow L^p(B_R \setminus B_r; K), \qquad 1 \leq p \leq \infty,$$

are compact.

Remark 2.3. For $R \gg 1$, the embedding

$$W^{1,2}_{\text{rad}}(B_R; V) \hookrightarrow W^{1,2}(B_R)$$

is continuous. That last result can be obtained by proceeding exactly as in [24, Lemma 4].

Using the above lemmas, the authors in [3] (see also [2]) have obtained the following crucial embedding result.

Lemma 2.4 ([3, Lemma 2.4])**.** *Assume that* $(V, K) \in \mathcal{K}$*. Then the embeddings* $E \hookrightarrow L^q(\mathbb{R}^2; K)$ *are compact for all* $2 \leq q < \infty$*.*

With the aid of classical Trudinger–Moser inequality (1.3) and that one involving singular weights obtained by Adimurthi and K. Sandeep in [1, Theorem 2.1] (this used in 2-D), by using Lemmas 2.1 and 2.4, the authors in [3] established the following Trudinger–Moser inequality in the functional space E.

Theorem 2.5 ([3, Theorem 1.3])**.** *Assume that* $(V, K) \in \mathcal{K}$*. Then, for any* $u \in E$ *and* $\alpha > 0$*, we have that* $\left(e^{\alpha u^2} - 1\right) \in L^1(\mathbb{R}^2; K)$*. Moreover, if* $\alpha < \lambda := \min\{4\pi, 4\pi(1 + \frac{b_0}{2})\}$*, there holds*

$$\sup_{u \in E:\ \|u\| \leq 1} \int_{\mathbb{R}^2} K(|x|)\left(e^{\alpha u^2} - 1\right) \mathrm{d}x < \infty. \tag{2.1}$$

An immediate consequence of Theorem 2.5 is the following:

Corollary 2.6. *Under the assumptions of Theorem 2.5, if* $u \in E$ *is such that* $\|u\| \leq M < \sqrt{\frac{\lambda}{\alpha}}$*, then there exists a constant* $C = C(M, \alpha) > 0$ *independent of* u *such that*

$$\int_{\mathbb{R}^2} K(|x|)\left(e^{\alpha u^2} - 1\right) \mathrm{d}x \leq C.$$

3 Variational formulation

Since we are interested in solutions (u, ϕ) such that u is nontrivial nonnegative, it is convenient to define $f(s) = 0$ for all $s \leq 0$. Let $\alpha > \alpha_0$ and $q \geq 2$. From (f_0) and (f_1), for any given $\varepsilon > 0$, there exists $b_1 > 0$ such that

$$|F(s)| \leq \frac{\varepsilon}{2}s^2 + b_1|s|^q\left(e^{\alpha s^2} - 1\right), \qquad \forall s \in \mathbb{R}. \tag{3.1}$$

Given $u \in E$, by (3.1) it yields

$$\int_{\mathbb{R}^2} K(|x|)F(u)\,\mathrm{d}x \leq \frac{\varepsilon}{2}\int_{\mathbb{R}^2} K(|x|)u^2\,\mathrm{d}x + b_1\int_{\mathbb{R}^2} K(|x|)|u|^q\left(e^{\alpha u^2} - 1\right)\mathrm{d}x.$$

From Lemma 2.4, the first integral in right-hand side is finite. Now, let $r_1, r_2 > 1$ be such that $\frac{1}{r_1} + \frac{1}{r_2} = 1$. Hölder's inequality, Lemma 2.4 and (2.1) imply that

$$\int_{\mathbb{R}^2} K(|x|)|u|^q \left(e^{\alpha u^2} - 1\right) \mathrm{d}x \leq \left(\int_{\mathbb{R}^2} K(|x|)|u|^{qr_1} \mathrm{d}x\right)^{\frac{1}{r_1}} \left(\int_{\mathbb{R}^2} K(|x|) \left(e^{\alpha r_2 u^2} - 1\right) \mathrm{d}x\right)^{\frac{1}{r_2}},$$

which is finite, where we have used the elementary inequality

$$(e^s - 1)^r \leq e^{rs} - 1, \tag{3.2}$$

for all $r \geq 1, s \geq 0$. Thereby, the energy functional $J : E \times \mathcal{D} \to \mathbb{R}$ associated to system (1.2) and given by

$$\begin{aligned} J(u, \phi) := &\frac{1}{2} \int_{\mathbb{R}^2} \left(|\nabla u|^2 + \left[m^2 - (\omega + \phi)^2\right] V(|x|) u^2\right) \mathrm{d}x \\ &- \frac{1}{8\pi} \int_{\mathbb{R}^2} |\nabla \phi|^2 \,\mathrm{d}x - \frac{\beta}{16\pi} \int_{\mathbb{R}^2} |\nabla \phi|^4 \,\mathrm{d}x - \int_{\mathbb{R}^2} K(|x|) F(u) \,\mathrm{d}x \end{aligned}$$

is well-defined. Using standard arguments, one can easily show that $J \in C^1(E \times \mathcal{D}, \mathbb{R})$ and with the partial derivatives given by

$$J_u(u, \phi)v = \int_{\mathbb{R}^2} \left(\nabla u \cdot \nabla v + [m^2 - (\omega + \phi)^2] V(|x|) uv - K(|x|) f(u) v\right) \mathrm{d}x$$

and

$$J_\phi(u, \phi)\eta = -\int_{\mathbb{R}^2} \left(\frac{1}{4\pi}\left((1 + \beta|\nabla \phi|^2)\nabla \phi \cdot \nabla \eta\right) + V(|x|)(\phi + \omega)u^2 \eta\right) \mathrm{d}x,$$

for $v \in E$ and $\eta \in \mathcal{D}$. Consequently, the critical points $(u, \phi) \in E \times \mathcal{D}$ of J satisfy (1.5) and (1.6) for all $v \in E$ and $\eta \in \mathcal{D}$.

The functional J has got a strong indefiniteness (unbounded both from below and from above on infinite dimensional subspace). For this reason the usual tools of the critical point theory cannot be used in a direct way. So to avoid this difficulty we will need the following technical result which proof is based in the ideas introduced by [13, Lemma 3] and [21, Lemma 2.3].

Lemma 3.1. *For any fixed $u \in E$, there exists a unique critical point $\phi = \phi_u \in \mathcal{D}$ for the functional*

$$\mathcal{E}_u(\phi) := \int_{\mathbb{R}^2} \left[\frac{1}{8\pi}|\nabla \phi|^2 + \frac{\beta}{16\pi}|\nabla \phi|^4 + \left(\omega + \frac{1}{2}\phi\right) V(|x|)\phi u^2\right] \mathrm{d}x$$

defined on $\mathcal{D}$ (i.e., $\mathcal{E}_u$ is the energy functional associated to the second equation in (1.2)). Moreover:

1. *$\phi_u \leq 0$ and, if $u(x) \neq 0$, $-\omega \leq \phi_u(x)$;*
2. *if u is radially symmetric, then ϕ_u is radial too.*

Proof. We consider the minimizing argument on $\mathcal{E}_u$. Obviously, the functional $\mathcal{E}_u$ is well-defined on $\mathcal{D}$. Furthermore, it is strictly convex, coercive and weakly lower semi-continuous. Indeed, the coercivity of $\mathcal{E}_u$ on $\mathcal{D}$ is the following fact that

$$\begin{aligned} \mathcal{E}_u(\phi) &= \int_{\mathbb{R}^2} \left[\frac{1}{8\pi}|\nabla \phi|^2 + \frac{\beta}{16\pi}|\nabla \phi|^4 + \frac{1}{2}(\omega + \phi)^2 V(|x|)u^2 - \frac{1}{2}\omega^2 V(|x|)u^2\right] \mathrm{d}x \\ &\geq \int_{\mathbb{R}^2} \left[\frac{1}{8\pi}|\nabla \phi|^2 + \frac{\beta}{16\pi}|\nabla \phi|^4\right] \mathrm{d}x - \frac{\omega^2}{2} \int_{\mathbb{R}^2} V(|x|)u^2 \mathrm{d}x. \end{aligned}$$

The convexity and weakly lower semi-continuity of $\mathcal{E}_u$ on $\mathcal{D}$ is obviously true. Hence, there is a unique minimizer ϕ_u of the functional $\mathcal{E}_u$ on $\mathcal{D}$, concluding the first part of the lemma. For the second part, since ϕ_u is a critical point of $\mathcal{E}_u$, we get

$$-\int_{\mathbb{R}^2}\left(\frac{1}{4\pi}\left((1+\beta|\nabla\phi_u|^2)\nabla\phi_u\cdot\nabla\eta\right)+V(|x|)(\phi_u+\omega)u^2\eta\right)\mathrm{d}x=0, \tag{3.3}$$

for all $\eta\in\mathcal{D}$. Then, if we take $\eta=\phi_u^+:=\max\{\phi_u,0\}$, that is, the positive part of ϕ_u, in (3.3), we obtain

$$\int_{\mathbb{R}^2}\left(|\nabla\phi_u^+|^2+\beta|\nabla\phi_u^+|^4\right)\mathrm{d}x=-4\pi\int_{\mathbb{R}^2}(\omega+\phi_u^+)\phi_u^+V(|x|)u^2\,\mathrm{d}x\le 0,$$

which implies that $\phi_u^+\equiv 0$ and, consequently, $\phi_u\le 0$. On the other hand, if we take $\eta=(\omega+\phi_u)^-:=\max\{-(\omega+\phi_u),0\}$, that is, the negative part of $\omega+\phi_u$, in (3.3), we get

$$\begin{aligned}\int_{\{x\in\mathbb{R}^2:\phi_u(x)\le-\omega\}}|\nabla\phi_u^-|^2\,\mathrm{d}x+&\int_{\{x\in\mathbb{R}^2:\phi_u(x)\le-\omega\}}\beta|\nabla\phi_u^-|^4\,\mathrm{d}x\\ &=-4\pi\int_{\{x\in\mathbb{R}^2:\phi_u(x)\le-\omega\}}V(|x|)[(\phi_u+\omega)^-]^2u^2\,\mathrm{d}x\le 0,\end{aligned}$$

so that $(\phi_u+\omega)^-\equiv 0$ where $u\neq 0$.

Finally, let $O(2)$ denote the group of rotations in $\mathbb{R}^2$. Then for every $g\in O(2)$ and $h:\mathbb{R}^2\to\mathbb{R}$, set $T_g(h)(x):=h(gx)$. It is well-known that

$$\Delta T_g(\phi_u)=T_g(\Delta\phi_u)\quad\text{and}\quad\Delta_4T_g(\phi_u)=T_g(\Delta_4\phi_u).$$

With this in mind, it is easy to verify that $\phi_{T_g(u)}$ and $T_g(\phi_u)$ are critical point of $\mathcal{E}_{T_g(u)}$. Hence, by the uniqueness of the critical point of $\mathcal{E}_{T_g(u)}$, we infer that

$$\phi_{T_g(u)}=T_g(\phi_u),$$

for all $g\in O(2)$. In particular, if u is radially symmetric, i.e., $u\in Y$ is a fixed point for the action T_g, ϕ_u is radial too and the result follows. This concludes the proof of the lemma. □

So, we can consider a C^1 functional $I:E\to\mathbb{R}$ defined by $I(u):=J(u,\phi_u)$, that is,

$$\begin{aligned}I(u)=&\frac{1}{2}\int_{\mathbb{R}^2}\left(|\nabla u|^2+\left[m^2-(\omega+\phi_u)^2\right]V(|x|)u^2\right)\mathrm{d}x\\ &-\frac{1}{8\pi}\int_{\mathbb{R}^2}|\nabla\phi_u|^2\,\mathrm{d}x-\frac{\beta}{16\pi}\int_{\mathbb{R}^2}|\nabla\phi_u|^4\,\mathrm{d}x-\int_{\mathbb{R}^2}K(|x|)F(u)\,\mathrm{d}x\end{aligned} \tag{3.4}$$

with Gâteaux derivative given by

$$\begin{aligned}I'(u)v=&\int_{\mathbb{R}^2}\left(\nabla u\cdot\nabla v+\left(m^2-\omega^2\right)V(|x|)uv-2V(|x|)\omega\phi_u uv-V(|x|)\phi_u^2uv\right)\mathrm{d}x\\ &-\int_{\mathbb{R}^2}K(|x|)f(u)v\,\mathrm{d}x,\end{aligned} \tag{3.5}$$

for all $v\in E$.

After using (3.3) with ϕ_u and through simple computation, we deduce

$$-\int_{\mathbb{R}^2}\left(|\nabla\phi_u|^2+\beta|\nabla\phi_u|^4\right)\mathrm{d}x=4\pi\int_{\mathbb{R}^2}(\omega+\phi_u)\phi_uV(|x|)u^2\,\mathrm{d}x. \tag{3.6}$$

Therefore, the reduced functional also takes the form

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^2} \left(|\nabla u|^2 + \left(m^2-\omega^2\right) V(|x|)u^2 + V(|x|)\phi_u^2 u^2\right) \mathrm{d}x \\ + \frac{1}{8\pi}\int_{\mathbb{R}^2} |\nabla\phi_u|^2 \,\mathrm{d}x + \frac{3\beta}{16\pi}\int_{\mathbb{R}^2} |\nabla\phi_u|^4 \,\mathrm{d}x - \int_{\mathbb{R}^2} K(|x|)F(u)\,\mathrm{d}x. \tag{3.7}$$

Throughout the rest of the paper, and according the convenience, we will use both forms (3.4) or (3.7). Now, following [6], a pair $(u,\phi) \in E \times \mathcal{D}$ is a critical point for J if and only if u is a critical point for I with $\phi = \phi_u$. Hence, we will look for its critical points. The next lemma shows that E actually is, in some sense, a natural constraint for finding weak solutions of problem (1.2). In fact, it is a symmetric criticality type result.

Lemma 3.2. *Assume that $(V,K) \in \mathcal{K}$ and the hypothesis (f_1) holds. Then, every critical point $u \in E$ of $I\colon E \to \mathbb{R}$ is a weak solution to problem (1.2), that is, satisfies (1.5) with $\phi = \phi_u$.*

Proof. We will show that if $u \in E$ satisfies (1.5) with $\phi = \phi_u$ and for all $v \in E$, then (1.5) holds also true for all $v \in Y$. Let $u \in E$. By Hölder's inequality, Lemma 2.4 and the growth assumption (f_1) on nonlinear term f yield a positive constant $C = C(\|u\|)$ such that

$$\left|\int_{\mathbb{R}^2} K(|x|)f(u)v\,\mathrm{d}x\right| \le C\|v\|, \qquad \forall v \in Y.$$

Thus, the linear functional $T_u : Y \to \mathbb{R}$ defined by

$$T_u(v) := \int_{\mathbb{R}^2} \left(\nabla u\cdot\nabla v + [m^2 - (\omega+\phi_u)^2]V(|x|)uv\right) \mathrm{d}x - \int_{\mathbb{R}^2} K(|x|)f(u)v\,\mathrm{d}x,$$

is well-defined and continuous on Y and so, by the Riesz Representation Theorem in the space Y with the inner product (1.4), there exists a unique $\tilde{u} \in Y$ such that $T_u(\tilde{u}) = \|\tilde{u}\|^2 = \|T_u\|_{Y'}$, where Y' denotes the dual space of Y. Then, by using change of variables, one has for each $v \in Y$

$$T_u(gv) = T_u(v) \quad \text{and} \quad \|gv\| = \|v\|, \quad \text{for all } g \in O(2),$$

whence, applying with $v = \tilde{u}$, one deduce, by uniqueness, $g\tilde{u} = \tilde{u}$, for all $g \in O(2)$, which means, $\tilde{u} \in E$. Hence, since $T_u(v) = 0$ for all $v \in E$, one has $T_u(\tilde{u}) = 0$, that is, $\|T_u\|_{Y'} = 0$ and therefore (1.5) with $\phi = \phi_u$ ensues. This concludes the proof of the lemma. □

In the next lemma, we show that the functional I satisfies the geometric conditions of the Mountain-Pass Theorem.

Lemma 3.3. *Suppose that $(V,K) \in \mathcal{K}$ and (f_0)–(f_2) hold. If $|m| > \omega > 0$, then*

1. *there exist some constants $\tau, \rho > 0$ such that $I(u) \ge \tau$ provided $\|u\| = \rho$;*
2. *there exists $v \in E$ satisfying $\|v\| > \rho$ and $I(v) < 0$.*

Proof. 1. From (3.1), we get

$$\int_{\mathbb{R}^2} K(|x|)F(u)\,\mathrm{d}x \le \frac{\varepsilon}{2}\int_{\mathbb{R}^2} K(|x|)u^2\,\mathrm{d}x + b_1\int_{\mathbb{R}^2} K(|x|)|u|^q\left(e^{\alpha u^2}-1\right)\mathrm{d}x.$$

Let $r_1, r_2 > 1$ be such that $\frac{1}{r_1} + \frac{1}{r_2} = 1$. By Hölder's inequality and (3.2), we infer

$$\int_{\mathbb{R}^2} K(|x|)|u|^q \left(e^{\alpha u^2} - 1\right) \mathrm{d}x \le \left(\int_{\mathbb{R}^2} K(|x|)\, |u|^{qr_1} \,\mathrm{d}x\right)^{\frac{1}{r_1}} \left(\int_{\mathbb{R}^2} K(|x|) \left(e^{\alpha r_2 u^2} - 1\right) \mathrm{d}x\right)^{\frac{1}{r_2}}$$
$$\le \|u\|^q_{qr_1;K} \left(\int_{\mathbb{R}^2} K(|x|) \left(e^{\alpha r_2 M^2 \left(\frac{u}{\|u\|}\right)^2} - 1\right) \mathrm{d}x\right)^{\frac{1}{r_2}}.$$

Choosing $r_2 > 1$ sufficiently close to 1 and $0 < M < \left(\frac{\lambda}{r_2\alpha}\right)^{\frac{1}{2}}$, then for $\|u\| \le M$, it follows from Corollary 2.6 that

$$\int_{\mathbb{R}^2} K(|x|) \left(e^{\alpha r_2 M^2 \left(\frac{u}{\|u\|}\right)^2} - 1\right) \mathrm{d}x \le C.$$

Hence, from Lemma 2.4, we deduce that

$$\int_{\mathbb{R}^2} K(|x|)F(u)\,\mathrm{d}x \le \frac{C_1\varepsilon}{2}\|u\|^2 - C_2\|u\|^q.$$

Consequently, since $|m| > \omega > 0$, by (3.7) we have

$$I(u) \ge \left(\frac{\min\{1, m^2 - \omega^2\}}{2} - \frac{C_1\varepsilon}{2}\right)\|u\|^2 - C_2\|u\|^q$$
$$= \left(\frac{\min\{1, m^2 - \omega^2\}}{2} - \frac{C_1\varepsilon}{2}\right)\rho^2 - C_2\rho^q$$

and, choosing $\varepsilon > 0$ sufficiently small such that $C_3 := \frac{\min\{1,m^2-\omega^2\}}{2} - \frac{C_1\varepsilon}{2} > 0$,

$$I(u) \ge C_3\rho^2 - C_2\rho^q.$$

Inasmuch $q > 2$, for $\rho > 0$ small enough, there exists $\tau > 0$ such that

$$I(u) \ge \tau, \quad \text{for any } u \in E \text{ with } \|u\| = \rho.$$

2. By the Ambrosetti–Rabinowitz type condition (f_2), for all $\delta > 0$, there exists a positive constant $C_4 = C_4(\delta)$ such that $F(s) \ge C_4|s|^\theta - \delta s^2$, for all $s \in \mathbb{R}$. Let $\varphi \in C^\infty_{0,\mathrm{rad}}(\mathbb{R}^2)$ be such that $\mathrm{supp}(\varphi)$ is a compact set of $\mathbb{R}^2$. Thus, by (3.4) and Lemma 2.4, we have

$$I(t\varphi) \le \frac{\max\{1, m^2\}}{2} t^2\|\varphi\|^2 - C_4 t^\theta \int_{\mathrm{supp}(\varphi)} K(|x|)|\varphi|^\theta \,\mathrm{d}x + \delta t^2 \int_{\mathrm{supp}(\varphi)} K(|x|)\varphi^2 \,\mathrm{d}x$$
$$\le \left(\frac{\max\{1, m^2\}}{2} + C_5\delta\right) t^2\|\varphi\|^2 - C_4 t^\theta \int_{\mathrm{supp}(\varphi)} K(|x|)|\varphi|^\theta \,\mathrm{d}x$$
$$\to -\infty, \quad \text{as } t \to +\infty,$$

since $\theta > 2$. Therefore, for t large enough and taking $v := t\varphi$ we conclude that $I(v) < 0$ and the lemma is proved. □

Next, we investigate the compactness conditions for the functional I. Recall that $(u_n) \subset E$ is a Palais–Smale, (P–S) for short, sequence at a level $c \in \mathbb{R}$ for the functional I if

$$I(u_n) \to c, \quad I'(u_n) \to 0, \text{ as } n \to +\infty,$$

where the second limit above occurs in the dual space E'. We say that I satisfies the Palais–Smale compactness condition if any (P–S) sequence has a convergent subsequence.

Lemma 3.4 (Boundedness). *Let $(u_n) \subset E$ be a (P–S) sequence at a level $c \in \mathbb{R}$ for the functional I. Then (u_n) is bounded in E.*

Proof. Let $(u_n) \subset E$ be a (P–S) sequence at a level $c \in \mathbb{R}$ for the functional I. In order to check that (u_n) is bounded in E, there are two cases to be considered: either $\theta > 4$ or $2 < \theta \leq 4$ and $\theta - 2 > 2\omega^2$.

Case 1: $\theta > 4$. Combining (3.5), (3.6), (3.7) and (f_2) together we can estimate

$$
\begin{aligned}
&\theta(c+1) + o_n(1)\|u_n\| \\
&\quad \geq \theta I(u_n) - I'(u_n)u_n \\
&\quad = \left(\frac{\theta}{2} - 1\right) \int_{\mathbb{R}^2} \left(|\nabla u_n|^2 + \left(m^2 - \omega^2\right) V(|x|) u_n^2\right) \mathrm{d}x \\
&\qquad + \left(\frac{\theta}{2} + 1\right) \int_{\mathbb{R}^2} K(|x|) \phi_{u_n}^2 u_n^2 \, \mathrm{d}x + 2 \int_{\mathbb{R}^2} K(|x|) \omega \phi_{u_n} u_n^2 \, \mathrm{d}x \\
&\qquad + \frac{\theta}{8\pi} \int_{\mathbb{R}^2} |\nabla \phi_{u_n}|^2 \, \mathrm{d}x + \frac{3\beta\theta}{16\pi} \int_{\mathbb{R}^2} |\nabla \phi_{u_n}|^4 \, \mathrm{d}x + \int_{\mathbb{R}^2} K(|x|) [f(u_n) u_n - \theta F(u_n)] \, \mathrm{d}x \\
&\quad \geq \left(\frac{\theta}{2} - 1\right) \int_{\mathbb{R}^2} \left(|\nabla u_n|^2 + \left(m^2 - \omega^2\right) V(|x|) u_n^2\right) \mathrm{d}x + 2 \int_{\mathbb{R}^2} K(|x|) (\phi_{u_n} + \omega) \phi_{u_n} u_n^2 \, \mathrm{d}x \\
&\qquad + \frac{\theta}{8\pi} \int_{\mathbb{R}^2} |\nabla \phi_{u_n}|^2 \, \mathrm{d}x + \frac{3\beta\theta}{16\pi} \int_{\mathbb{R}^2} |\nabla \phi_{u_n}|^4 \, \mathrm{d}x \\
&\quad = \left(\frac{\theta}{2} - 1\right) \int_{\mathbb{R}^2} \left(|\nabla u_n|^2 + \left(m^2 - \omega^2\right) V(|x|) u_n^2\right) \mathrm{d}x + \left(\frac{\theta}{8\pi} - \frac{1}{2\pi}\right) \int_{\mathbb{R}^2} |\nabla \phi_{u_n}|^2 \, \mathrm{d}x \\
&\qquad + \left(\frac{3\beta\theta}{16\pi} - \frac{\beta}{2\pi}\right) \int_{\mathbb{R}^2} |\nabla \phi_{u_n}|^4 \, \mathrm{d}x \\
&\quad \geq \frac{\max\{\theta - 2, m^2 - \omega^2\}}{2} \|u_n\|^2.
\end{aligned}
$$

Before passing to the next case, we need first to rewrite $\theta I(u)$ as follows. By (3.4) and (3.6), we can write

$$
\begin{aligned}
\theta I(u_n) &= \frac{\theta}{2} \int_{\mathbb{R}^2} \left(|\nabla u_n|^2 + \left(m^2 - \omega^2\right) V(|x|) u_n^2\right) \mathrm{d}x - \theta \int_{\mathbb{R}^2} V(|x|) \omega \phi_{u_n} u_n^2 \, \mathrm{d}x \\
&\quad - \frac{\theta}{2} \int_{\mathbb{R}^2} V(|x|) \phi_{u_n}^2 u_n^2 \, \mathrm{d}x - \frac{\theta}{8\pi} \int_{\mathbb{R}^2} |\nabla \phi_{u_n}|^2 \, \mathrm{d}x - \frac{\beta\theta}{16\pi} \int_{\mathbb{R}^2} |\nabla \phi_{u_n}|^4 \, \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^2} K(|x|) \theta F(u_n) \, \mathrm{d}x \\
&= \frac{\theta}{2} \int_{\mathbb{R}^2} \left(|\nabla u_n|^2 + \left(m^2 - \omega^2\right) V(|x|) u_n^2\right) \mathrm{d}x - \frac{\theta}{2} \int_{\mathbb{R}^2} V(|x|) \omega \phi_{u_n} u_n^2 \, \mathrm{d}x \\
&\quad + \frac{\beta\theta}{16\pi} \int_{\mathbb{R}^2} |\nabla \phi_{u_n}|^4 \, \mathrm{d}x - \int_{\mathbb{R}^2} K(|x|) \theta F(u_n) \, \mathrm{d}x.
\end{aligned}
$$

Now, we are able to treat the next case.

Case 2: $2 < \theta \leq 4$ and $\theta - 2 > 2\omega^2$. By using $\theta I(u_n)$ rewritten above, (3.5) and (f_2), we can estimate

$$
\begin{aligned}
&\theta(c+1) + o_n(1)\|u_n\| \\
&\quad \geq \theta I(u_n) - I'(u_n)u_n \\
&\quad = \left(\frac{\theta}{2} - 1\right) \int_{\mathbb{R}^2} \left(|\nabla u_n|^2 + \left(m^2 - \omega^2\right) V(|x|) u_n^2\right) \mathrm{d}x + \int_{\mathbb{R}^2} V(|x|) \phi_{u_n}^2 u_n^2 \, \mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&- \left(\frac{\theta}{2} - 2\right) \int_{\mathbb{R}^2} V(|x|)\omega\phi_{u_n} u_n^2 \,\mathrm{d}x + \frac{\beta\theta}{16\pi} \int_{\mathbb{R}^2} |\nabla \phi_{u_n}|^4 \,\mathrm{d}x \\
&+ \int_{\mathbb{R}^2} K(|x|)[f(u_n)u_n - \theta F(u_n)] \,\mathrm{d}x \\
\geq &\left(\frac{\theta}{2} - 1\right) \int_{\mathbb{R}^2} (|\nabla u_n|^2 + m^2 V(|x|)u_n^2) \,\mathrm{d}x - \omega^2 \int_{\mathbb{R}^2} V(|x|)u_n^2 \,\mathrm{d}x \\
\geq &\left(\frac{\max\{\theta - 2, m^2\}}{2} - \omega^2\right) \|u_n\|^2.
\end{aligned}
$$

In any case, we infer that (u_n) stays bounded in E, concluding the proof of the lemma. □

In view of the mountain-pass geometry of I assured by Lemma 3.3, we introduce the mountain pass level

$$
c_\mu := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) \geq \tau > 0,
$$

where the set of paths is defined as

$$
\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0 \text{ and } I(\gamma(1)) < 0\}.
$$

With the purpose to verify that I satisfies the Palais–Smale condition in certain levels of energy we will need the following upper bound for the mountain-pass level c_μ:

Lemma 3.5 (Level estimate). *Suppose that (f_3) is satisfied with*

$$
\mu \geq \mu_0 := \max\left\{\mu_1, \left[\frac{2\alpha_0\theta(\vartheta - 2)\|K\|_{L^1(B_1)}}{\lambda\vartheta(\theta - 2)}\right]^{\frac{\vartheta-2}{2}} \left(\frac{2\mu_1}{\vartheta}\right)^{\frac{\vartheta}{2}}\right\},
$$

where $\mu_1 = \frac{\vartheta \max\{1,m^2\}\left(4\pi + \|V\|_{L^1(B_2)}\right)}{2\|K\|_{L^1(B_1)}}$. *Then*

$$
c_\mu < \frac{\lambda}{2\alpha_0}\left(\frac{1}{2} - \frac{1}{\theta}\right). \tag{3.8}
$$

Proof. We shall consider a cut-off function $\varphi_0 \in C_0^\infty(\mathbb{R}^2)$ verifying

$$
0 \leq \varphi_0 \leq 1 \quad \text{in } \mathbb{R}^2, \qquad \varphi_0 \equiv 1 \quad \text{in } \overline{B}_1, \qquad \varphi_0 \equiv 0 \quad \text{in } B_2^c \quad \text{and} \quad |\nabla\varphi_0| \leq 1 \quad \text{in } \mathbb{R}^2.
$$

From (3.4) and (f_3), we get

$$
\begin{aligned}
I(\varphi_0) &\leq \frac{\max\{1, m^2\}}{2} \int_{B_2} (|\nabla\varphi_0|^2 + V(|x|)\varphi_0^2) \,\mathrm{d}x - \frac{\mu_1}{\vartheta} \int_{B_2} K(|x|)|\varphi_0|^\vartheta \,\mathrm{d}x \\
&< \frac{\max\{1, m^2\}}{2} \left(4\pi + \|V\|_{L^1(B_2)}\right) - \frac{\mu_1}{\vartheta}\|K\|_{L^1(B_1)} = 0,
\end{aligned}
$$

since $\mu_1 = \frac{\vartheta \max\{1,m^2\}\left(4\pi + \|V\|_{L^1(B_2)}\right)}{2\|K\|_{L^1(B_1)}}$. In particular,

$$
\frac{\max\{1, m^2\}}{2} \int_{B_2} (|\nabla\varphi_0|^2 + V(|x|)\varphi_0^2) \,\mathrm{d}x < \frac{\mu_1}{\vartheta}\|K\|_{L^1(B_1)}. \tag{3.9}
$$

According to the definition of c_μ, (3.4), (3.9) and straightforward manipulations, we deduce that

$$\begin{aligned} c_\mu &\leq \max_{t\geq 0}\left[\frac{\max\{1,m^2\}}{2}t^2\int_{B_2}(|\nabla\varphi_0|^2+V(|x|)\varphi_0^2)\,\mathrm{d}x - t^\vartheta\frac{\mu}{\vartheta}\int_{B_2}K(|x|)|\varphi_0|^\vartheta\,\mathrm{d}x\right] \\ &< \max_{t\geq 0}\left[\frac{\mu_1}{\vartheta}\|K\|_{L^1(B_1)}t^2 - \frac{\mu}{\vartheta}\|K\|_{L^1(B_1)}t^\vartheta\right] \\ &\leq \frac{\|K\|_{L^1(B_1)}}{\vartheta}\max_{t\geq 0}\left[\mu_1 t^2-\mu t^\vartheta\right] \\ &= \frac{\|K\|_{L^1(B_1)}}{\vartheta}(\vartheta-2)\left(\frac{2}{\mu}\right)^{\frac{2}{\vartheta-2}}\left(\frac{\mu_1}{\vartheta}\right)^{\frac{\vartheta}{\vartheta-2}}. \end{aligned} \tag{3.10}$$

Thus, if

$$\mu \geq \left[\frac{2\alpha_0\theta(\vartheta-2)\|K\|_{L^1(B_1)}}{\lambda\vartheta(\theta-2)}\right]^{\frac{\vartheta-2}{2}}\left(\frac{2\mu_1}{\vartheta}\right)^{\frac{\vartheta}{2}},$$

we immediately arrive at estimate (3.8), concluding the proof of the lemma. □

Corollary 3.6 (Behavior of the minimax level)**.** *The minimax level vanishes, i.e., $c_\mu \to 0$ as $\mu \to +\infty$.*

Proof. This can be easily checked as a byproduct from the proof of Lemma 3.5, specifically estimate (3.10). □

Taking into account Lemma 3.3, we may apply the Mountain-Pass Theorem without the Palais–Smale compactness condition (see [5]) to guarantee the existence of a (P–S) sequence (u_n) in E at the level c_μ. To obtain the existence of nontrivial solutions to (1.2), the following technical result will be useful and plays a crucial role in the proof of Theorem 1.2.

Lemma 3.7. *The sequence $(u_n) \subset E$ obtained above satisfies*

$$\sup_{n\geq 1}\|f(u_n)\|_{2;K} < +\infty. \tag{3.11}$$

Proof. We begin the proof estimating the quantity $\theta I(u_n)$. For this aim, similarly was done in the proof of Lemma 3.4, we also divide our proof into two cases about θ as follows.

Case 1: $\theta > 4$.

$$\begin{aligned} \theta I(u_n) &= \theta I(u_n) - I'(u_n)u_n + o_n(1) \\ &\geq \frac{\max\{\theta-2, m^2-\omega^2\}}{2}\|u_n\|^2 + o_n(1) \to \theta c_\mu, \quad \text{as } n\to+\infty. \end{aligned}$$

Hence, invoking the level estimate (3.8) and Corollary 3.6, for any $\mu > \mu_0$, it follows that

$$\frac{\theta c_\mu}{\frac{\max\{\theta-2,m^2-\omega^2\}}{2}} < \frac{\lambda}{2\alpha_0}.$$

Case 2: $2 < \theta \leq 4$ and $\theta - 2 > 2\omega^2$.

$$\begin{aligned} \theta I(u_n) &= \theta I(u_n) - I'(u_n)u_n + o_n(1) \\ &\geq \left(\frac{\max\{\theta-2, m^2\}}{2} - \omega^2\right)\|u_n\|^2 + o_n(1) \to \theta c_\mu, \quad \text{as } n\to+\infty. \end{aligned}$$

Again, by virtue of (3.8) and Corollary 3.6, for any $\mu > \mu_0$, it follows that

$$\frac{\theta c_\mu}{\frac{\max\{\theta-2,m^2\}}{2} - \omega^2} < \frac{\lambda}{2\alpha_0}.$$

Thereby, in any case, we deduce that

$$\limsup_{n\to+\infty} \|u_n\|^2 < \frac{\lambda}{2\alpha_0},$$

and in view of Trudinger–Moser type inequality (2.1) we conclude that

$$\sup_{n\geq 1} \int_{\mathbb{R}^2} K(|x|) \left(e^{2\alpha_0 u_n^2} - 1\right) \mathrm{d}x < +\infty. \tag{3.12}$$

On the other hand, by (f_0) and (f_1), and using the fact that $2\alpha_0 > \alpha_0$, there exists a positive constant C_1 such that

$$|f(u_n)|^2 \leq C_1 \left(u_n^2 + e^{2\alpha_0 u_n^2} - 1\right).$$

Therefore, having in mind that (u_n) is bounded in $L^2(\mathbb{R}^2; K)$ and (3.12), our lemma immediately follows. □

4 Proof of the main results

In this section, we will prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. Let $(u_n) \subset E$ be the (P–S) sequence at the level c_μ. From Lemma 3.4, (u_n) is bounded in E, which implies the weak convergence $u_n \rightharpoonup u_0$ in E. We shall prove that, up to a subsequence, $u_n \to u_0$ strongly in E and $(u_0, \phi_{u_0}) \in E \times \mathcal{D}$ is a weak solution of (1.2). Set

$$\mathcal{I}_n^1 := \int_{\mathbb{R}^2} K(|x|) f(u_n)(u_n - u_0) \,\mathrm{d}x \tag{4.1}$$

and

$$\mathcal{I}_n^2 = \int_{\mathbb{R}^2} K(|x|)\phi_{u_n} u_n (u_n - u_0) \,\mathrm{d}x, \quad \mathcal{I}_n^3 = \int_{\mathbb{R}^2} K(|x|)\phi_{u_n}^2 u_n (u_n - u_0) \,\mathrm{d}x. \tag{4.2}$$

We claim that $\mathcal{I}_n^1$, $\mathcal{I}_n^2$, $\mathcal{I}_n^3 \to 0$, as $n \to +\infty$. Let us to check these convergences in the following steps:

Step 1: $\mathcal{I}_n^1 = o_n(1)$, as $n \to +\infty$. In fact, by Hölder's inequality

$$|\mathcal{I}_n^1| \leq \|f(u_n)\|_{2;K} \|u_n - u_0\|_{2;K}.$$

The compact embedding $E \hookrightarrow L^2(\mathbb{R}^2; K)$ implies that $u_n \to u_0$ strongly in $L^2(\mathbb{R}^2; K)$. Consequently,

$$\|u_n - u_0\|_{2;K} \to 0, \text{ as } n \to +\infty,$$

and from (3.11) we get the first convergence.

Step 2: $\mathcal{I}_n^2,\ \mathcal{I}_n^3 \to 0$, as $n \to +\infty$. In fact, combining Hölder's inequality, Lemmas 2.4, 3.1 and the boundedness of (u_n) in E, we have

$$\begin{aligned}|\mathcal{I}_n^2| &\le \int_{\mathbb{R}^2} K(|x|)|\phi_{u_n}||u_n||u_n - u_0|\,\mathrm{d}x \\ &\le \left(\int_{\mathbb{R}^2} K(|x|)\phi_{u_n}^2 {u_n}^2\,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} K(|x|)(u_n - u_0)^2\,\mathrm{d}x\right)^{\frac{1}{2}} \\ &\le \omega\|u_n\|_{2;K}\|u_n - u_0\|_{2;K} \le \omega C_1\|u_n\|\|u_n - u_0\|_{2;K} \\ &\le C_2\|u_n - u_0\|_{2;K} \to 0, \text{ as } n \to +\infty,\end{aligned}$$

since, again by Lemma 2.4, $u_n \to u_0$ strongly in $L^2(\mathbb{R}^2; K)$. Analogously, $\mathcal{I}_n^3 \to 0$, as $n \to +\infty$. Thus, from (4.1), (4.2) and having in mind that

$$\lim_{n\to\infty} I'(u_n)(u_n - u_0) = 0,$$

it leads to

$$\int_{\mathbb{R}^2} \left(\nabla u_n \cdot \nabla(u_n - u_0) + (m^2 - \omega^2)V(|x|)u_n(u_n - u_0)\right)\mathrm{d}x = o_n(1).$$

Now, as an immediate consequence of the weak convergence $u_n \rightharpoonup u_0$ in E, we have

$$\int_{\mathbb{R}^2} (\nabla u_0 \cdot \nabla(u_n - u_0) + V(|x|)u_0(u_n - u_0))\,\mathrm{d}x = o_n(1).$$

Combining that last identities, we conclude that $u_n \to u_0$ strongly in E. Since I and I' are continuous, then

$$I'(u_n) = o_n(1) \to I'(u_0) = 0 \ \text{ and } \ I(u_n) \to I(u_0) = c_\mu > 0,$$

proving that u_0 is a nontrivial critical point of the functional I and, consequently, (u_0, ϕ_{u_0}) is a solution of (1.2). Finally, it remains to check that u_0 is nonnegative. But, it just suffices to observe that $I'(u_0)(u_0^-) = 0$ which leads to $\|u_0^-\|^2 = 0$ and therefore $u_0 = u_0^+ \ge 0$. This completes the proof. □

To finish the paper, we give the end of our proof.

Proof of Theorem 1.3. Our goal is to show that (u_0, ϕ_{u_0}) is a ground state solution, that is, is a solution which minimizes the functional J among all the nontrivial solutions of (1.2), namely, $J(u_0, \phi_{u_0}) \le J(u, \phi)$ for any nontrivial solution (u, ϕ) of (1.2). In this direction, this aim will carry out by considering a minimization problem where the constraint is defined by the Nehari manifold. By a ground state solution of system (1.2) we mean a nontrivial solution $(\tilde{u}, \phi_{\tilde{u}}) \in E \times \mathcal{D}$ of (1.2) such that

$$I(\tilde{u}) = \min\{I(u) : u \in E \setminus \{0\} \text{ is a critical point of } I\}.$$

So, let

$$M_\mu := \min_{u\in\mathcal{N}} I(u),$$

where $\mathcal{N}$ is the Nehari manifold

$$\mathcal{N} := \{u \in E \setminus \{0\} : I'(u)u = 0\}.$$

For this aim, it is sufficient to prove that $c_\mu \le M_\mu$. The Nehari manifold $\mathcal{N}$ is closely linked to the behavior of the function of the form $h_u : t \to I(tu)$ for $t > 0$. Such map is known as fibering map. Let $u \in \mathcal{N}$, from (3.4), we find

$$h'_u(t) = t\int_{\mathbb{R}^2} \left(|\nabla u_n|^2 + \left(m^2 - \omega^2\right) V(|x|)u_n^2\right) \mathrm{d}x - 2t\int_{\mathbb{R}^2} V(|x|)\omega\phi_u u^2\,\mathrm{d}x$$
$$- t\int_{\mathbb{R}^2} V(|x|)\phi_u^2 u^2\,\mathrm{d}x - \int_{\mathbb{R}^2} K(|x|)f(tu)u\,\mathrm{d}x.$$

Since $I'(u)u = 0$, as a direct consequence, we obtain

$$h'_u(t) = t\int_{\mathbb{R}^2} K(|x|)\left[\frac{f(u)}{u} - \frac{f(tu)}{tu}\right] u^2\,\mathrm{d}x,$$

for $t > 0$. Taking into account that $f(s)/s$ is increasing for $s > 0$, we infer that $h'_u(t) > 0$ for $t \in (0,1)$ and $h'_u(t) < 0$ for $t \in (1,\infty)$. Hence, after observing $h'_u(1) = 0$, we conclude that $I(u) = \max_{t\ge 0} I(tu)$. Setting $\gamma(t) := tt_0u$, for $t \in [0,1]$, where t_0 is such that $I(t_0u) < 0$, we have $\gamma \in \Gamma$, and so

$$c_\mu \le \max_{t\in[0,1]} I(\gamma(t)) \le \max_{t\ge 0} I(tu) = I(u).$$

Thereby, since $u \in \mathcal{N}$ is arbitrary $c_\mu \le M_\mu$. This implies that (u_0, ϕ_{u_0}) is a ground state solution for (1.2) and, therefore, the proof of Theorem 1.3 is finished. □

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