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Uniqueness in some higher order elliptic boundary value problems in n dimensional domains

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This paper is dedicated to Jerry Lee Lewis on the occasion of his 75th birthday

Abstract

We develop maximum principles for several P functions which are defined on solutions to equations of fourth and sixth order (including a equation which arises in plate theory and bending of cylindrical shells).
As a consequence, we obtain uniqueness results for fourth and sixth order boundary value problems in arbitrary n dimensional domains.

Keywords: P function method, higher order, elliptic, plate theory.

Mathematics Subject Classifications: 35B50, 35G15, 35J40.

1 Introduction

This paper represents the n dimensional analogue of Schaefer's paper [9] and is concerned with uniqueness results for boundary value problems of fourth and sixth order.

Schaefer [9] investigated the uniqueness of the solution for the boundary value problems

$$\begin{cases} \Delta^3 u - a(x)\Delta^2 u + b(x)\Delta u - c(x)u = f & \text{in } \Omega \subset \mathbb{R}^n \\ u = g,\ \Delta u = h,\ \Delta^2 u = i & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

and

$$\begin{cases} \Delta^2 u - \varphi(x)\Delta u + \rho(x)u = f & \text{in } \Omega \subset \mathbb{R}^n \\ u = g,\ \Delta u = h & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $a, b, \geq 0$, $c > 0$ are constants, $\varphi \equiv 0$, $\rho > 0$ in the bounded domain Ω, $n = 2$ and the curvature of the boundary is strictly positive.

Our aim here is to remove via the P function method dimension and geometry conditions (convexity and smoothness) with, of course, further conditions on the coefficients a, b, c and ρ.

Finally, we deal with a equation that arises in plate theory and in bending of cylindrical shells. We prove the uniqueness result for the corresponding homogeneous boundary value problem without the hypothesis that the plate has a convex shape.

A word on notations. For simplicity, we shall say that a function Φ satisfies a generalized maximum principle in Ω, if either there exists a constant $k \in \mathbb{R}$ such that $\Phi \equiv k$ in Ω or Φ does not attain a nonnegative maximum in Ω. Throughout the paper Ω and diamΩ denote respectively a bounded domain in $\mathbb{R}^n$, the diameter of Ω.

2 Some useful results

We first involve second order operators and establish some useful results. The results are not only useful for our purposes but also yielding other results for partial differential equations (see [3], [7]).

We consider the problem of determining a smooth function w (a positive supersolution), which satisfies

$$\mathrm{L}w \equiv \Delta w + \gamma(x)w \leq 0 \quad \text{in } \Omega, \tag{2.1}$$

$$w > 0 \quad \text{in } \overline{\Omega}. \tag{2.2}$$

The problem of determining such a function is of interest only if γ takes positive values or both positive and negative values. If $\gamma \leq 0$ in Ω then, the function $w \equiv c$, where c is a positive constant, satisfies (2.1) and (2.2).

Lemma 2.1. *Suppose that $\gamma \geq 0$ in Ω and that*

$$\sup_{\Omega} \gamma < \frac{4n+4}{(\operatorname{diam}\Omega)^2}. \tag{2.3}$$

Then, there exists a function $w_1 \in C^\infty(\mathbb{R}^n)$ satisfying (2.1) and (2.2).

If Ω lies in a slab of width d and if

$$\sup_{\Omega} \gamma < \frac{\pi^2}{d^2}, \tag{2.4}$$

there exists a function $w_2 \in C^\infty(\overline{\Omega})$ satisfying (2.1) and (2.2).

Proof. By virtue of Jung's theorem (see [5] or [1], Theorem 11.5.8, p. 357), we may suppose without loss of generality that Ω is embedded in the ball (the smallest ball containing Ω): $\Omega \subset B_{(n/2n+2)^{\frac{1}{2}}\operatorname{diam}\Omega} = \{x \mid x_1^2 + \cdots + x_n^2 < \frac{n(\operatorname{diam}\Omega)^2}{2n+2}\}$.

We define the function

$$w_1(x) = 1 - \alpha(x_1^2 + \cdots + x_n^2) \quad \text{in } \overline{\Omega}, \tag{2.5}$$

where the positive constant α is to be determined.

By calculations we get

$$\mathrm{L}w_1 \leq -2\alpha n + \sup_{\Omega} \gamma \quad \text{in } \Omega.$$

By choosing

$$\alpha = \sup_{\Omega} \gamma / 2n,$$

we obtain $\mathrm{L}w_1 \leq 0$ in Ω.

To insure that $w_1 > 0$ in $\overline{\Omega}$, we must have $\alpha(x_1^2 + \cdots + x_n^2) < 1$ in $\overline{\Omega}$. Hence the inequality (2.3) must be valid.

If Ω lies in a slab of width d, then the result follows from Lemma 21.11, p.158, [8]. Here

$$w_2 = \cos\frac{\pi(2x_i - d)}{2(d+\varepsilon)}\prod_{j=1}^{n}\cosh(\varepsilon x_j),$$

for some $i \in \{1, \dots, n\}$, where $\varepsilon > 0$ is small. $\square$

Combining our Lemma and Theorem 10, [7], p.73, we get the following useful result:

THEOREM 2.1. *Let $u \in C^2(\Omega)\cap C^0(\overline{\Omega})$ satisfy the inequality* $\mathrm{L}u \equiv \Delta u + \gamma(x)u \geq 0$, *where $\gamma \geq 0$ in Ω. Suppose that (2.3) holds.*

Then, the function u/w_1 satisfies a generalized maximum principle in Ω.

Similarly, if we impose the restriction (2.4), we obtain that u/w_2 satisfies a generalized maximum principle in Ω.

Comments

1. A broad class of domains satisfy $\Omega \subset B_{\mathrm{diam}\Omega/2}$. For these domains $C(n, \mathrm{diam}\Omega) = (4n+4)/(\mathrm{diam}\Omega)^2$ may be replaced by $C_1(n, \mathrm{diam}\Omega) = 8n/(\mathrm{diam}\Omega)^2$.

2. We may improve the constant $C_1(n, \mathrm{diam}\Omega)$ if $\Omega = \{x \in \mathbb{R}^n \mid 0 < R <\mid x \mid< R+\varepsilon\}$, where $\varepsilon > 0$ is sufficiently small. We define $w_3(x) = R + \varepsilon- \mid x \mid +\delta$, where δ is any positive constant. Since

$$\mathrm{L}w_3 \leq 2(n-1)/\mathrm{diam}\Omega + (\varepsilon+\delta)\sup_{\Omega}\gamma \quad \text{in } \Omega,$$

we get that w_3 is a supersolution under the restriction

$$\sup_{\Omega}\gamma < \frac{2(n-1)}{(\varepsilon+\delta)\mathrm{diam}\Omega}.$$

For sufficiently small ε we have $C_2(n, \mathrm{diam}\Omega) = 2(n-1)/(\varepsilon+\delta)\mathrm{diam}\Omega > C_1(n, \mathrm{diam}\Omega)$.

3. A method for determining a function having properties (2.1) and (2.2) was given in [7], p. 73–74 . The authors proved that if

$$\sup_{\Omega}\gamma < \frac{4}{d^2e^2}, \tag{2.6}$$

then there exists a function w_4 fulfilling (2.1) and (2.2). Here $\mathrm{L}w \equiv \Delta w + \gamma(x)w$, $\gamma \geq 0$ in Ω and Ω is supposed to lie in a strip of width d. Note that (2.6) is the inequality in the footnote of p. 74.

Of course, our Lemma is sharper that this result. For a more general result concerning the construction of supersolutions see [3].

4. We have to impose some restrictions to γ. Otherwise, as the following example shows, the maximum principle (Theorem 2.1) is false. The function $u(x, y) = \sin x \sin y$ satisfies $u = 0$ on $\partial\Omega$ and is solution of the equation $\Delta u + 2u = 0$ in $\Omega = (0, \pi) \times (0, \pi)$. Of course, (2.3) does not hold.

3 Maximum principles and uniqueness results for sixth order equations

We now tackle the uniqueness for the boundary value problem (1.1). For the sake of simplicity we consider four cases.

Case 1. $a, b, c > 0$.

We deal with classical solutions (i.e. $u \in C^6(\Omega) \cap C^4(\overline{\Omega})$) of

$$\Delta^3 u - a(x)\Delta^2 u + b(x)\Delta u - c(x)u = 0 \quad \text{in } \Omega \subset \mathrm{I\!R}^n, n \geq 2. \tag{3.7}$$

The uniqueness results can be inferred from the following maximum principles.

Lemma 3.1. *Let u be a classical solution of (3.7).*
i). Suppose that

$$\frac{a(b+c)^2}{b^2(a-1)} < \frac{8n+8}{(\operatorname{diam}\Omega)^2}, \tag{3.8}$$

holds, where $a > 1, b, c$ are constants. We consider the function P_1 given by

$$\mathrm{P}_1 = (a\Delta^2 u + bu)^2 + ab(a-1)(\Delta u)^2 + b^2(a-1)u^2.$$

Then, the function P_1/w_1 satisfies a generalized maximum principle in Ω.

ii). Suppose that

$$\sup_{\Omega} \frac{(a+c)^2}{a(b-1)} < \frac{8n+8}{(\operatorname{diam}\Omega)^2}, \tag{3.9}$$

$$b \in C^2(\Omega)\,, b > 1 \quad \text{in } \overline{\Omega},\ \Delta(1/(\mathrm{b}-1)) \leq 0 \quad \text{in } \Omega \tag{3.10}$$

holds.

If

$$\mathrm{P}_2 = (\Delta^2 u + u)^2 + (b-1)(\Delta u)^2 + (b-1)u^2$$

then, the function P_2/w_1 satisfies a generalized maximum principle in Ω.

If $a = c$ in Ω then, P_2 attains its maximum value on $\partial\Omega$ (the restriction (3.9) is not needed).

iii). Suppose that

$$\sup_{\Omega} \frac{c^2}{2a} + 1 < \frac{4n+4}{(\operatorname{diam}\Omega)^2}, \tag{3.11}$$

$$b \in C^2(\Omega)\,, b > 0,\ \Delta(1/b) \leq 0 \quad \text{in } \Omega, \tag{3.12}$$

$$b\left(\frac{c^2}{2a} + 1\right) \geq 1 \quad \text{in } \Omega \tag{3.13}$$

holds, where $a > 0$ in $\overline{\Omega}$, and c is of arbitrary sign in Ω.

If

$$\mathrm{P}_3 = (\Delta^2 u)^2 + b(\Delta u)^2 + u^2$$

then, the function P_3/w_1 satisfies a generalized maximum principle in Ω.

Proof. i). By computation and using equation (3.7) we have in Ω

$$\begin{aligned}\Delta\big((a\Delta^2 u+bu)^2\big) &\geq 2(a\Delta^2 u+bu)(a\Delta^3 u+b\Delta u)\\ &= 2\big(a^3(\Delta^2 u)^2+abcu^2+a^2(b+c)u\Delta^2 u+ab(1-a)\Delta u\Delta^2 u\\ &\quad +b^2(1-a)u\Delta u\big),\end{aligned}$$

$$\Delta\big(ab(a-1)(\Delta u)^2\big)\geq 2ab(a-1)\Delta u\Delta^2 u,$$
$$\Delta\big(b^2(a-1)u^2\big)\geq 2b^2(a-1)u\Delta u.$$

That means that

$$\begin{aligned}\Delta \mathrm{P}_1 &\geq 2a\big(a^2(\Delta^2 u)^2+a(b+c)u\Delta^2 u+bcu^2\big)\\ &= 2a\left(a\Delta^2 u+\frac{b+c}{2}u\right)^2+2a\left(bc-\frac{(b+c)^2}{4}\right)u^2\\ &\geq -\frac{a(b+c)^2}{2}u^2.\end{aligned}\tag{3.14}$$

Hence P_1 satisfies the differential inequality

$$\Delta \mathrm{P}_1+\frac{a(b+c)^2}{2b^2(a-1)}\mathrm{P}_1\geq 0 \quad \text{in } \Omega.$$

Since (3.8) holds, we can use the maximum principle (Theorem 2.1) to obtain the desired result.

ii). A computation shows that

$$\begin{aligned}\Delta\big((\Delta^2 u+u)^2\big) &\geq 2(\Delta^2 u+u)(a\Delta^2 u+(1-b)\Delta u+cu)\\ &\geq 2a\left((\Delta^2 u)^2+\frac{a+c}{a}u\Delta^2 u+\frac{c}{a}u^2\right)+2(1-b)(\Delta u\Delta^2 u+u\Delta u).\end{aligned}$$

By (3.10) and the arithmetic - geometric mean inequality we get

$$\Delta\big((b-1)(\Delta u)^2\big)\geq 2(b-1)\Delta u\Delta^2 u,$$
$$\Delta\big((b-1)u^2\big)\geq 2(b-1)u\Delta u.$$

Adding, we obtain that P_2 satisfies

$$\Delta \mathrm{P}_2\geq 2a\left(\Delta^2 u+\frac{a+c}{2a}u\right)^2+2\left(c-\frac{(a+c)^2}{4a}\right)u^2 \quad \text{in } \Omega.$$

Hence

$$\Delta \mathrm{P}_2+\frac{(a+c)^2}{2a(b-1)}\mathrm{P}_2\geq 0 \quad \text{in } \Omega,$$

and the proof follows.

iii). The proof follows by similar reasoning.

□

Case 2. $a, c > 0, b = 0$.

LEMMA 3.2. *Let u be a classical solution of (3.7).*
i). Suppose that

$$\sup_{\Omega} \frac{1}{a}\left(c + \frac{(c+1)^2}{4(a-1)}\right) < \frac{2n+2}{(\operatorname{diam}\Omega)^2}, \tag{3.15}$$

$$a \in C^2(\Omega)\,, a > 1 \quad \text{in}\,\overline{\Omega},\ \Delta(1/\mathrm{a}) \leq 0 \quad \text{in}\,\Omega, \tag{3.16}$$

$$c \in C^2(\Omega)\,, c > 0,\ \Delta(1/c) \leq 0 \quad \text{in}\,\Omega \tag{3.17}$$

holds.

We consider the function P_4 *given by*

$$\mathrm{P}_4 = (\Delta^2 u - \Delta u)^2 + c(\Delta u - u)^2 + a(\Delta u)^2.$$

Then, the function P_4/w_1 *satisfies a generalized maximum principle in* Ω.

ii). Suppose that

$$\sup_{\Omega} \frac{c}{a-c-1} < \frac{2n+2}{(\operatorname{diam}\Omega)^2}, \tag{3.18}$$

$$a, c \in C^2(\Omega)\,, a - c - 1 > 0 \quad \text{in}\,\overline{\Omega},\ \Delta(1/(\mathrm{a}-\mathrm{c}-1)) \leq 0 \quad \text{in}\,\Omega, \tag{3.19}$$

and (3.17) holds.

We consider the function P_5 *given by*

$$\mathrm{P}_5 = (\Delta^2 u - \Delta u)^2 + c(\Delta u - u)^2 + (a - c - 1)(\Delta u)^2.$$

Then, the function P_5/w_1 *satisfies a generalized maximum principle in* Ω.

Proof. i). It is easily verified that P_4 satisfies in Ω

$$\begin{aligned} \Delta\mathrm{P}_4 &\geq 2(a-1)(\Delta^2 u)^2 + 2(c+1)\Delta u \Delta^2 u - 2c(\Delta u)^2 \\ &= 2(c+1)\left(\sqrt{\frac{a-1}{c+1}}\Delta^2 u + \frac{1}{2}\sqrt{\frac{c+1}{a-1}}\Delta u\right)^2 - 2c(\Delta u)^2 - \frac{2(c+1)^2}{4(a-1)}(\Delta u)^2 \end{aligned}$$

Hence

$$\Delta\mathrm{P}_4 + \frac{2}{a}\left(c + \frac{(c+1)^2}{4(a-1)}\right)\mathrm{P}_4 \geq 0 \quad \text{in}\,\Omega,$$

and the proof follows.

ii). The proof follows by similar reasoning.

□

Case 3. $b, c > 0, a = 0$.

Lemma 3.3. *Let u be a classical solution of (3.7).*
Suppose that

$$\max\{2, \sup_{\Omega} \frac{1}{b}, \sup_{\Omega} c^2\} < \frac{4n+4}{(\operatorname{diam}\Omega)^2}, \tag{3.20}$$

and (3.12) holds.

Then, the function P_3/w_1 *satisfies a generalized maximum principle in* Ω.

The proof is achieved by arguing exactly as in Lemma 3.1.

Case 4. $a = b = 0$.

Lemma 3.4. *Let u be a classical solution of (3.7), where c satisfies (3.17).*
If the relation

$$\max\{1, \sup_{\Omega} c\} < \frac{2n+2}{(\operatorname{diam}\Omega)^2}. \tag{3.21}$$

holds, then the function P_6/w_1 *satisfies a generalized maximum principle in* Ω. *Here*

$$\mathrm{P}_6 = (\Delta^2 u)^2 + c(\Delta u)^2 + cu^2.$$

Similarly, if

$$c \in C^2(\Omega)\,, c > 0 \quad \text{in}\,\overline{\Omega},\ \ \Delta \mathrm{c} \leq 0 \quad \text{in}\,\Omega, \tag{3.22}$$

then, the function P_7/w_1 *satisfies a generalized maximum principle in* Ω. *Here*

$$\mathrm{P}_7 = \frac{1}{c}(\Delta^2 u)^2 + (\Delta u)^2 + u^2.$$

The proof is achieved by arguing exactly as in Lemma 3.1.

We now conclude the uniqueness result.

Theorem 3.1. *There is at most one classical solution of the boundary value problem (1.1), where a, b and c satisfy the conditions of Lemma 3.1, Lemma 3.2, Lemma 3.3 or Lemma 3.4.*

Proof. Suppose that (3.8) is satisfied. We define $u = u_1 - u_2$, where u_1 and u_2 are solutions of (1.1). Then u satisfies the equation (3.7) and

$$u = \Delta u = \Delta^2 u = \Delta^3 u = 0 \quad \text{on}\ \partial\Omega. \tag{3.23}$$

Hence by Lemma 3.1 either there exists a constant $k \in \mathbb{R}$ such that

$$\frac{\mathrm{P}_1}{w_1} \equiv k \quad \text{in}\,\Omega, \tag{3.24}$$

or

$$\frac{\mathrm{P}_1}{w_1} \quad \text{does not attain a maximum in}\,\Omega. \tag{3.25}$$

In the first case the function P_1/w_1 is smooth and hence (3.24) holds in $\overline{\Omega}$. By the boundary conditions (3.23) we have $\mathrm{P}_1 = 0$ on $\partial\Omega$, i.e., k=0. It follows that $\mathrm{P}_1 = 0$ in Ω, which means $u \equiv 0$ in Ω. Hence $u_1 = u_2$ in Ω.

We are left to check the condition (3.25), i.e.,

$$\max_{\overline{\Omega}} \frac{\mathrm{P}_1}{w_1} = \max_{\partial\Omega} \frac{\mathrm{P}_1}{w_1}.$$

By the boundary conditions (3.23) we have

$$0 \leq \max_{\overline{\Omega}} \frac{\mathrm{P}_1}{w_1} = 0,$$

i.e., $u_1 = u_2$ in Ω.

We can argue similarly if we are under the hypotheses of Lemma 3.2, Lemma 3.3 or Lemma 3.4.

□

Comments.

1. Our uniqueness results extend Theorem 1, [9] to the n dimensional and nonconstant coefficient case without the convexity restriction imposed to Ω. In our paper, the removal was achieved by using P functions without gradient terms.

2. We could also derive P functions containing gradient terms. This kind of functions would have led us to weaker uniqueness results.

For example, the function

$$\mathrm{P}_8 = \frac{1}{2}(\Delta^2 u)^2 + \frac{b}{2}(\Delta u)^2 + c(|\nabla u|^2 - u\Delta u) + \frac{c(2c+nc)}{nb}u^2.$$

satisfies the inequality

$$\Delta \mathrm{P}_8 + \frac{2(2c+nc)}{nb}\mathrm{P}_8 \geq 0 \quad \text{in } \Omega.$$

Hence P_8/w_1 attains its maximum value on $\partial\Omega$ (unless $\mathrm{P}_8 < 0$ in Ω), if a=0 and

$$\frac{2c+nc}{nb} < \frac{2n+2}{(\operatorname{diam}\Omega)^2}.$$

We note that this maximum principle can be used to obtain gradient bounds for the solution of (3.7) (the method is similar to the method presented in Section 4 and hence will be omitted).

3. We note that the case $a, b, c > 0$ and n arbitrary was also treated in [4]. We see that our results cannot be deduced from results in [4]. Moreover, we are able here to treat the cases $a = 0$ or $b = 0$ or $a = b = 0$.

4. If $b = c$ and $a > 1$ (see relation (3.14)), then Lemma 3.1 holds without the assumption (3.8). This particular result can be deduced from Theorem 2, [4].

5. Different uniqueness results for boundary value problems of sixth order have been obtained in [2].

6. The sign condition on the coefficients a and b is needed. The following example shows that if $a < 0$, then the uniqueness result (Theorem 3.1) is violated.

The boundary value problem

$$\begin{cases} \Delta^3 u + 3\Delta^2 u + \Delta u - 2u = 0 & \text{in } \Omega = (0,\pi) \times (0,\pi) \\ u = \Delta u = \Delta^2 u = \Delta^3 u = 0 & \text{on } \partial\Omega, \end{cases}$$

has (at least) the solutions $u_1(x, y) \equiv 0$ and $u_2(x, y) = \sin x \sin y$ in Ω.

4 A maximum principle and an uniqueness result for a fourth order equation

Finally, we deal with the following equation

$$\Delta^2 u + k_1 u + k_2 u^3 = 0 \quad \text{in}\, \Omega \subset \mathrm{I\!R}^{\mathrm{n}},\ \mathrm{n} \geq 2, \tag{4.26}$$

where $k_1, k_2 > 0$ are constants.
The equation (4.26) arises in the plate theory and in the bending of cylindrical shells [10].

The next maximum principle will be used to obtain solution and gradient bounds for the equation (4.26).

Lemma 4.1. *Let u be a classical solution of (4.26).*

Then the function

$$\mathrm{P}_9 = (\Delta u)^2 + \frac{k_2}{2} u^4 + k_1 u^2$$

attains its maximum value on $\partial\Omega$.

Proof. Since

$$\Delta\big((\Delta u)^2\big) \geq 2\Delta u(-k_1 u - k_2 u^3) = -2k_1 u\Delta u - 2k_2 u^3 \Delta u,$$

$$\frac{k_2}{2}\Delta u^4 \geq k_2 u^2 \Delta u^2 \geq 2k_2 u^3 \Delta u,$$

$$k_1 \Delta u^2 \geq 2k_1 u \Delta u,$$

we get

$$\Delta \mathrm{P}_9 \geq 0 \quad \text{in}\, \Omega,$$

and the proof follows by the classical maximum principle. $\square$

Theorem 4.1. *If u satisfies (4.26) then, we have the following bounds*

a).

$$\max_{\overline{\Omega}} |u| \leq \sqrt{\frac{1}{k_1}}\left(\max_{\partial\Omega} |\Delta u| + \sqrt{\frac{k_2}{2}} \max_{\partial\Omega} u^2 + \sqrt{k_1}\max_{\partial\Omega}|u|\right), \tag{4.27}$$

where $n \geq 2$.

b).

$$\max_{\overline{\Omega}} |\nabla u|^2 \leq \max_{\partial\Omega} |\nabla u|^2 + \frac{3+k_1}{2}\max_{\partial\Omega} u^2 + \frac{k_2(1+k_1)}{4k_1}\max_{\partial\Omega} u^4 + \frac{1+2k_1}{2k_1}\max_{\partial\Omega}(\Delta u)^2, \tag{4.28}$$

where $n = 2$.

Proof. a). Case a). is a simple consequence of Lemma 4.1.

b). From Theorem 1, [10] we know that the the function $|\nabla u|^2 - u\Delta u$ attains it maximum value on $\partial\Omega$, which we may rewrite as

$$|\nabla u|^2 \leq \frac{u^2}{2} + \frac{(\Delta u)^2}{2} + \max_{\partial\Omega}\left(|\nabla u|^2 - u\Delta u\right) \quad \text{in}\, \Omega. \tag{4.29}$$

By Lemma 4.1 we get

$$\frac{u^2}{2} \le \max_{\partial\Omega} u^2 + \frac{k_2}{4k_1} \max_{\partial\Omega} u^4 + \frac{1}{2k_1} \max_{\partial\Omega} (\Delta u)^2 \quad \text{in } \Omega, \tag{4.30}$$

$$\frac{(\Delta u)^2}{2} \le \frac{k_1}{2} \max_{\partial\Omega} u^2 + \frac{k_2}{4} \max_{\partial\Omega} u^4 + \frac{1}{2} \max_{\partial\Omega} (\Delta u)^2 \quad \text{in } \Omega. \tag{4.31}$$

Combining inequalities (4.29), (4.30) and (4.31), we get the inequality (4.28).

□

The hypothesis that is assumed over and over again in plate theory is convexity. Under this assumption, Schaefer [10] proved the uniqueness for the solution of

$$\begin{cases} \Delta^2 u + k_1 u + k_2 u^3 = 0 & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.32}$$

where $\Omega \subset \mathbb{R}^2$ is a convex domain.

An application of our Lemma 4.1 shows that the convexity assumption is redundant. Moreover, the uniqueness result for solutions of (4.32) holds for $n > 2$.

The result reads as follows:

Theorem 4.2. *Let u be a classical solution of (4.32), where $\Omega \subset \mathbb{R}^n$ is an arbitrary domain. Then $u \equiv 0$ in Ω.*

Comments.

1. Some maximum principles and their applications for general equations of fourth and six order have been given in [11] and [6]. Unfortunately, it is difficult to apply their results in the study of uniqueness results.
2. If $n \ge 3$ we can still obtain gradient bounds for solutions of (4.26).

We must use the function

$$\mathrm{P}_{10} = 2(\Delta u)^2 + k_2 u^4 + \left(2k_1 + \frac{3}{2}\right) u^2 + |\nabla u|^2 - u\Delta u.$$

By the inequality,

$$\sum_{i,j=1}^{n} \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2 \ge \frac{1}{n} (\Delta u)^2$$

(which holds in n dimensions) we see the function P_{10} satisfies

$$\Delta \mathrm{P}_{10} + \mathrm{P}_{10} \ge (u + \Delta u)^2 \ge 0 \quad \text{in } \Omega.$$

Hence, by Theorem 2.1 we obtain (under the restriction $(\operatorname{diam}\Omega)^2 < 4n + 4$)

$$\mathrm{P}_{10} \le w_1 \max_{\partial\Omega} \frac{\mathrm{P}_{10}}{w_1} \le \frac{\max_{\partial\Omega} \mathrm{P}_{10}}{\min_{\partial\Omega} w_1} \quad \text{in } \Omega, \tag{4.33}$$

unless $P_{10} < 0$ in Ω.

If $P_{10} < 0$ in Ω we have

$$|\nabla u|^2 - u\Delta u \leq 0 \quad \text{in } \Omega,$$

i.e.

$$|\nabla u|^2 \leq \frac{u^2}{2} + \frac{(\Delta u)^2}{2} \quad \text{in } \Omega. \tag{4.34}$$

Using an argument similar to that we have used in Theorem 4.1 and combining the inequalities (4.33) and (4.34), we get a gradient bound.

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