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General solutions to four classes of nonlinear difference equations and some of their representations

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Abstract. We present general solutions to four classes of nonlinear difference equations, as well as some representations of the general solutions for two of the classes in terms of specially chosen solutions to linear homogeneous difference equations with constant coefficients which are naturally associated to the equations of the classes. Our main results considerably generalize some very special ones in recent literature, and present concrete methods for solving the equations.

Keywords: difference equation, solvable difference equation, representation of general solution.

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1 Introduction

In this paper $\mathbb{N}$ denotes the set of all positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ the set of all integers, $\mathbb{R}$ the set of all real numbers, whereas the set of all complex numbers is denoted by $\mathbb{C}$.

One of the oldest problems regarding difference equations is their solvability. De Moivre solved the following equation

$$y_{n+2} = ay_{n+1} + by_n, \quad n \in \mathbb{N}, \tag{1.1}$$

where $b \neq 0$ [6], by showing that

$$y_n = \frac{(y_2 - t_2y_1)t_1^{n-1} + (t_1y_1 - y_2)t_2^{n-1}}{t_1 - t_2}, \quad n \in \mathbb{N}, \tag{1.2}$$

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is the general solution to equation (1.1) when $a^2 + 4b \neq 0$. The corresponding general solution to the difference equation in the case $a^2 = -4b$, as well as the general solutions to the linear homogeneous difference equation of the third order, in all possible cases, can be found in [8], which motivated further investigations of linear difference equations.

Some investigations of solvability of nonhomogeneous linear difference equations can be found in [13], where, among other things, it was solved the linear first-order difference equation, that is, the equation

$$x_{n+1} = p_n x_n + q_n, \quad n \in \mathbb{N}_0,$$

(another method for solving the equation was presented in [14]; note that the equation is with variable coefficients and that its special cases frequently appear in the literature, see, e.g., [5,9,11,12,16–19,28,30,33,34,37]). As a consequence of the obtained formula for the general solution to the equation it was obtained that in the case $p_n = p$, $q_n = q$, $n \in \mathbb{N}_0$, the general solution is

$$x_n = p^n x_0 + q \sum_{j=0}^{n-1} p^j, \tag{1.3}$$

for $n \in \mathbb{N}_0$, from which are easily obtained the corresponding general solutions in the cases $p \neq 1$ and $p = 1$.

Investigation of solvability of some classes of nonlinear difference equations can be found in [14] yet, from which it is clear that the solvability of the following difference equation

$$x_{n+1} = \frac{ax_n + b}{cx_n + d}, \quad n \in \mathbb{N}_0, \tag{1.4}$$

has been essentially known at the time. For more information on the difference equation and some of its applications see, e.g., [1,4,15,18,31,36,39,42].

A lack of new important and general solvability methods for difference equations and systems of difference equations caused a turn in their investigations to some other topics, which can be noticed in the literature from the previous century (see, for example, [9,11,15,16,19]).

Recent development of computers enabled researchers to do some calculations and to make some conjectures concerning solvability easier, what can be seen in recent literature. However, many of the recent papers on solvability hardly use any theory producing some issues, which have been mentioned and discussed in some of our recent papers [31,36–40,42]. Note that, as a topic of wide interest, solvability and their applications appear frequently in popular mathematical literature (see, e.g., [1,12,17,18,28]).

A paper on a special case of the difference equation

$$x_{n+2} = \frac{x_n}{\alpha + \beta x_{n+1} x_n}, \quad n \in \mathbb{N}_0, \tag{1.5}$$

presenting some closed-form formulas for their solutions, motivated us to explain them theoretically in 2004. This, among other things, motivated some further investigations on solvability of related difference equations and systems of difference equations (see, e.g., [27,30,37] and the references therein; see also [3]). Not long before it started some investigations of many concrete classes of systems of difference equations [20–25], motivating us to study solvability of the systems corresponding to some solvable difference equations (see, e.g, [5,32,37,41]). Recall that there are numerous applications of solvable difference equations and systems in many research areas (see, e.g., [4,6,9–12,14,15,17,18,28,29,33–35]). Let us also mention that beside general solutions, invariants for difference equations and systems of difference equations are of some importance (see, e.g., [21–23,26] and the related references therein).

The Fibonacci sequence is the solution to the following special case of difference equation (1.1)

$$x_{n+2} = x_{n+1} + x_n, \quad n \in \mathbb{N}, \tag{1.6}$$

such that $x_1 = x_2 = 1$. Since from (1.6) we have $x_n = x_{n+2} - x_{n+1}$, we see that it can be also calculated for $n \leq 0$. The sequence is denoted by f_n, and by using formula (1.2) it is obtained the corresponding closed-form formula for the sequence. Moreover, from formula (1.2) we have that every solution to equation (1.6) can be represented in the following form

$$x_n = x_1 f_{n-2} + x_2 f_{n-1}, \quad n \in \mathbb{Z}. \tag{1.7}$$

Representation (1.7) is well-known and it is a basic one (see, e.g., [2]). For some other results on the Fibonacci sequences, including many identities and relations involving the sequence, as well as their various applications in mathematics, see, e.g., [2,12,17,43].

There are some recent papers which give representations of solutions to very special cases of some solvable difference equations in terms of the Fibonacci sequence. Bearing in mind that practically none of these papers is based on a mathematical theory and that they use only some simple inductive arguments (if they present any one), we have done some research in this direction. For example, in [31,36,38,39,42] we explained the theoretical background lying behind some of such difference equations. Paper [40] explains a representation for the general solution to a second-order difference equation in terms of a related sequence which is a solution to a linear homogeneous difference equation of third order.

One of the papers which gives representations of solutions to some nonlinear difference equations is [7], where seven difference equations are considered or mentioned. Some theoretical explanations for three out of the seven difference equations were given in [38].

Here we present some methods for getting the general solutions to the other four difference equations considered in [7]. In fact, we consider here four classes of difference equations which include the four difference equations respectively as their very special cases. Besides, for two of these four classes we present some representations of their general solutions in terms of specially chosen solutions to some linear homogeneous difference equations with constant coefficients which are naturally associated to the equations of the classes.

Now we list these four difference equations, as well as the corresponding closed-form formulas for their general solutions presented in [7]. Before this, to avoid any possible confusion, we would like to point out that the definition of the Fibonacci sequence given in [7] is different from the standard one appearing in the literature, which we also use here. Namely, they defined the Fibonacci sequence $(F_n)_{n\geq -2}$ as the solution to equation (1.6) such that $x_{-2} = 0$ and $x_{-1} = 1$, which means that $F_n = f_{n+2}$, that is, their Fibonacci sequence is the standard one with the indices shifted for two.

Equation 1. The first difference equation is the following:

$$x_{n+1} = \frac{x_{n-1}x_{n-2}}{x_n + x_{n-2}}, \quad n \in \mathbb{N}_0, \tag{1.8}$$

for which it is said in [7] that its general solution is given by the following two formulas

$$x_{2n} = x_0 \prod_{j=1}^{n} \frac{x_0 f_{2j-1} + x_{-2} f_{2j}}{x_0 f_{2j} + x_{-2} f_{2j+1}}, \tag{1.9}$$

$$x_{2n+1} = x_{-1}\prod_{j=0}^{n}\frac{x_0 f_{2j}+x_{-2}f_{2j+1}}{x_0 f_{2j+1}+x_{-2}f_{2j+2}}, \tag{1.10}$$

for $n \in \mathbb{N}_0$.

Equation 2. The second difference equation is the following:

$$x_{n+1} = \frac{x_n x_{n-1}}{x_n + x_{n-2}}, \quad n \in \mathbb{N}_0, \tag{1.11}$$

for which it is said in [7] that its general solution is given by the following two formulas

$$x_{2n} = x_0\prod_{j=1}^{n}\frac{x_0}{2x_0 j + x_{-2}}, \tag{1.12}$$

$$x_{2n+1} = x_{-1}\prod_{j=0}^{n}\frac{x_0}{x_0(2j+1)+x_{-2}}, \tag{1.13}$$

for $n \in \mathbb{N}_0$.

Equation 3. The third difference equation is the following:

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1} + x_{n-2}}, \quad n \in \mathbb{N}_0, \tag{1.14}$$

for which it is said in [7] that its general solution is given by the following two formulas

$$x_{2n} = \frac{x_0 x_{-1} x_{-2}}{(f_n x_{-1} + f_{n+1}x_{-2})(f_n x_0 + f_{n+1}x_{-1})}, \tag{1.15}$$

$$x_{2n+1} = \frac{x_0 x_{-1} x_{-2}}{(f_{n+1} x_{-1} + f_{n+2}x_{-2})(f_n x_0 + f_{n+1}x_{-1})}, \tag{1.16}$$

for $n \in \mathbb{N}_0$.

Equation 4. The fourth difference equation is the following:

$$x_{n+1} = \frac{x_n x_{n-1}}{x_{n-1} + x_{n-2}}, \quad n \in \mathbb{N}_0, \tag{1.17}$$

for which it is said in [7] that its general solution is given by the following two formulas

$$x_{2n} = x_0\prod_{j=0}^{n-1}\frac{x_0 x_{-1}}{((j+1)x_{-1}+x_{-2})((j+1)x_0+x_{-1})}, \tag{1.18}$$

$$x_{2n+1} = \frac{(x_0 x_{-1})^{n+1}}{\prod_{j=0}^{n}((j+1)x_{-1}+x_{-2})\prod_{j=0}^{n-1}((j+1)x_0+x_{-1})}, \tag{1.19}$$

for $n \in \mathbb{N}_0$.

Remark 1.1. Closed-form formulas given in (1.9), (1.10), (1.12), (1.13), (1.15), (1.16), (1.18) and (1.19), are not proved in [7]. There was simply said that difference equations (1.8), (1.11), (1.14) and (1.17) can be treated similarly as in the previous cases, where some closed-form formulas for the general solutions to two difference equations are proved by induction. This means that [7] does not give any explanation for getting any of the closed-form formulas mentioned there including the formulas (1.9), (1.10), (1.12), (1.13), (1.15), (1.16), (1.18) and (1.19), that is, [7] does not present any constructive method for getting them.

Let $g : \mathcal{D}_g \to \mathbb{R}$, where $\mathcal{D}_g \subseteq \mathbb{R}$, be a bijection from the domain $\mathcal{D}_g$ of function g onto the range $g(\mathcal{D}_g)$.

Here we consider four generalizations of difference equations (1.8), (1.11), (1.14) and (1.17). More specifically, here we consider the following generalization of equation (1.8)

$$x_{n+1} = g^{-1}\left(\frac{g(x_{n-1})g(x_{n-2})}{ag(x_n) + bg(x_{n-2})}\right), \quad n \in \mathbb{N}_0; \tag{1.20}$$

the following generalization of equation (1.11)

$$x_{n+1} = g^{-1}\left(\frac{g(x_n)g(x_{n-1})}{ag(x_n) + bg(x_{n-2})}\right), \quad n \in \mathbb{N}_0; \tag{1.21}$$

the following generalization of equation (1.14)

$$x_{n+1} = g^{-1}\left(\frac{g(x_n)g(x_{n-2})}{ag(x_{n-1}) + bg(x_{n-2})}\right), \quad n \in \mathbb{N}_0; \tag{1.22}$$

and the following generalization of equation (1.17)

$$x_{n+1} = g^{-1}\left(\frac{g(x_n)g(x_{n-1})}{ag(x_{n-1}) + bg(x_{n-2})}\right), \quad n \in \mathbb{N}_0, \tag{1.23}$$

where $a, b \in \mathbb{R}$.

By using the method of transformation, which is based on some suitably chosen changes of variables which transform original difference equations to some known solvable ones, we show that difference equations (1.20)–(1.23) are solvable in closed form. Besides, the representation formulas (1.9), (1.10), (1.15) and (1.16), are considerably extended, by presenting infinitely many other related representations to the general solution to equations (1.20) and (1.22), respectively. This shows that not only the representation in terms of the Fibonacci sequence is one of many possible representations, but also that more or less the sequence is chosen arbitrary. Since many other nonlinear difference equations can be solved by using equation (1.1), it follows that the general solution to some of them can be also represented in terms of any suitably chosen solution to equation (1.1). Hence, from the point of view of solvability, the choice of the Fibonacci sequence in such representations does not have some advantages with the respect to the other ones.

2 An auxiliary result

This section quotes an important and interesting auxiliary result which is employed in the analysis of solvability of difference equation (1.20) which follows in the next section.

Motivated by representation (1.7) of the general solution to equation (1.6) and some other representations which include the Fibonacci sequence (see also [31] and the related references therein), and since the choice of the Fibonacci sequence in the representations looked a bit artificially, we came up with an idea to find other sequences which can be used in similar representations of the general solution to the difference equation.

The following result recently proved in [38], gives an answer to the problem of representing the general solution to equation (1.1) in terms of a specially chosen solution to the equation.

Lemma 2.1. *Assume that $b \neq 0$ and $(s_n(\vec{v}))_{n\in\mathbb{N}_0}$ is the solution to equation (1.1) such that*

$$s_0 = v_0 \quad \text{and} \quad s_1 = v_1. \tag{2.1}$$

Then the following representation

$$y_n = c_1 s_{n+1}(\vec{v}) + c_2 s_n(\vec{v}), \quad n \in \mathbb{N}_0, \tag{2.2}$$

holds for each solution $(y_n)_{n\in\mathbb{N}_0}$ to the equation if and only if

$$v_1^2 \neq v_0(av_1 + bv_0). \tag{2.3}$$

Further, if (2.3) holds, then

$$y_n = \frac{(v_1y_0 - v_0y_1)s_{n+1}(\vec{v}) + (v_1y_1 - av_1y_0 - bv_0y_0)s_n(\vec{v})}{v_1^2 - av_0v_1 - bv_0^2}, \quad n \in \mathbb{N}_0, \tag{2.4}$$

for every solution $(y_n)_{n\in\mathbb{N}_0}$ to the equation.

Remark 2.2. Note that (2.3) does not hold if and only if $v_0 = v_1 = 0$ or when the quantity v_1/v_0 is equal to one of the roots of the characteristic polynomial associated with equation (1.1).

3 Main results

In this section we conduct a detailed analysis related to solvability of each of difference equations (1.20)–(1.23), which leads to the formulations of the main results in this paper.

The first step is to transform each of the equations in a form of some rational difference equations. To do this, first note that since g is one-to-one function it is possible to use the following change of variables

$$y_n = g(x_n), \quad n \geq -2, \tag{3.1}$$

in any of the equations from which it follows that equation (1.20) is transformed to the following one

$$y_{n+1} = \frac{y_{n-1}y_{n-2}}{ay_n + by_{n-2}}, \quad n \in \mathbb{N}_0; \tag{3.2}$$

equation (1.21) is transformed to the following one

$$y_{n+1} = \frac{y_n y_{n-1}}{ay_n + by_{n-2}}, \quad n \in \mathbb{N}_0; \tag{3.3}$$

equation (1.22) is transformed to the following one

$$y_{n+1} = \frac{y_n y_{n-2}}{ay_{n-1} + by_{n-2}}, \quad n \in \mathbb{N}_0; \tag{3.4}$$

whereas equation (1.23) is transformed to the following one

$$y_{n+1} = \frac{y_n y_{n-1}}{ay_{n-1} + by_{n-2}}, \quad n \in \mathbb{N}_0, \tag{3.5}$$

where $a, b \in \mathbb{R}$.

Remark 3.1. Before we continue with our analysis, note that in the case when $a = 0$ or $b = 0$ the above four difference equations are reduced to some simple product-type ones (for some results on product-type difference equations and systems of difference equations see, for example, [32,41] and the related references therein). From the analyses that follow we will see that the case $b = 0$ need not be considered separately, whereas the case $a = 0$ should be considered separately in the cases of equations (3.2) and (3.4).

The next step is to transform each of the difference equations (3.2)–(3.5) to a known solvable one (in the case of equations (3.2)–(3.5) it will be some special cases of the bilinear difference equations, that is, of equation (1.4), or to some of their cousins with interlacing indices).

By using the change of variables

$$z_n = \frac{y_n}{y_{n-2}}, \quad n \in \mathbb{N}_0, \tag{3.6}$$

equation (3.2) is transformed to

$$z_{n+1} = \frac{1}{az_n + b}, \quad n \in \mathbb{N}_0, \tag{3.7}$$

whereas equation (3.3) is transformed to

$$z_{n+1} = \frac{z_n}{az_n + b}, \quad n \in \mathbb{N}_0. \tag{3.8}$$

By using the change of variables

$$z_n = \frac{y_n}{y_{n-1}}, \quad n \geq -1, \tag{3.9}$$

equation (3.4) is transformed to

$$z_{n+1} = \frac{1}{az_{n-1} + b}, \quad n \in \mathbb{N}_0, \tag{3.10}$$

whereas equation (3.5) is transformed to

$$z_{n+1} = \frac{z_{n-1}}{az_{n-1} + b}, \quad n \in \mathbb{N}_0. \tag{3.11}$$

Now we consider each of the difference equations (3.7), (3.8), (3.10) and (3.11) separately.

An analysis of solvability of equation (3.7). By using the change of variables

$$z_n = \frac{u_{n-1}}{u_n}, \quad n \in \mathbb{N}_0, \tag{3.12}$$

equation (3.7) is transformed to

$$u_{n+1} = bu_n + au_{n-1}, \tag{3.13}$$

for $n \in \mathbb{N}_0$.

There are two cases to be considered: 1) $a \neq 0$; 2) $a = 0$.

Case $a \neq 0$. Since parameter a is different from zero, we can employ Lemma 2.1 to equation (3.13), from which we obtain the following representation formula for its general solution

$$u_n = \frac{(v_0u_{-1} - v_{-1}u_0)\widehat{s}_{n+1}(\vec{v}) + (v_0u_0 - bv_0u_{-1} - av_{-1}u_{-1})\widehat{s}_n(\vec{v})}{v_0^2 - bv_0v_{-1} - av_{-1}^2}, \tag{3.14}$$

for $n \geq -1$, and for every solution $(u_n)_{n\geq -1}$ to the equation, where $(\widehat{s}_n(\vec{v}))_{n\geq -1}$ is the solution to equation (3.13) such that

$$\widehat{s}_{-1}(\vec{v}) = v_{-1} \quad \text{and} \quad \widehat{s}_0(\vec{v}) = v_0, \tag{3.15}$$

and where v_{-1} and v_0 are such that

$$v_0^2 \neq bv_0v_{-1} + av_{-1}^2. \tag{3.16}$$

Combining (3.12) and (3.14), and then using (3.6) with $n = 0$, we have

$$\begin{aligned} z_n &= \frac{(v_0u_{-1} - v_{-1}u_0)\widehat{s}_n(\vec{v}) + (v_0u_0 - bv_0u_{-1} - av_{-1}u_{-1})\widehat{s}_{n-1}(\vec{v})}{(v_0u_{-1} - v_{-1}u_0)\widehat{s}_{n+1}(\vec{v}) + (v_0u_0 - bv_0u_{-1} - av_{-1}u_{-1})\widehat{s}_n(\vec{v})} \\ &= \frac{(v_0z_0 - v_{-1})\widehat{s}_n(\vec{v}) + (v_0 - (bv_0 + av_{-1})z_0)\widehat{s}_{n-1}(\vec{v})}{(v_0z_0 - v_{-1})\widehat{s}_{n+1}(\vec{v}) + (v_0 - (bv_0 + av_{-1})z_0)\widehat{s}_n(\vec{v})} && (3.17) \\ &= \frac{(v_0y_0 - v_{-1}y_{-2})\widehat{s}_n(\vec{v}) + (v_0y_{-2} - (bv_0 + av_{-1})y_0)\widehat{s}_{n-1}(\vec{v})}{(v_0y_0 - v_{-1}y_{-2})\widehat{s}_{n+1}(\vec{v}) + (v_0y_{-2} - (bv_0 + av_{-1})y_0)\widehat{s}_n(\vec{v})}, && (3.18) \end{aligned}$$

for $n \in \mathbb{N}_0$.

From (3.6) and (3.18), we have

$$y_n = y_{n-2}\frac{(v_0y_0 - v_{-1}y_{-2})\widehat{s}_n(\vec{v}) + (v_0y_{-2} - (bv_0 + av_{-1})y_0)\widehat{s}_{n-1}(\vec{v})}{(v_0y_0 - v_{-1}y_{-2})\widehat{s}_{n+1}(\vec{v}) + (v_0y_{-2} - (bv_0 + av_{-1})y_0)\widehat{s}_n(\vec{v})}, \tag{3.19}$$

for $n \in \mathbb{N}_0$.

Hence we have

$$\begin{aligned} y_{2n} &= y_{2n-2}\frac{(v_0y_0 - v_{-1}y_{-2})\widehat{s}_{2n}(\vec{v}) + (v_0y_{-2} - (bv_0 + av_{-1})y_0)\widehat{s}_{2n-1}(\vec{v})}{(v_0y_0 - v_{-1}y_{-2})\widehat{s}_{2n+1}(\vec{v}) + (v_0y_{-2} - (bv_0 + av_{-1})y_0)\widehat{s}_{2n}(\vec{v})} \\ y_{2n+1} &= y_{2n-1}\frac{(v_0y_0 - v_{-1}y_{-2})\widehat{s}_{2n+1}(\vec{v}) + (v_0y_{-2} - (bv_0 + av_{-1})y_0)\widehat{s}_{2n}(\vec{v})}{(v_0y_0 - v_{-1}y_{-2})\widehat{s}_{2n+2}(\vec{v}) + (v_0y_{-2} - (bv_0 + av_{-1})y_0)\widehat{s}_{2n+1}(\vec{v})}, \end{aligned}$$

for $n \in \mathbb{N}_0$, from which it follows that

$$y_{2n} = y_0\prod_{j=1}^{n}\frac{(v_0y_0 - v_{-1}y_{-2})\widehat{s}_{2j}(\vec{v}) + (v_0y_{-2} - (bv_0 + av_{-1})y_0)\widehat{s}_{2j-1}(\vec{v})}{(v_0y_0 - v_{-1}y_{-2})\widehat{s}_{2j+1}(\vec{v}) + (v_0y_{-2} - (bv_0 + av_{-1})y_0)\widehat{s}_{2j}(\vec{v})}, \tag{3.20}$$

$$y_{2n+1} = y_{-1}\prod_{j=0}^{n}\frac{(v_0y_0 - v_{-1}y_{-2})\widehat{s}_{2j+1}(\vec{v}) + (v_0y_{-2} - (bv_0 + av_{-1})y_0)\widehat{s}_{2j}(\vec{v})}{(v_0y_0 - v_{-1}y_{-2})\widehat{s}_{2j+2}(\vec{v}) + (v_0y_{-2} - (bv_0 + av_{-1})y_0)\widehat{s}_{2j+1}(\vec{v})}, \tag{3.21}$$

for $n \in \mathbb{N}_0$.

Using (3.1) in (3.20) and (3.21), as well as the fact that g is a bijection, we obtain

$$x_{2n} = g^{-1}\left(g(x_0)\prod_{j=1}^{n}\frac{(v_0g(x_0) - v_{-1}g(x_{-2}))\widehat{s}_{2j}(\vec{v}) + (v_0g(x_{-2}) - (bv_0 + av_{-1})g(x_0))\widehat{s}_{2j-1}(\vec{v})}{(v_0g(x_0) - v_{-1}g(x_{-2}))\widehat{s}_{2j+1}(\vec{v}) + (v_0g(x_{-2}) - (bv_0 + av_{-1})g(x_0))\widehat{s}_{2j}(\vec{v})}\right), \tag{3.22}$$

$$x_{2n+1} = g^{-1}\left(g(x_{-1})\prod_{j=0}^{n}\frac{(v_0g(x_0)-v_{-1}g(x_{-2}))\widehat{s}_{2j+1}(\vec{v})+(v_0g(x_{-2})-(bv_0+av_{-1})g(x_0))\widehat{s}_{2j}(\vec{v})}{(v_0g(x_0)-v_{-1}g(x_{-2}))\widehat{s}_{2j+2}(\vec{v})+(v_0g(x_{-2})-(bv_0+av_{-1})g(x_0))\widehat{s}_{2j+1}(\vec{v})}\right), \tag{3.23}$$

for $n \in \mathbb{N}_0$.

Case $a = 0$. Since $a = 0$, then from (3.2) we see that every well-defined solution to the equation satisfies the following relation

$$y_{n+1} = \frac{y_{n-1}}{b}, \quad n \in \mathbb{N}_0,$$

from which it easily follows that

$$y_{2m} = \frac{y_0}{b^m}, \tag{3.24}$$

$$y_{2m+1} = \frac{y_{-1}}{b^{m+1}}, \tag{3.25}$$

for every $m \in \mathbb{N}_0$.

Using (3.1) in relations (3.24) and (3.25), as well as the fact that g is a bijection, we obtain

$$x_{2m} = g^{-1}\left(\frac{g(x_0)}{b^m}\right), \tag{3.26}$$

$$x_{2m+1} = g^{-1}\left(\frac{g(x_{-1})}{b^{m+1}}\right), \tag{3.27}$$

for $m \in \mathbb{N}_0$.

Hence, from the consideration conducted above we have that the following theorem holds.

Theorem 3.2. *Consider equation* (1.20)*, where* $a, b \in \mathbb{R}$*,* $g : \mathcal{D}_g \to \mathbb{R}$ *is a bijection. Then, the equation is solvable and the following statements hold.*

(a) If $a \neq 0$*, then every well-defined solution* $(x_n)_{n\geq -2}$ *to equation* (1.20) *can be represented by formulas* (3.22) *and* (3.23)*, where* $(\widehat{s}_n(\vec{v}))_{n\geq -1}$ *is the solution to equation* (3.13) *satisfying the initial conditions in* (3.15) *with* $v_{-1}, v_0 \in \mathbb{R}$ *such that condition* (3.16) *holds.*

(b) If $a = 0$*, then the general solution to equation* (1.20) *is given by formulas* (3.26) *and* (3.27)*.*

The following corollary concerns equation (1.8).

Corollary 3.3. *Equation* (1.8) *is solvable in closed form and its general solution is given by formulas* (1.9) *and* (1.10)*.*

Proof. If in Theorem 3.2 we take $g(t) = t$, $t \in \mathbb{R}$, $a = b = 1$, and the sequence $(\widehat{s}_n(\vec{v}))_{n\geq -1}$ is chosen to be the solution to the corresponding equation (3.13) such that $v_{-1} = 1$ and $v_0 = 0$, then we see that $s_n = f_n$, for every $n \geq -1$, from which along with formulas (3.22) and (3.23), the formulas (1.9) and (1.10) are easily obtained by some simple calculations. □

An analysis of solvability of equation (3.8)*.* By using the following change of variables

$$z_n = \frac{1}{u_n}, \quad n \in \mathbb{N}_0, \tag{3.28}$$

equation (3.8) is transformed to

$$u_{n+1} = bu_n + a, \tag{3.29}$$

for $n \in \mathbb{N}_0$.

Equation (3.29) is the linear first-order difference equation with constant coefficient. Hence, by using formula (1.3) we see that its general solution is given by

$$u_n = b^n u_0 + a \sum_{j=0}^{n-1} b^j,$$

for $n \in \mathbb{N}_0$, from which it follows that

$$u_n = b^n u_0 + a \frac{b^n - 1}{b - 1}, \tag{3.30}$$

for $n \in \mathbb{N}_0$, when $b \neq 1$, and

$$u_n = u_0 + an, \tag{3.31}$$

for $n \in \mathbb{N}_0$, when $b = 1$.

If $b \neq 1$, then by combining relations (3.28) and (3.30), and then employing (3.6) with $n = 0$, it follows that

$$z_n = \frac{(b-1)z_0}{(b-1+az_0)b^n - az_0} \tag{3.32}$$

$$= \frac{(b-1)y_0}{((b-1)y_{-2} + ay_0)b^n - ay_0}, \tag{3.33}$$

for $n \in \mathbb{N}_0$.

If $b = 1$, then by combining relations (3.28) and (3.31), and using (3.6) with $n = 0$, we have

$$z_n = \frac{z_0}{az_0 n + 1} \tag{3.34}$$

$$= \frac{y_0}{ay_0 n + y_{-2}}, \tag{3.35}$$

for $n \in \mathbb{N}_0$.

Hence, if $b \neq 1$, then from (3.6) and (3.33) we have

$$y_n = y_{n-2} \frac{(b-1)y_0}{((b-1)y_{-2} + ay_0)b^n - ay_0},$$

for $n \in \mathbb{N}_0$, that is,

$$y_{2n} = y_{2n-2} \frac{(b-1)y_0}{((b-1)y_{-2} + ay_0)b^{2n} - ay_0},$$

$$y_{2n+1} = y_{2n-1} \frac{(b-1)y_0}{((b-1)y_{-2} + ay_0)b^{2n+1} - ay_0},$$

for $n \in \mathbb{N}_0$, from which it follows that

$$y_{2n} = y_0 \prod_{j=1}^{n} \frac{(b-1)y_0}{((b-1)y_{-2} + ay_0)b^{2j} - ay_0}, \tag{3.36}$$

$$y_{2n+1} = y_{-1} \prod_{j=0}^{n} \frac{(b-1)y_0}{((b-1)y_{-2} + ay_0)b^{2j+1} - ay_0}, \tag{3.37}$$

for $n \in \mathbb{N}_0$.

Using (3.1) in (3.36) and (3.37), we obtain

$$x_{2n} = g^{-1}\left(g(x_0)\prod_{j=1}^{n}\frac{(b-1)g(x_0)}{((b-1)g(x_{-2})+ag(x_0))b^{2j}-ag(x_0)}\right), \tag{3.38}$$

$$x_{2n+1} = g^{-1}\left(g(x_{-1})\prod_{j=0}^{n}\frac{(b-1)g(x_0)}{((b-1)g(x_{-2})+ag(x_0))b^{2j+1}-ag(x_0)}\right), \tag{3.39}$$

for $n \in \mathbb{N}_0$.

If $b = 1$, then from (3.6) and (3.35) we have

$$y_n = y_{n-2}\frac{y_0}{ay_0 n + y_{-2}},$$

for $n \in \mathbb{N}_0$, that is,

$$y_{2n} = y_{2n-2}\frac{y_0}{2ay_0 n + y_{-2}},$$

$$y_{2n+1} = y_{2n-1}\frac{y_0}{ay_0(2n+1) + y_{-2}},$$

for $n \in \mathbb{N}_0$, from which it follows that

$$y_{2n} = y_0\prod_{j=1}^{n}\frac{y_0}{2ay_0 j + y_{-2}}, \tag{3.40}$$

$$y_{2n+1} = y_{-1}\prod_{j=0}^{n}\frac{y_0}{ay_0(2j+1) + y_{-2}}, \tag{3.41}$$

for $n \in \mathbb{N}_0$.

Using (3.1) in (3.40) and (3.41), we obtain

$$x_{2n} = g^{-1}\left(g(x_0)\prod_{j=1}^{n}\frac{g(x_0)}{2ag(x_0)j + g(x_{-2})}\right), \tag{3.42}$$

$$x_{2n+1} = g^{-1}\left(g(x_{-1})\prod_{j=0}^{n}\frac{g(x_0)}{ag(x_0)(2j+1) + g(x_{-2})}\right), \tag{3.43}$$

for $n \in \mathbb{N}_0$.

Hence, from the above consideration we have that the following theorem holds.

Theorem 3.4. *Consider equation (1.21), where $a, b \in \mathbb{R}$, and $g : \mathcal{D}_g \to \mathbb{R}$ is a bijection. Then, the equation is solvable and the following statements hold.*

(a) If $b \neq 1$, then the general solution to the equation is given by formulas (3.38) and (3.39).

(b) If $b = 1$, then the general solution to the equation is given by formulas (3.42) and (3.43).

The following corollary concerns equation (1.11).

Corollary 3.5. *Equation (1.11) is solvable in closed form and its general solution is given by formulas (1.12) and (1.13).*

Proof. If in Theorem 3.4 we take $g(t) = t$, $t \in \mathbb{R}$, $a = b = 1$, the formulas (1.12) and (1.13) are easily obtained from formulas (3.42) and (3.43). □

An analysis of solvability of equation (3.10). First note that equation (3.10) is a difference equation with interlacing indices of order two (for the definition of the notion and some explanations related to it see [37,39]). This means that the following two sequences

$$\widehat{z}_m^{(i)} := z_{2m+i}, \quad m \in \mathbb{N}_0,\ i = -1, 0,$$

are solutions to difference equation (3.7).

Hence, in the case when $a \neq 0$, by using formula (3.17), we have

$$\widehat{z}_m^{(i)} = \frac{(v_0\widehat{z}_0^{(i)} - v_{-1})\widehat{s}_m(\vec{v}) + (v_0 - (bv_0 + av_{-1})\widehat{z}_0^{(i)})\widehat{s}_{m-1}(\vec{v})}{(v_0\widehat{z}_0^{(i)} - v_{-1})\widehat{s}_{m+1}(\vec{v}) + (v_0 - (bv_0 + av_{-1})z_0^{(i)})\widehat{s}_m(\vec{v})},$$

for $m \in \mathbb{N}_0$, $i = -1, 0$, where $(\widehat{s}_m(\vec{v}))_{n \geq -1}$ is the solution to equation (3.13) satisfying the initial conditions (3.15), and $v_{-1}, v_0 \in \mathbb{R}$ are such that condition (3.16) holds, from which along with (3.9) it follows that

$$y_{2m} = y_{2m-1}\frac{(v_0y_0 - v_{-1}y_{-1})\widehat{s}_m(\vec{v}) + (v_0y_{-1} - (bv_0 + av_{-1})y_0)\widehat{s}_{m-1}(\vec{v})}{(v_0y_0 - v_{-1}y_{-1})\widehat{s}_{m+1}(\vec{v}) + (v_0y_{-1} - (bv_0 + av_{-1})y_0)\widehat{s}_m(\vec{v})}, \tag{3.44}$$

$$y_{2m-1} = y_{2m-2}\frac{(v_0y_{-1} - v_{-1}y_{-2})\widehat{s}_m(\vec{v}) + (v_0y_{-2} - (bv_0 + av_{-1})y_{-1})\widehat{s}_{m-1}(\vec{v})}{(v_0y_{-1} - v_{-1}y_{-2})\widehat{s}_{m+1}(\vec{v}) + (v_0y_{-2} - (bv_0 + av_{-1})y_{-1})\widehat{s}_m(\vec{v})}, \tag{3.45}$$

for $m \in \mathbb{N}_0$.

From (3.44) and (3.45) it follows that

$$\begin{aligned} y_{2m} = y_{2m-2}&\frac{(v_0y_0 - v_{-1}y_{-1})\widehat{s}_m(\vec{v}) + (v_0y_{-1} - (bv_0 + av_{-1})y_0)\widehat{s}_{m-1}(\vec{v})}{(v_0y_0 - v_{-1}y_{-1})\widehat{s}_{m+1}(\vec{v}) + (v_0y_{-1} - (bv_0 + av_{-1})y_0)\widehat{s}_m(\vec{v})} \\ &\times \frac{(v_0y_{-1} - v_{-1}y_{-2})\widehat{s}_m(\vec{v}) + (v_0y_{-2} - (bv_0 + av_{-1})y_{-1})\widehat{s}_{m-1}(\vec{v})}{(v_0y_{-1} - v_{-1}y_{-2})\widehat{s}_{m+1}(\vec{v}) + (v_0y_{-2} - (bv_0 + av_{-1})y_{-1})\widehat{s}_m(\vec{v})} \end{aligned} \tag{3.46}$$

and

$$\begin{aligned} y_{2m-1} = y_{2m-3}&\frac{(v_0y_{-1} - v_{-1}y_{-2})\widehat{s}_m(\vec{v}) + (v_0y_{-2} - (bv_0 + av_{-1})y_{-1})\widehat{s}_{m-1}(\vec{v})}{(v_0y_{-1} - v_{-1}y_{-2})\widehat{s}_{m+1}(\vec{v}) + (v_0y_{-2} - (bv_0 + av_{-1})y_{-1})\widehat{s}_m(\vec{v})} \\ &\times \frac{(v_0y_0 - v_{-1}y_{-1})\widehat{s}_{m-1}(\vec{v}) + (v_0y_{-1} - (bv_0 + av_{-1})y_0)\widehat{s}_{m-2}(\vec{v})}{(v_0y_0 - v_{-1}y_{-1})\widehat{s}_m(\vec{v}) + (v_0y_{-1} - (bv_0 + av_{-1})y_0)\widehat{s}_{m-1}(\vec{v})}, \end{aligned} \tag{3.47}$$

for $m \in \mathbb{N}_0$.

From (3.46) and some simple calculations we obtain

$$\begin{aligned} y_{2m} &= y_0\prod_{j=1}^{m}\frac{(v_0y_0 - v_{-1}y_{-1})\widehat{s}_j(\vec{v}) + (v_0y_{-1} - (bv_0 + av_{-1})y_0)\widehat{s}_{j-1}(\vec{v})}{(v_0y_0 - v_{-1}y_{-1})\widehat{s}_{j+1}(\vec{v}) + (v_0y_{-1} - (bv_0 + av_{-1})y_0)\widehat{s}_j(\vec{v})} \\ &\quad \times \prod_{j=1}^{m}\frac{(v_0y_{-1} - v_{-1}y_{-2})\widehat{s}_j(\vec{v}) + (v_0y_{-2} - (bv_0 + av_{-1})y_{-1})\widehat{s}_{j-1}(\vec{v})}{(v_0y_{-1} - v_{-1}y_{-2})\widehat{s}_{j+1}(\vec{v}) + (v_0y_{-2} - (bv_0 + av_{-1})y_{-1})\widehat{s}_j(\vec{v})} \\ &= y_0\frac{(v_0y_0 - v_{-1}y_{-1})\widehat{s}_1(\vec{v}) + (v_0y_{-1} - (bv_0 + av_{-1})y_0)\widehat{s}_0(\vec{v})}{(v_0y_0 - v_{-1}y_{-1})\widehat{s}_{m+1}(\vec{v}) + (v_0y_{-1} - (bv_0 + av_{-1})y_0)\widehat{s}_m(\vec{v})} \\ &\quad \times \frac{(v_0y_{-1} - v_{-1}y_{-2})\widehat{s}_1(\vec{v}) + (v_0y_{-2} - (bv_0 + av_{-1})y_{-1})\widehat{s}_0(\vec{v})}{(v_0y_{-1} - v_{-1}y_{-2})\widehat{s}_{m+1}(\vec{v}) + (v_0y_{-2} - (bv_0 + av_{-1})y_{-1})\widehat{s}_m(\vec{v})}, \end{aligned} \tag{3.48}$$

for $m \in \mathbb{N}_0$.

From (3.47) and some simple calculations we obtain

$$\begin{aligned} y_{2m-1} &= y_{-1}\prod_{j=1}^{m}\frac{(v_0y_{-1}-v_{-1}y_{-2})\widehat{s}_j(\vec{v})+(v_0y_{-2}-(bv_0+av_{-1})y_{-1})\widehat{s}_{j-1}(\vec{v})}{(v_0y_{-1}-v_{-1}y_{-2})\widehat{s}_{j+1}(\vec{v})+(v_0y_{-2}-(bv_0+av_{-1})y_{-1})\widehat{s}_j(\vec{v})} \\ &\quad\times\prod_{j=1}^{m}\frac{(v_0y_0-v_{-1}y_{-1})\widehat{s}_{j-1}(\vec{v})+(v_0y_{-1}-(bv_0+av_{-1})y_0)\widehat{s}_{j-2}(\vec{v})}{(v_0y_0-v_{-1}y_{-1})\widehat{s}_j(\vec{v})+(v_0y_{-1}-(bv_0+av_{-1})y_0)\widehat{s}_{j-1}(\vec{v})} \\ &= y_{-1}\frac{(v_0y_{-1}-v_{-1}y_{-2})\widehat{s}_1(\vec{v})+(v_0y_{-2}-(bv_0+av_{-1})y_{-1})\widehat{s}_0(\vec{v})}{(v_0y_{-1}-v_{-1}y_{-2})\widehat{s}_{m+1}(\vec{v})+(v_0y_{-2}-(bv_0+av_{-1})y_{-1})\widehat{s}_m(\vec{v})} \\ &\quad\times\frac{(v_0y_0-v_{-1}y_{-1})\widehat{s}_0(\vec{v})+(v_0y_{-1}-(bv_0+av_{-1})y_0)\widehat{s}_{-1}(\vec{v})}{(v_0y_0-v_{-1}y_{-1})\widehat{s}_m(\vec{v})+(v_0y_{-1}-(bv_0+av_{-1})y_0)\widehat{s}_{m-1}(\vec{v})}, \end{aligned} \tag{3.49}$$

for $m \in \mathbb{N}_0$.

Using (3.1) in (3.48) and (3.49), we obtain

$$x_{2m} = g^{-1}\left(g(x_0)\frac{(v_0g(x_0)-v_{-1}g(x_{-1}))\widehat{s}_1(\vec{v})+(v_0g(x_{-1})-(bv_0+av_{-1})g(x_0))\widehat{s}_0(\vec{v})}{(v_0g(x_0)-v_{-1}g(x_{-1}))\widehat{s}_{m+1}(\vec{v})+(v_0g(x_{-1})-(bv_0+av_{-1})g(x_0))\widehat{s}_m(\vec{v})} \times\frac{(v_0g(x_{-1})-v_{-1}g(x_{-2}))\widehat{s}_1(\vec{v})+(v_0g(x_{-2})-(bv_0+av_{-1})g(x_{-1}))\widehat{s}_0(\vec{v})}{(v_0g(x_{-1})-v_{-1}g(x_{-2}))\widehat{s}_{m+1}(\vec{v})+(v_0g(x_{-2})-(bv_0+av_{-1})g(x_{-1}))\widehat{s}_m(\vec{v})}\right) \tag{3.50}$$

$$x_{2m-1} = g^{-1}\left(g(x_{-1})\frac{(v_0g(x_{-1})-v_{-1}g(x_{-2}))\widehat{s}_1(\vec{v})+(v_0g(x_{-2})-(bv_0+av_{-1})g(x_{-1}))\widehat{s}_0(\vec{v})}{(v_0g(x_{-1})-v_{-1}g(x_{-2}))\widehat{s}_{m+1}(\vec{v})+(v_0g(x_{-2})-(bv_0+av_{-1})g(x_{-1}))\widehat{s}_m(\vec{v})} \times\frac{(v_0g(x_0)-v_{-1}g(x_{-1}))\widehat{s}_0(\vec{v})+(v_0g(x_{-1})-(bv_0+av_{-1})g(x_0))\widehat{s}_{-1}(\vec{v})}{(v_0g(x_0)-v_{-1}g(x_{-1}))\widehat{s}_m(\vec{v})+(v_0g(x_{-1})-(bv_0+av_{-1})g(x_0))\widehat{s}_{m-1}(\vec{v})}\right), \tag{3.51}$$

for $m \in \mathbb{N}_0$.

If $a = 0$, then equation (3.4) becomes

$$y_{n+1} = \frac{y_n}{b}, \quad n \in \mathbb{N}_0,$$

from which it follows that

$$y_n = \frac{y_0}{b^n}, \quad n \in \mathbb{N}_0. \tag{3.52}$$

Using (3.1) in (3.52), we obtain

$$x_n = g^{-1}\left(\frac{g(x_0)}{b^n}\right), \tag{3.53}$$

for $n \in \mathbb{N}_0$.

Hence, from the above conducted analysis we have that the following theorem holds.

Theorem 3.6. *Consider equation (1.22), where $a, b \in \mathbb{R}$ and $g : \mathcal{D}_g \to \mathbb{R}$ is a bijection. Then, the equation is solvable and the following statements hold.*

(a) If $a \neq 0$, then every well-defined solution $(x_n)_{n\geq -2}$ to equation (1.22) can be represented by formulas (3.50) and (3.51), where $(\widehat{s}_n(\vec{v}))_{n\geq -1}$ is the solution to equation (3.13) satisfying the initial conditions in (3.15) with $v_{-1}, v_0 \in \mathbb{R}$ such that condition (3.16) holds.

(b) If $a = 0$, then the general solution to equation (1.22) is given by formula (3.53).

The following corollary concerns equation (1.14).

Corollary 3.7. *Equation* (1.14) *is solvable in closed form and its general solution is given by formulas* (1.15) *and* (1.16)*.*

Proof. If in Theorem 3.6 we take $g(t) = t$, $t \in \mathbb{R}$, $a = b = 1$, and the sequence $(\widehat{s}_n(\vec{v}))_{n\ge -1}$ is chosen to be the solution to the corresponding equation (3.13) such that $v_{-1} = 1$ and $v_0 = 0$, then we see that $s_n = f_n$, for every $n \ge -1$, from which along with formulas (3.50) and (3.51), the formulas (1.15) and (1.16) are easily obtained by some simple calculations. □

An analysis of solvability of equation (3.11). First note that equation (3.11) is a difference equation with interlacing indices of order two. This means that the sequences

$$\widetilde{z}_m^{(i)} := z_{2m+i}, \quad m \in \mathbb{N}_0,\ i = -1, 0,$$

are two solutions to difference equation (3.8).

Hence, if $b \ne 1$, then by using formula (3.32), it follows that

$$\widetilde{z}_m^{(i)} = \frac{(b-1)\widetilde{z}_0^{(i)}}{(b-1+a\widetilde{z}_0^{(i)})b^m - a\widetilde{z}_0^{(i)}},$$

for $m \in \mathbb{N}_0$, or equivalently

$$z_{2m} = \frac{(b-1)y_0}{((b-1)y_{-1}+ay_0)b^m - ay_0}, \tag{3.54}$$

$$z_{2m-1} = \frac{(b-1)y_{-1}}{((b-1)y_{-2}+ay_{-1})b^m - ay_{-1}}, \tag{3.55}$$

for $m \in \mathbb{N}_0$, where we have used (3.9) with $n = 0$ and $n = -1$, respectively.

From (3.9), (3.54) and (3.55), we have

$$y_{2m} = y_{2m-1}\frac{(b-1)y_0}{((b-1)y_{-1}+ay_0)b^m - ay_0},$$

$$y_{2m-1} = y_{2m-2}\frac{(b-1)y_{-1}}{((b-1)y_{-2}+ay_{-1})b^m - ay_{-1}},$$

for $m \in \mathbb{N}_0$, from which it easily follows that

$$y_{2m} = y_{2m-2}\frac{(b-1)y_0}{((b-1)y_{-1}+ay_0)b^m - ay_0}\,\frac{(b-1)y_{-1}}{((b-1)y_{-2}+ay_{-1})b^m - ay_{-1}}, \tag{3.56}$$

$$y_{2m-1} = y_{2m-3}\frac{(b-1)y_{-1}}{((b-1)y_{-2}+ay_{-1})b^m - ay_{-1}}\,\frac{(b-1)y_0}{((b-1)y_{-1}+ay_0)b^{m-1} - ay_0}, \tag{3.57}$$

for $m \in \mathbb{N}_0$.

Relations (3.56) and (3.57) are simple product-type recurrent relations, from which we obtain

$$y_{2m} = y_0\prod_{j=1}^{m}\frac{(b-1)y_0}{((b-1)y_{-1}+ay_0)b^j - ay_0}\,\frac{(b-1)y_{-1}}{((b-1)y_{-2}+ay_{-1})b^j - ay_{-1}}, \tag{3.58}$$

$$y_{2m-1} = y_{-1}\prod_{j=1}^{m}\frac{(b-1)y_{-1}}{((b-1)y_{-2}+ay_{-1})b^j - ay_{-1}}\,\frac{(b-1)y_0}{((b-1)y_{-1}+ay_0)b^{j-1} - ay_0}, \tag{3.59}$$

for $m \in \mathbb{N}_0$.

Using (3.1) in (3.58) and (3.59), we obtain

$$x_{2m} = g^{-1}\left(g(x_0)\prod_{j=1}^{m}\frac{(b-1)g(x_0)}{((b-1)g(x_{-1})+ag(x_0))b^j - ag(x_0)}\cdot\right.$$
$$\left.\times\frac{(b-1)g(x_{-1})}{((b-1)g(x_{-2})+ag(x_{-1}))b^j - ag(x_{-1})}\right), \tag{3.60}$$
$$x_{2m-1} = g^{-1}\left(g(x_{-1})\prod_{j=1}^{m}\frac{(b-1)g(x_{-1})}{((b-1)g(x_{-2})+ag(x_{-1}))b^j - ag(x_{-1})}\right.$$
$$\left.\times\frac{(b-1)g(x_0)}{((b-1)g(x_{-1})+ag(x_0))b^{j-1} - ag(x_0)}\right), \tag{3.61}$$

for $m \in \mathbb{N}_0$.

If $b = 1$, then by using formula (3.34), it follows that

$$\tilde{z}_m^{(i)} = \frac{\tilde{z}_0^{(i)}}{a\tilde{z}_0^{(i)}m + 1}, \tag{3.62}$$

for $m \in \mathbb{N}_0$, or equivalently

$$z_{2m} = \frac{y_0}{ay_0m + y_{-1}}, \tag{3.63}$$
$$z_{2m-1} = \frac{y_{-1}}{ay_{-1}m + y_{-2}}, \tag{3.64}$$

for $m \in \mathbb{N}_0$, where we have used (3.9) with $n = 0$ and $n = -1$, respectively

From (3.9), (3.63) and (3.64), we have

$$y_{2m} = y_{2m-1}\frac{y_0}{ay_0m + y_{-1}},$$
$$y_{2m-1} = y_{2m-2}\frac{y_{-1}}{ay_{-1}m + y_{-2}},$$

for $m \in \mathbb{N}_0$, from which it follows that

$$y_{2m} = y_{2m-2}\frac{y_0}{ay_0m + y_{-1}}\frac{y_{-1}}{ay_{-1}m + y_{-2}},$$
$$y_{2m-1} = y_{2m-3}\frac{y_{-1}}{ay_{-1}m + y_{-2}}\frac{y_0}{ay_0(m-1) + y_{-1}}$$

for $m \in \mathbb{N}_0$, and consequently

$$y_{2m} = y_0\prod_{j=1}^{m}\frac{y_0}{ay_0j + y_{-1}}\frac{y_{-1}}{ay_{-1}j + y_{-2}}, \tag{3.65}$$
$$y_{2m-1} = y_{-1}\prod_{j=1}^{m}\frac{y_{-1}}{ay_{-1}j + y_{-2}}\frac{y_0}{ay_0(j-1) + y_{-1}}, \tag{3.66}$$

for $m \in \mathbb{N}_0$.

Using (3.1) in (3.65) and (3.66), we obtain

$$x_{2m} = g^{-1}\left(g(x_0)\prod_{j=1}^{m}\frac{g(x_0)}{ag(x_0)j + g(x_{-1})}\frac{g(x_{-1})}{ag(x_{-1})j + g(x_{-2})}\right), \tag{3.67}$$

$$x_{2m-1} = g^{-1}\left(g(x_{-1})\prod_{j=1}^{m}\frac{g(x_{-1})}{ag(x_{-1})j+g(x_{-2})}\frac{g(x_0)}{ag(x_0)(j-1)+g(x_{-1})}\right), \tag{3.68}$$

for $m \in \mathbb{N}_0$.

Hence, from the above conducted consideration we have that the following theorem holds.

Theorem 3.8. *Consider equation* (1.23)*, where* $a, b \in \mathbb{R}$*,* $a \neq 0$*, and* $g : \mathcal{D}_g \to \mathbb{R}$ *is a bijection. Then, the equation is solvable and the following statements hold.*

(a) If $b \neq 1$*, then the general solution to the equation is given by formulas* (3.60) *and* (3.61)*.*

(b) If $b = 1$*, then the general solution to the equation is given by formulas* (3.67) *and* (3.68)*.*

The following corollary concerns equation (1.17).

Corollary 3.9. *Equation* (1.17) *is solvable in closed form and its general solution is given by formulas* (1.18) *and* (1.19)*.*

Proof. If in Theorem 3.8 we take $g(t) = t$, $t \in \mathbb{R}$, $a = b = 1$, the formulas (1.18) and (1.19) are easily obtained from formulas (3.67) and (3.68). □

Remark 3.10. Note that Corollaries 3.3, 3.5, 3.7 and 3.9 show that the formulas (1.9) and (1.10), (1.12) and (1.13), (1.15) and (1.16), (1.18) and (1.19), listed in [7], are really general solutions to the equations (1.8), (1.11), (1.14), (1.17), respectively.

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