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A global bifurcation theorem for a multiparameter positone problem and its application to the one-dimensional perturbed Gelfand problem

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Abstract. We study the global bifurcation and exact multiplicity of positive solutions for

$$\begin{cases} u''(x) + \lambda f_\varepsilon(u) = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases}$$

where $\lambda > 0$ is a bifurcation parameter, $\varepsilon \in \Theta$ is an evolution parameter, and $\Theta \equiv (\sigma_1, \sigma_2)$ is an open interval with $0 \leq \sigma_1 < \sigma_2 \leq \infty$. Under some suitable hypotheses on f_ε, we prove that there exists $\varepsilon_0 \in \Theta$ such that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve is S-shaped for $\sigma_1 < \varepsilon < \varepsilon_0$ and is monotone increasing for $\varepsilon_0 \leq \varepsilon < \sigma_2$. We give an application to prove global bifurcation of bifurcation curves for the one-dimensional perturbed Gelfand problem.

Keywords: global bifurcation, multiparameter problem, S-shaped bifurcation curve, exact multiplicity, positive solution.

2010 Mathematics Subject Classification: 34B18, 74G35.

1 Introduction

We study the global bifurcation and exact multiplicity of positive solutions for the multiparameter positone problem

$$\begin{cases} u''(x) + \lambda f_\varepsilon(u) = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a bifurcation parameter, $\varepsilon \in \Theta$ is an evolution parameter, $\Theta \equiv (\sigma_1, \sigma_2)$ is an open interval with $0 \leq \sigma_1 < \sigma_2 \leq \infty$, and nonlinearity $f_\varepsilon \in C^3[0, \infty)$. We first define some

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functions needed below:

$$F_\varepsilon(u) = \int_0^u f_\varepsilon(t)dt, \qquad \text{where } \varepsilon \in \Theta \text{ and } u > 0, \tag{1.2}$$

$$I_1(\varepsilon,\alpha,u) = F_\varepsilon(\alpha) - F_\varepsilon(u), \qquad \text{where } \varepsilon \in \Theta \text{ and } \alpha > u > 0, \tag{1.3}$$

$$I_2(\varepsilon,\alpha,u) = \alpha f_\varepsilon(\alpha) - u f_\varepsilon(u), \qquad \text{where } \varepsilon \in \Theta \text{ and } \alpha > u > 0, \tag{1.4}$$

$$I_3(\varepsilon,\alpha,u) = \alpha^2 f_\varepsilon'(\alpha) - u^2 f_\varepsilon'(u), \qquad \text{where } \varepsilon \in \Theta \text{ and } \alpha > u > 0,$$

$$I_4(\varepsilon,\alpha,u) = \alpha^3 f_\varepsilon''(\alpha) - u^3 f_\varepsilon''(u), \qquad \text{where } \varepsilon \in \Theta \text{ and } \alpha > u > 0.$$

We assume that f_ε satisfies hypotheses (F1)–(F6) as follows:

(F1) For any fixed $\varepsilon \in \Theta$, there exists a positive number γ_ε such that $f_\varepsilon(0) > 0$ (positone), $f_\varepsilon(u) > 0$ on $(0,\infty)$, $f_\varepsilon''(u) > 0$ on $[0,\gamma_\varepsilon)$, $f_\varepsilon''(u) < 0$ on $(\gamma_\varepsilon,\infty)$ and $f_\varepsilon''(\gamma_\varepsilon) = 0$. Moreover, $\lim_{u\to\infty}(f_\varepsilon(u)/u) = 0$.

(F2) For any fixed $u > 0$, $f_\varepsilon(u)$ is a continuously differentiable, strictly decreasing function of $\varepsilon \in \Theta$.

(F3) There exist two positive numbers $\tilde{\varepsilon},\bar{\varepsilon} \in (\sigma_1,\sigma_2)$ such that $\tilde{\varepsilon} < \bar{\varepsilon}$ and the following conditions (i)–(iii) hold:

(i) $f_\varepsilon(\gamma_\varepsilon) - \gamma_\varepsilon f_\varepsilon'(\gamma_\varepsilon) \ge 0$ for $\bar{\varepsilon} \le \varepsilon < \sigma_2$.

(ii) For $\sigma_1 < \varepsilon < \bar{\varepsilon}$, the function $G_\varepsilon(u) \equiv \int_0^u t^3 f_\varepsilon''(t)dt$ has a positive zero κ_ε in $(0,\infty)$.

(iii) For $\sigma_1 < \varepsilon \le \tilde{\varepsilon}$, there exists a number $\rho_\varepsilon \in (0,\kappa_\varepsilon]$ such that

$$H_\varepsilon(u) \equiv \int_0^u t f_\varepsilon(t) - t^2 f_\varepsilon'(t)dt \begin{cases} = 0 & \text{if } u = \rho_\varepsilon, \\ < 0 & \text{if } \rho_\varepsilon < u \le \kappa_\varepsilon. \end{cases}$$

(F4) For $\sigma_1 < \varepsilon < \bar{\varepsilon}$,

$$\gamma_\varepsilon < \eta_\varepsilon \equiv \begin{cases} \rho_\varepsilon & \text{if } \sigma_1 < \varepsilon \le \tilde{\varepsilon}, \\ \kappa_\varepsilon & \text{if } \tilde{\varepsilon} < \varepsilon < \bar{\varepsilon}, \end{cases}$$

and

$$\begin{aligned} K(\varepsilon,u,v) \equiv\ & -8(I_1)^2(I_2) - 16(I_1)^2(I_3) - 4(I_1)^2(I_4) \\ & + 24(I_1)(I_2)^2 + 18(I_1)(I_2)(I_3) - 15(I_2)^3 \\ & > 0 \quad \text{for } u \in [\gamma_\varepsilon,\eta_\varepsilon] \text{ and } 0 < v < u. \end{aligned}$$

(F5) For $\sigma_1 < \varepsilon < \bar{\varepsilon}$, there exists a number $\omega_\varepsilon \in (\eta_\varepsilon,\infty]$ such that

$$3\left(\frac{\partial}{\partial\varepsilon}I_1\right)(I_2) - 2\left(\frac{\partial}{\partial\varepsilon}I_1\right)(I_1) - 2\left(\frac{\partial}{\partial\varepsilon}I_2\right)(I_1) > 0 \quad \text{for } 0 < v < u < \omega_\varepsilon.$$

Furthermore, ω_ε is a decreasing function on $[\tilde{\varepsilon},\bar{\varepsilon})$.

(F6) For $\tilde{\varepsilon} \le \varepsilon < \bar{\varepsilon}$,

$$2I_1(\varepsilon,\omega_\varepsilon,u) - I_2(\varepsilon,\omega_\varepsilon,u) > 0 \quad \text{for } 0 < u < \omega_\varepsilon.$$

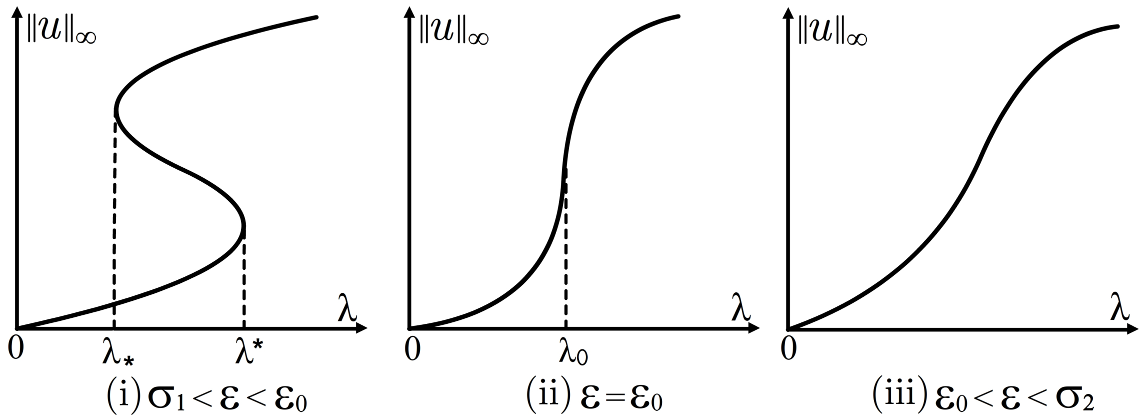

Figure 1.1: Global bifurcation of bifurcation curves S_ε of (1.1) with varying $\varepsilon \in \Theta = (\sigma_1, \sigma_2)$.

For any $\varepsilon \in \Theta$, on the $(\lambda, \|u\|_\infty)$-plane, we study the shape and structure of bifurcation curves S_ε of positive solutions of (1.1), defined by

$$S_\varepsilon \equiv \{(\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of (1.1)}\}.$$

We say that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve S_ε is S-shaped if S_ε is a continuous curve and there exist two positive numbers $\lambda_* < \lambda^*$ such that S_ε has *exactly two* turning points at some points $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ and $(\lambda_*, \|u_{\lambda_*}\|_\infty)$, and

(i) $\lambda_* < \lambda^*$ and $\|u_{\lambda^*}\|_\infty < \|u_{\lambda_*}\|_\infty$,

(ii) at $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ the bifurcation curve S_ε turns to the *left*,

(iii) at $(\lambda_*, \|u_{\lambda_*}\|_\infty)$ the bifurcation curve S_ε turns to the *right*.

See Fig. 1.1 (i).

In this paper, we mainly study the global bifurcation of bifurcation curves S_ε with varying $\varepsilon \in \Theta = (\sigma_1, \sigma_2)$. In Theorem 2.1 for (1.1) stated below, assuming that $f_\varepsilon \in C^3[0, \infty)$ satisfies hypotheses (F1)–(F6), we prove that there exists $\varepsilon_0 \in \Theta$ such that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve S_ε is S-shaped when $\sigma_1 < \varepsilon < \varepsilon_0$ and is monotone increasing when $\varepsilon_0 \le \varepsilon < \sigma_2$, see Fig. 1.1. In Theorem 2.3 stated behind, we give an application of Theorem 2.1 for (1.1) to the famous one-dimensional *perturbed Gelfand problem*:

$$\begin{cases} u''(x) + \lambda f_\varepsilon(u) = 0, \ -1 < x < 1, \ u(-1) = u(1) = 0, \\ f_\varepsilon(u) = \exp\left(\frac{u}{1+\varepsilon u}\right), \end{cases} \tag{1.5}$$

where $\lambda > 0$ is the Frank–Kamenetskii parameter or ignition parameter, $\varepsilon > 0$ is the *reciprocal* activation energy parameter, $u(x)$ is the dimensionless temperature, and the reaction term $f_\varepsilon(u)$ in (1.5) is the temperature dependence obeying the simple Arrhenius reaction-rate law in irreversible chemical reaction kinetics, see, e.g., Gelfand [5] and Boddington et al. [2]. This is the one-dimensional case of a problem arising in the study of (steady state) solid fuel ignition models in thermal combustion theory, cf. [1,4,6].

For (1.5), it has been a long-standing conjecture on the global bifurcation of bifurcation curves S_ε with varying $\varepsilon > 0$, see e.g. [8, Conjecture 1]. Also see [3,6,8,12,13,16,19]. Very recently, by developing some new time-map techniques and applying Sturm's theorem, Huang and Wang [8] gave a rigorous proof of this conjecture for (1.5). Their main result is stated in the next theorem.

Theorem 1.1 ([8, Theorem 4])**.** *Consider (1.5) with varying $\varepsilon > 0$. Then the bifurcation curve S_ε starts at the origin and tends to infinity as $\lambda \to \infty$, and there exists a positive critical bifurcation value ε_0 ($\approx 1/4.069 \approx 0.245$) < 0.25 such that the following assertions (i)–(iii) hold:*

(i) (See Fig. 1.1 (i).) For $0 < \varepsilon < \varepsilon_0$, the bifurcation curve S_ε is S-shaped on the $(\lambda, \|u\|_\infty)$-plane. More precisely, there exist two positive numbers $\lambda_* < \lambda^*$ such that (1.5) has exactly three positive solutions for $\lambda_* < \lambda < \lambda^*$, exactly two positive solutions for $\lambda = \lambda_*$ and $\lambda = \lambda^*$, and exactly one positive solution for $0 < \lambda < \lambda_*$ and $\lambda > \lambda^*$. Furthermore, all positive solutions u_λ are nondegenerate except that u_{λ_*} and u_{λ^*} are degenerate.

(ii) (See Fig. 1.1 (ii).) For $\varepsilon = \varepsilon_0$, the bifurcation curve S_{ε_0} is monotone increasing on the $(\lambda, \|u\|_\infty)$-plane. More precisely, (1.5) has exactly one positive solution for all $\lambda > 0$. Furthermore, all positive solutions u_λ are nondegenerate except that u_{λ_0} is a cusp type degenerate solution for some $\lambda = \lambda_0 > 0$.

(iii) (See Fig. 1.1 (iii).) If $\varepsilon > \varepsilon_0$, the bifurcation curve S_ε is monotone increasing on the $(\lambda, \|u\|_\infty)$-plane. More precisely, (1.5) has exactly one positive solution for all $\lambda > 0$. Furthermore, all positive solutions u_λ are nondegenerate.

Note that the definitions of degenerate and nondegenerate positive solutions and cusp type degenerate solution are defined later in Section 3.

Under somewhat different hypotheses to (F1)–(F6), the authors [9, Theorem 2.1] studied the global bifurcation and exact multiplicity of positive solutions for (1.1) and obtained the same results in Theorem 2.1. The hypotheses in [9, Theorem 2.1] can apply to a class of polynomial nonlinearities

$$f_\varepsilon(u) = -\varepsilon u^p + bu^2 + cu + d, \qquad p \geq 3,\ \varepsilon, b, d > 0,\ c \geq 0,$$

see [9, Theorem 2.1 and hypotheses (H1)–(H5)] for details. But the hypotheses in [9, Theorem 2.1] do not apply to (1.5) with $f_\varepsilon(u) = \exp\left(\frac{u}{1+\varepsilon u}\right)$. Cf. [9, Theorem 2.1 and hypotheses (H1)–(H5)] with Theorem 2.1 under (F1)–(F6).

The paper is organized as follows. Section 2 contains statements of the main results (Theorems 2.1–2.4). Section 3 contains several lemmas needed to prove the main results. Section 4 contains the proofs of the main results.

2 Main results

The main results in this paper are the next Theorems 2.1–2.4, in particular, Theorems 2.1 and 2.3. In Theorem 2.1, we prove the global bifurcation of bifurcation curves S_ε and hence we are able to determine exact multiplicity of positive solutions by $\varepsilon \in \Theta$ and $\lambda > 0$, see Fig. 1.1. In Theorem 2.3, we apply Theorem 2.1 to prove the global bifurcation of bifurcation curves S_ε for the one-dimension perturbed Gelfand problem (1.5).

Theorem 2.1 (See Fig. 1.1)**.** *Consider (1.1) with varying $\varepsilon \in \Theta = (\sigma_1, \sigma_2)$ where $0 \leq \sigma_1 < \sigma_2 \leq \infty$. Assume that $f \in C^3[0, \infty)$ satisfies (F1)–(F6). Then the bifurcation curve S_ε starts at the origin and tends to infinity as $\lambda \to \infty$, and there exists a positive critical bifurcation value $\varepsilon_0 \in (\tilde{\varepsilon}, \bar{\varepsilon})$ such that the following assertions (i)–(iii) hold:*

(i) *(See Fig. 1.1 (i).) For $\sigma_1 < \varepsilon < \varepsilon_0$, the bifurcation curve S_ε is S-shaped on the $(\lambda, \|u\|_\infty)$-plane. More precisely, there exist two positive numbers $\lambda_* < \lambda^*$ such that (1.1) has exactly three positive solutions for $\lambda_* < \lambda < \lambda^*$, exactly two positive solutions for $\lambda = \lambda_*$ and $\lambda = \lambda^*$, and exactly one positive solution for $0 < \lambda < \lambda_*$ and $\lambda > \lambda^*$. Furthermore, all positive solutions u_λ are nondegenerate except that u_{λ_*} and u_{λ^*} are degenerate.*

(ii) *(See Fig. 1.1 (ii).) For $\varepsilon = \varepsilon_0$, the bifurcation curve S_{ε_0} is monotone increasing on the $(\lambda, \|u\|_\infty)$-plane. More precisely, (1.1) has exactly one positive solution u_λ for all $\lambda > 0$. Furthermore, all positive solutions u_λ are nondegenerate except that u_{λ_0} is a degenerate solution for some $\lambda = \lambda_0 > 0$. In addition, u_{λ_0} is a cusp type degenerate solution if, for any fixed $u > 0$, $f'_\varepsilon(u)$ is continuously differentiable at $\varepsilon = \varepsilon_0$.*

(iii) *(See Fig. 1.1 (iii).) For $\varepsilon_0 < \varepsilon < \sigma_2$, the bifurcation curve S_ε is monotone increasing on the $(\lambda, \|u\|_\infty)$-plane. More precisely, (1.1) has exactly one positive solution u_λ for all $\lambda > 0$. Furthermore, all positive solutions u_λ are nondegenerate.*

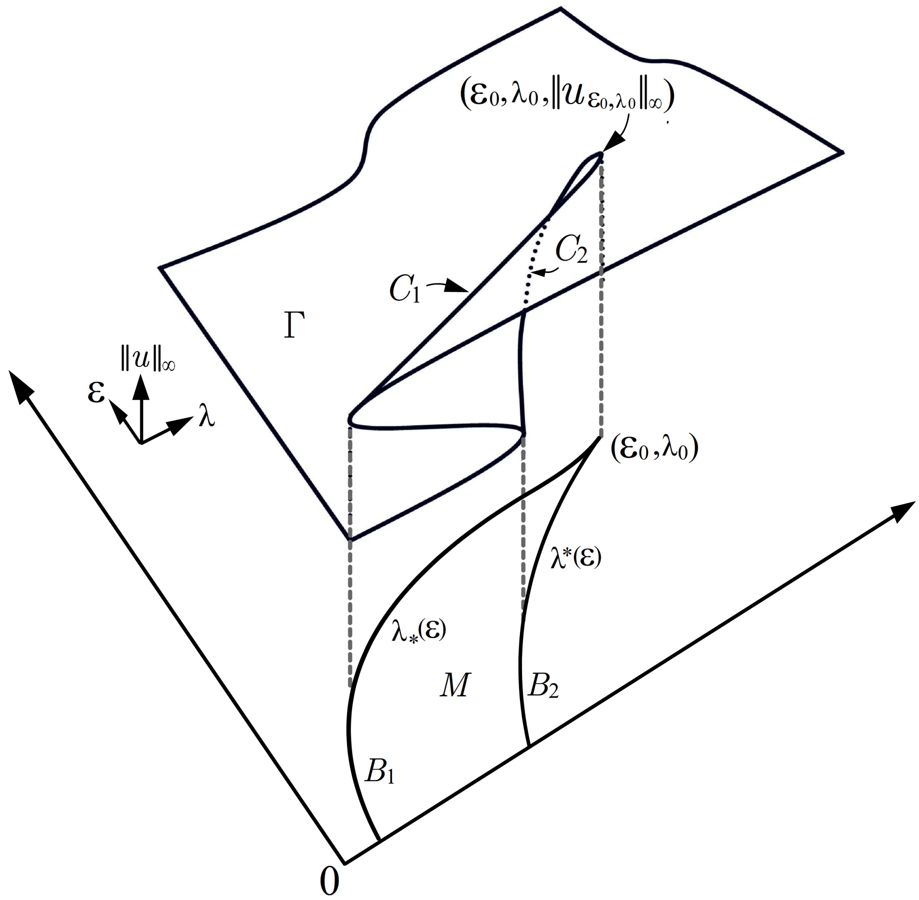

Figure 2.1: The bifurcation surface Γ with the fold curve $C_\Gamma = C_1 \cup C_2$, and the projection of C_Γ onto F_q. $B_\Gamma = B_1 \cup B_2 \cup \{(\varepsilon_0, \lambda_0)\}$ is the bifurcation set.

We next study, in the $(\varepsilon, \lambda, \|u\|_\infty)$-space, the shape and structure of the *bifurcation surface* Γ of (1.1), defined by

$$\Gamma \equiv \{(\varepsilon, \lambda, \|u_{\varepsilon,\lambda}\|_\infty) : \varepsilon, \lambda > 0 \text{ and } u_{\varepsilon,\lambda} \text{ is a positive solution of (1.1)}\}$$

which has the appearance of a folded surface with the *fold curve*

$$C_\Gamma \equiv \{(\varepsilon, \lambda, \|u_{\varepsilon,\lambda}\|_\infty : \varepsilon \in \Theta,\ \lambda > 0 \text{ and } u_{\varepsilon,\lambda} \text{ is a } \textit{degenerate} \text{ positive solution of (1.1)}\}.$$

See Fig. 2.1. Let F_q denote the first quadrant of the (ε, λ)-parameter plane. We also study, on F_q, the *bifurcation set* of (1.1)

$$B_\Gamma \equiv \{(\varepsilon, \lambda) : \varepsilon \in \Theta,\ \lambda > 0 \text{ and } u_{\varepsilon,\lambda} \text{ is a } \textit{degenerate} \text{ positive solution of (1.1)}\}.$$

By Theorem 2.1, we know that the bifurcation set $B_\Gamma = B_1 \cup B_2 \cup \{(\varepsilon_0, \lambda_0)\}$, where

$$B_1 \equiv \{(\varepsilon, \lambda_*(\varepsilon)) : \sigma_1 < \varepsilon < \varepsilon_0\} \quad \text{and} \quad B_2 \equiv \{(\varepsilon, \lambda^*(\varepsilon)) : \sigma_1 < \varepsilon < \varepsilon_0\}.$$

We define the set

$$M \equiv \{(\varepsilon, \lambda) : \sigma_1 < \varepsilon < \varepsilon_0 \text{ and } \lambda_*(\varepsilon) < \lambda < \lambda^*(\varepsilon)\}.$$

We analyze the structure of the bifurcation set B_Γ of (1.1) in the next theorem.

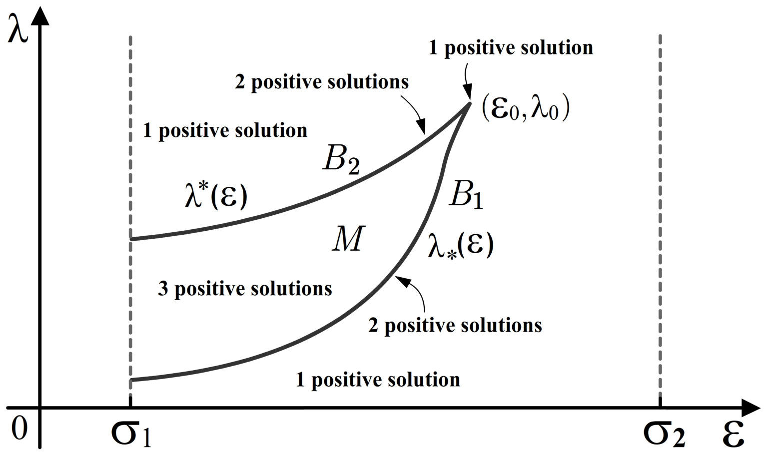

Figure 2.2: The graph of the bifurcation set $B_\Gamma = B_1 \cup B_2 \cup \{(\varepsilon_0, \lambda_0)\}$. $(\varepsilon_0, \lambda_0)$ is a cusp point of B_Γ.

Theorem 2.2 (See Fig. 2.2)**.** *Consider* (1.1) *with* $\varepsilon \in \Theta = (\sigma_1, \sigma_2)$ *where* $0 \le \sigma_1 < \sigma_2 \le \infty$. *Assume that* $f_\varepsilon \in C^3[0,\infty)$ *satisfies (F1)–(F6),* ω_ε *is a increasing function on* $(\sigma_1, \tilde{\varepsilon}]$*, and there exists a function* $\beta_\varepsilon \in [\rho_\varepsilon, \kappa_\varepsilon]$ *on* $(\sigma_1, \tilde{\varepsilon})$ *such that* β_ε *is decreasing on* (σ_1, ε') *and* $(\varepsilon', \tilde{\varepsilon})$ *for some* $\varepsilon' \in (\sigma_1, \tilde{\varepsilon})$ *respectively. Then* (1.1) *has exactly two positive solutions for* $(\varepsilon, \lambda) \in B_\Gamma \setminus \{(\varepsilon_0, \lambda_0)\}$*, exactly three positive solutions for* $(\varepsilon, \lambda) \in M$*, and exactly one positive solution for* $(\varepsilon, \lambda) \notin (B_\Gamma \setminus \{(\varepsilon_0, \lambda_0)\}) \cup M$*. Moreover,* $\lambda_*(\varepsilon)$ *and* $\lambda^*(\varepsilon)$ *are both continuous, strictly increasing functions on* $(\sigma_1, \varepsilon_0)$ *and satisfy*

$$0 \le \lim_{\varepsilon \to \sigma_1^+} \lambda_*(\varepsilon) \le \lim_{\varepsilon \to \sigma_1^+} \lambda^*(\varepsilon) < \lambda_0 = \lim_{\varepsilon \to \varepsilon_0^-} \lambda^*(\varepsilon) = \lim_{\varepsilon \to \varepsilon_0^-} \lambda_*(\varepsilon).$$

In addition, $\lim_{\varepsilon \to \sigma_1^+} \lambda_*(\varepsilon) < \lim_{\varepsilon \to \sigma_1^+} \lambda^*(\varepsilon)$ *if* $\lim_{\varepsilon \to \sigma_1^+} \rho_\varepsilon < \lim_{\varepsilon \to \sigma_1^+} \omega_\varepsilon$.

Theorem 2.3. *Consider* (1.5) *with varying* $\varepsilon \in (0, \infty)$*. Then the bifurcation curve* S_ε *starts at the origin and tends to infinity as* $\lambda \to \infty$*, and there exists a positive critical bifurcation value* ε_0 (≈ 0.245) *satisfying* $0.243 \approx \tilde{\varepsilon} < \varepsilon_0 < \bar{\varepsilon} \equiv 0.25$*, where* $\tilde{\varepsilon} = 1/\tilde{a}$ *and* $\tilde{a} \approx 4.107$ *is defined in [7, (1.4)] such that all the results in Theorem 1.1 (i)–(iii) hold.*

Theorem 2.4 (See Fig. 2.2). *Consider* (1.5) *with* $\varepsilon > 0$. *Then* (1.5) *has exactly two positive solutions for* $(\varepsilon, \lambda) \in B_\Gamma \setminus \{(\varepsilon_0, \lambda_0)\}$, *exactly three positive solutions for* $(\varepsilon, \lambda) \in M$, *and exactly one positive solution for* $(\varepsilon, \lambda) \notin (B_\Gamma \setminus \{(\varepsilon_0, \lambda_0)\}) \cup M$. *Moreover,* $\lambda_*(\varepsilon)$ *and* $\lambda^*(\varepsilon)$ *are both continuous, strictly increasing functions on* $(\sigma_1, \varepsilon_0)$ *and satisfy*

$$0 = \lim_{\varepsilon \to 0^+} \lambda_*(\varepsilon) < \lambda_\infty = \lim_{\varepsilon \to 0^+} \lambda^*(\varepsilon) < \lambda_0 = \lim_{\varepsilon \to \varepsilon_0^-} \lambda^*(\varepsilon) = \lim_{\varepsilon \to \varepsilon_0^-} \lambda_*(\varepsilon) \ (\approx 2.286),$$

where

$$\lambda_\infty \equiv \max_{\alpha \in (0,\infty)} \frac{1}{2e^\alpha} \left[\ln \left(2e^\alpha + 2\sqrt{e^\alpha (e^\alpha - 1)} - 1 \right) \right]^2 \approx 0.878.$$

3 Lemmas

To prove Theorem 2.1, we need the next Lemmas 3.1–3.11. We simply modify the time-map techniques used in [8, 9, 11, 18] without applying Sturm's theorem for Theorem 1.1 ([8, Theorem 4]). The time map formula we apply to study (1.1) takes the form as follows:

$$\sqrt{\lambda} = \frac{1}{\sqrt{2}} \int_0^\alpha [F_\varepsilon(\alpha) - F_\varepsilon(u)]^{-1/2} du \equiv T_\varepsilon(\alpha) \quad \text{for } \alpha > 0 \ \text{ if } \varepsilon \in \Theta = (\sigma_1, \sigma_2), \tag{3.1}$$

where $F_\varepsilon(u)$ is defined by (1.2), see Laetsch [14]. Observe that positive solutions $u_{\varepsilon,\lambda}$ for (1.1) correspond to

$$\|u_{\varepsilon,\lambda}\|_\infty = \alpha \quad \text{and} \quad T_\varepsilon(\alpha) = \sqrt{\lambda}. \tag{3.2}$$

Thus, studying of the exact number of positive solutions of (1.1) for fixed $\varepsilon \in \Theta$ is equivalent to studying the shape of the time map $T_\varepsilon(\alpha)$ on $(0, \infty)$, cf. [8,9,11,18]. In this section we always assume that $f_\varepsilon \in C^3[0, \infty)$ satisfies (F1)–(F6). Notice that, since $f_\varepsilon \in C^3[0, \infty)$, it can be proved that $T_\varepsilon(\alpha)$ is a thrice differentiable function of $\alpha > 0$ for $\varepsilon \in \Theta$. The proof is easy but tedious and consequently we omit it.

In addition, we recall that a positive solution u_λ of (1.1) is *degenerate* if $T'_\varepsilon(\|u_\lambda\|_\infty) = 0$ and is *nondegenerate* if $T'_\varepsilon(\|u_\lambda\|_\infty) \neq 0$. Also, a *degenerate* positive solution u_λ of (1.1) is of *cusp type* if $T''_\varepsilon(\|u_\lambda\|_\infty) = 0$ and $T'''_\varepsilon(\|u_\lambda\|_\infty) \neq 0$, see [16, p. 497] and [17, p. 214].

By (3.2), Theorem 2.1 follows if $\lim_{\alpha \to 0^+} T_\varepsilon(\alpha) = 0$ and $\lim_{\alpha \to \infty} T_\varepsilon(\alpha) = \infty$, and there exists $\varepsilon_0 \in (\tilde{\varepsilon}, \bar{\varepsilon}) \subset \Theta = (\sigma_1, \sigma_2)$ such that the following assertions (M1)–(M3) hold (See Fig. 3.1):

(M1) For $\sigma_1 < \varepsilon < \varepsilon_0$, $T_\varepsilon(\alpha)$ has exactly two critical points, a local maximum at some α_M and a local minimum at some α_m $(> \alpha_M)$, on $(0, \infty)$.

(M2) For $\varepsilon = \varepsilon_0$, $T'_{\varepsilon_0}(\alpha) > 0$ for $\alpha \in (0, \infty) \setminus \{\alpha_0\}$, and $T'_{\varepsilon_0}(\alpha_0) = 0$. In addition, $T''_{\varepsilon_0}(\alpha_0) = 0$ and $T'''_{\varepsilon_0}(\alpha_0) \neq 0$ if, for any fixed $u > 0$, $f'_\varepsilon(u)$ is continuously differentiable at $\varepsilon = \varepsilon_0$.

(M3) For $\varepsilon_0 < \varepsilon < \sigma_2$, $T'_\varepsilon(\alpha) > 0$ for $\alpha \in (0, \infty)$.

The main difficulty to obtain the above assertions (M1)–(M3) is to prove the *exact* number of critical points of the time map $T_\varepsilon(\alpha)$ on $(0, \infty)$ for all $\varepsilon \in \Theta = (\sigma_1, \sigma_2)$. Notice that by [15, Proposition 1.1.2], we see that if $f_\varepsilon \in C^3[0, \infty)$, then $T_\varepsilon(\alpha) \in C^3(0, \infty)$. By (3.1), we compute that

$$T'_\varepsilon(\alpha) = \frac{1}{2\sqrt{2}\alpha} \int_0^\alpha \frac{\theta(\alpha) - \theta(u)}{[F_\varepsilon(\alpha) - F_\varepsilon(u)]^{3/2}} du \quad \text{for } \alpha > 0, \tag{3.3}$$

where $\theta(u) \equiv 2F_\varepsilon(u) - u f_\varepsilon(u)$.

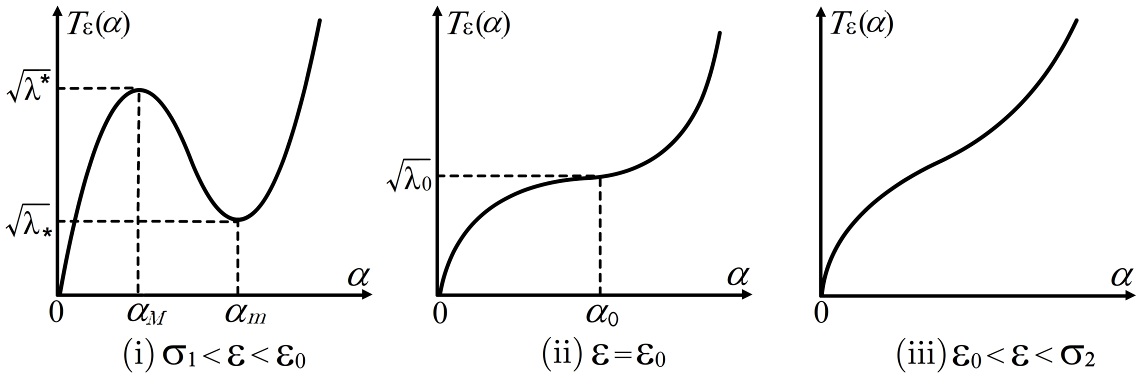

Figure 3.1: Graphs of $T_\varepsilon(\alpha)$ on $(0,\infty)$ with varying $\varepsilon \in \Theta = (\sigma_1, \sigma_2)$.

Lemma 3.1. *Consider* (1.1)*. For any fixed* $\varepsilon \in \Theta = (\sigma_1, \sigma_2)$ *with* $0 \le \sigma_1 < \sigma_2 \le \infty$*, the following assertions (i)–(ii) hold:*

(i) $\lim_{\alpha \to 0^+} T_\varepsilon(\alpha) = 0$ *and* $\lim_{\alpha \to \infty} T_\varepsilon(\alpha) = \infty$.

(ii) For $\varepsilon \in \Theta$*, either* $T_\varepsilon(\alpha)$ *is strictly increasing on* $(0, \gamma_\varepsilon]$*, or* $T_\varepsilon(\alpha)$ *is strictly increasing and then strictly decreasing on* $(0, \gamma_\varepsilon]$.

Proof. By (F1), we obtain that $f_\varepsilon(0) > 0$ on $[0,\infty)$ and $\lim_{u \to \infty}(f_\varepsilon(u)/u) = 0$. Thus assertion (i) follows by [14, Theorems 2.6 and 2.9]. By (F1) again, $f_\varepsilon''(u) > 0$ on $[0, \gamma_\varepsilon)$ and $f_\varepsilon''(\gamma_\varepsilon) = 0$, then assertion (ii) follows by [14, Theorem 3.2].

The proof of Lemma 3.1 is complete. □

Lemma 3.2. *Consider* (1.1) *with* $\varepsilon \in \Theta = (\sigma_1, \sigma_2)$ *where* $0 \le \sigma_1 < \sigma_2 \le \infty$*. For any fixed* $\alpha > 0$*,* $T_\varepsilon(\alpha)$ *is a continuous, strictly increasing function of* $\varepsilon \in \Theta$.

Proof. By (F2), for any fixed $u > 0$, $f_\varepsilon(u)$ is a continuous function of $\varepsilon \in \Theta$. Thus $T_\varepsilon(\alpha)$ is a continuous function of $\varepsilon \in \Theta$ by [14, Theorem 2.4]. By (F2) again, for any fixed $u > 0$, $f_{\varepsilon_1}(u) > f_{\varepsilon_2}(u)$ if $\sigma_1 < \varepsilon_1 < \varepsilon_2 < \sigma_2$. By (3.1), we directly obtain that $T_{\varepsilon_1}(\alpha) < T_{\varepsilon_2}(\alpha)$ if $\sigma_1 < \varepsilon_1 < \varepsilon_2 < \sigma_2$.

The proof of Lemma 3.2 is complete. □

Lemma 3.3. *Consider* (1.1) *with* $\sigma_1 < \varepsilon < \bar{\varepsilon}$*. Then* $\kappa_\varepsilon > \gamma_\varepsilon$ *and* κ_ε *is a continuous function of* ε *on* $(\sigma_1, \bar{\varepsilon})$*. Furthermore,*

$$G_\varepsilon(u) \begin{cases} > 0 & \text{if } 0 < u < \kappa_\varepsilon, \\ = 0 & \text{if } u = \kappa_\varepsilon, \\ < 0 & \text{if } u > \kappa_\varepsilon. \end{cases} \tag{3.4}$$

Proof. By (F1), we compute and observe that

$$G_\varepsilon(0) = 0 \quad \text{and} \quad G_\varepsilon'(u) \left(= \frac{\partial G_\varepsilon(u)}{\partial u} \right) = u^3 f_\varepsilon''(u) \begin{cases} > 0 & \text{if } 0 < u < \gamma_\varepsilon, \\ = 0 & \text{if } u = \gamma_\varepsilon, \\ < 0 & \text{if } u > \gamma_\varepsilon. \end{cases} \tag{3.5}$$

So for $\sigma_1 < \varepsilon < \bar{\varepsilon}$, by (F3) (ii), we observe that $G_\varepsilon(u)$ has a unique positive zero $\kappa_\varepsilon\,(> \gamma_\varepsilon)$ on $(0,\infty)$ such that (3.4) holds. Since $G_\varepsilon'(\kappa_\varepsilon) < 0$ by (3.5) and by the Implicit Function Theorem, κ_ε is a continuous function of ε on $(\sigma_1, \bar{\varepsilon})$.

The proof of Lemma 3.3 is complete. □

Lemma 3.4. *Consider (1.1) with $\varepsilon \in \Theta = (\sigma_1, \sigma_2)$ where $0 \le \sigma_1 < \sigma_2 \le \infty$. Then one of the following assertions (i)–(ii) holds:*

(i) $\theta'(u) > 0$ *for* $u > 0$ *and* $u \neq \gamma_\varepsilon$.

(ii) There exist two positive numbers $p_1(\varepsilon) < p_2(\varepsilon)$, dependent on ε, such that $p_1(\varepsilon) < \gamma_\varepsilon < p_2(\varepsilon)$ and

$$\theta'(u) = f_\varepsilon(u) - u f_\varepsilon'(u) \begin{cases} > 0 & \text{for } u \in (0, p_1(\varepsilon)) \cup (p_2(\varepsilon), \infty), \\ = 0 & \text{for } u \in \{p_1(\varepsilon), p_2(\varepsilon)\}, \\ < 0 & \text{for } u \in (p_1(\varepsilon), p_2(\varepsilon)). \end{cases} \tag{3.6}$$

Furthermore, if $\alpha \in (p_1(\varepsilon), p_2(\varepsilon)]$ satisfying $\theta(\alpha) \ge 0$, then there exists $\bar{\alpha} \in [0, p_1(\varepsilon))$ such that $\theta(\bar{\alpha}) = \theta(\alpha)$. See Fig. 3.2.

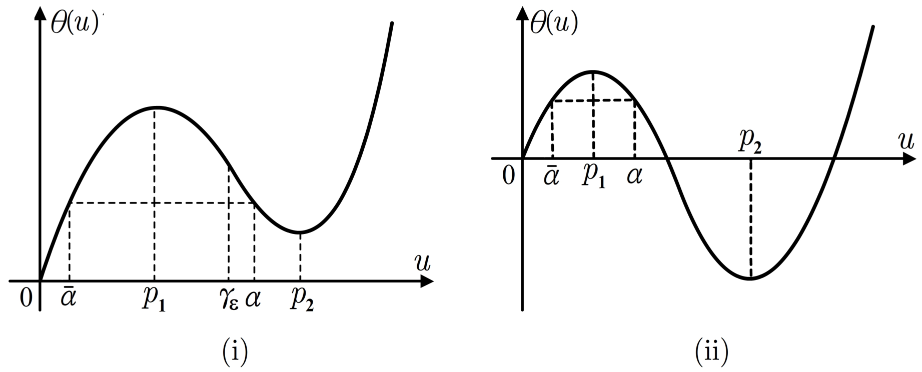

Figure 3.2: Graphs of $\theta(u)$ on $[0, \infty)$. (i) $\theta(u) \ge 0$ for all $u > 0$. (ii) $\theta(u) < 0$ for some $u > 0$.

Proof. By (F1), we observe that

$$\theta''(u) = -u^2 f_\varepsilon''(u) \begin{cases} < 0 & \text{if } 0 < u < \gamma_\varepsilon, \\ = 0 & \text{if } u = \gamma_\varepsilon, \\ > 0 & \text{if } u > \gamma_\varepsilon. \end{cases} \tag{3.7}$$

Assume that $\theta'(\gamma_\varepsilon) \ge 0$. It is easy to see that assertion (i) holds by (3.7). Assume that $\theta'(\gamma_\varepsilon) < 0$. Clearly, $\theta'(0) = f_\varepsilon(0) > 0$ by (F1). We assert that

$$\lim_{u \to \infty} \theta'(u) > 0. \tag{3.8}$$

So by (3.7) and (3.8), there exist two positive numbers $p_1(\varepsilon) < p_2(\varepsilon)$ such that $p_1(\varepsilon) < \gamma_\varepsilon < p_2(\varepsilon)$ and (3.6) holds. If $\alpha \in (p_1(\varepsilon), p_2(\varepsilon)]$ satisfying $\theta(\alpha) \ge 0$, then there exists $\bar{\alpha} \in [0, p_1(\varepsilon))$ such that $\theta(\bar{\alpha}) = \theta(\alpha)$. See Fig. 3.2 (i)–(ii). Next, we prove assertion (3.8). Let $v \in [\gamma_\varepsilon, \infty)$ be given. Since $\theta'(u)$ is strictly increasing for $u > \gamma_\varepsilon$ by (3.7), we observe that, for $u \ge v$,

$$\frac{f_\varepsilon(v)}{v} - \frac{f_\varepsilon(u)}{u} = \int_v^u \frac{d}{dt}\left(\frac{-f_\varepsilon(t)}{t}\right) dt = \int_v^u \frac{\theta'(t)}{t^2} dt < \theta'(u) \int_v^u \frac{1}{t^2} dt = \frac{u - v}{uv} \theta'(u).$$

So by (F1) and (F2), we see that

$$\lim_{u \to \infty} \theta'(u) \ge \lim_{u \to \infty} \left[\left(\frac{f_\varepsilon(v)}{v} - \frac{f_\varepsilon(u)}{u} \right) \left(\frac{uv}{u - v} \right) \right] = f_\varepsilon(v) > 0.$$

Thus (3.8) holds. Then assertion (ii) holds.

The proof of Lemma 3.4 is complete. □

Lemma 3.5. *Consider (1.1) with $\sigma_1 < \varepsilon \le \tilde{\varepsilon}$. Then ρ_ε is a continuous function of ε on $(\sigma_1, \tilde{\varepsilon}]$.*

Proof. Since $H_\varepsilon(0) = 0$ and $H'_\varepsilon(u) = u\theta'(u)$ for $u > 0$, and by (F3) (iii) and Lemma 3.4, we observe that $p_1(\varepsilon)$ and $p_2(\varepsilon)$ exist for $\sigma_1 < \varepsilon \le \tilde{\varepsilon}$. It follows that

$$\theta'(p_1(\varepsilon)) = \theta'(p_2(\varepsilon)) = 0 \quad \text{for } \sigma_1 < \varepsilon \le \tilde{\varepsilon}. \tag{3.9}$$

By integration by parts, (F3) (iii) and (3.4), we obtain that

$$0 = 2H_\varepsilon(\rho_\varepsilon) = \rho_\varepsilon^2\theta'(\rho_\varepsilon) + G_\varepsilon(\rho_\varepsilon) \ge \rho_\varepsilon^2\theta'(\rho_\varepsilon). \tag{3.10}$$

So by Lemma 3.4, we see that $p_1(\varepsilon) < \rho_\varepsilon \le p_2(\varepsilon)$ for $\sigma_1 < \varepsilon \le \tilde{\varepsilon}$, and

$$H'_\varepsilon(u) = u\theta'(u) \begin{cases} > 0 & \text{for } u \in (0, p_1(\varepsilon)) \cup (p_2(\varepsilon), \infty)\,, \\ = 0 & \text{for } u \in \{p_1(\varepsilon), p_2(\varepsilon)\}\,, \\ < 0 & \text{for } u \in (p_1(\varepsilon), p_2(\varepsilon))\,. \end{cases} \tag{3.11}$$

So by (3.11), we observe that ρ_ε is the unique zero of $H_\varepsilon(u)$ on $(0, p_2(\varepsilon)]$. By Lemma 3.4, we see that $p_1(\varepsilon) < \gamma_\varepsilon < p_2(\varepsilon)$ for $\sigma_1 < \varepsilon \le \tilde{\varepsilon}$. By (3.7), we further see that $\theta''(p_1(\varepsilon)) > 0$ and $\theta''(p_2(\varepsilon)) > 0$ for $\sigma_1 < \varepsilon \le \tilde{\varepsilon}$. So by the Implicit Function Theorem and (3.9), we obtain that $p_1(\varepsilon)$ and $p_2(\varepsilon)$ are continuous functions of ε on $(\sigma_1, \tilde{\varepsilon}]$. Let $\check{\varepsilon} \in (\sigma_1, \tilde{\varepsilon}]$ be given. We choose a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (\sigma_1, \tilde{\varepsilon}]/\{\check{\varepsilon}\}$ such that $\lim_{n\to\infty} \varepsilon_n = \check{\varepsilon}$. Since $p_1(\varepsilon_n) < \rho_{\varepsilon_n} < p_2(\varepsilon_n)$ for $n \in \mathbb{N}$ by (3.11), we see that

$$0 < p_1(\check{\varepsilon}) \le \liminf_{n\to\infty} \rho_{\varepsilon_n} \le \limsup_{n\to\infty} \rho_{\varepsilon_n} \le p_2(\check{\varepsilon}). \tag{3.12}$$

In addition, there exist two subsequences $\{\varepsilon_{1,n}\}_{n\in\mathbb{N}}$ and $\{\varepsilon_{2,n}\}_{n\in\mathbb{N}}$ of $\{\varepsilon_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \rho_{\varepsilon_{1,n}} = \liminf_{n\to\infty} \rho_{\varepsilon_n} \quad \text{and} \quad \lim_{n\to\infty} \rho_{\varepsilon_{2,n}} = \limsup_{n\to\infty} \rho_{\varepsilon_n}.$$

So by continuity of $H_\varepsilon(u)$ for u and ε, we observe that

$$H_{\check{\varepsilon}}(\liminf_{n\to\infty} \rho_{\varepsilon_n}) = \lim_{n\to\infty} H_{\varepsilon_{1,n}}(\rho_{\varepsilon_{1,n}}) = 0, \tag{3.13}$$

$$H_{\check{\varepsilon}}(\limsup_{n\to\infty} \rho_{\varepsilon_n}) = \lim_{n\to\infty} H_{\varepsilon_{2,n}}(\rho_{\varepsilon_{2,n}}) = 0. \tag{3.14}$$

So by (3.12)–(3.14), we further observe that $\limsup_{n\to\infty} \rho_{\varepsilon_n}$ and $\liminf_{n\to\infty} \rho_{\varepsilon_n}$ are two zeros of $H_{\check{\varepsilon}}(u)$ on $(0, p_2(\varepsilon)]$. Moreover,

$$\limsup_{n\to\infty} \rho_{\varepsilon_n} = \liminf_{n\to\infty} \rho_{\varepsilon_n} = \lim_{n\to\infty} \rho_{\varepsilon_n} = \rho_{\check{\varepsilon}}.$$

Thus the function ρ_ε is a continuous at $\varepsilon = \check{\varepsilon}$.

The proof of Lemma 3.5 is complete. □

Lemma 3.6. *Consider (1.1) with $\varepsilon \in \Theta = (\sigma_1, \sigma_2)$ where $0 \le \sigma_1 < \sigma_2 \le \infty$. Then the following assertions (i)–(iii) hold:*

(i) For $\bar{\varepsilon} \le \varepsilon < \sigma_2$, $T_\varepsilon'(\alpha) > 0$ for $\alpha > 0$.

(ii) For $\sigma_1 < \varepsilon < \bar{\varepsilon}$,

$$T_\varepsilon''(\alpha) + \frac{2}{\alpha}T_\varepsilon'(\alpha) > 0 \ \text{ for } \alpha \ge \kappa_\varepsilon. \tag{3.15}$$

Moreover, $T_\varepsilon(\alpha)$ has at most one critical point, a local minimum, on $[\kappa_\varepsilon, \infty)$.

(iii) For $\sigma_1 < \varepsilon \le \tilde{\varepsilon}$, $T_\varepsilon'(\alpha) < 0$ for $\rho_\varepsilon \le \alpha \le \kappa_\varepsilon$.

Proof. (I) We prove assertion (i). By (F3) (i) and (3.7), we observe that, for $\bar{\varepsilon} \le \varepsilon < \sigma_2$,

$$\theta'(u) > \theta'(\gamma_\varepsilon) = f_\varepsilon(\gamma_\varepsilon) - \gamma_\varepsilon f_\varepsilon'(\gamma_\varepsilon) \ge 0 \ \text{ for } u > 0 \ \text{ and } \ u \ne \gamma_\varepsilon.$$

It follows that $\theta(\alpha) - \theta(u) > 0$ for $\alpha > u > 0$. So by (3.3), we see that $T_\varepsilon'(\alpha) > 0$ for $\alpha > 0_\varepsilon$. So assertion (i) holds.

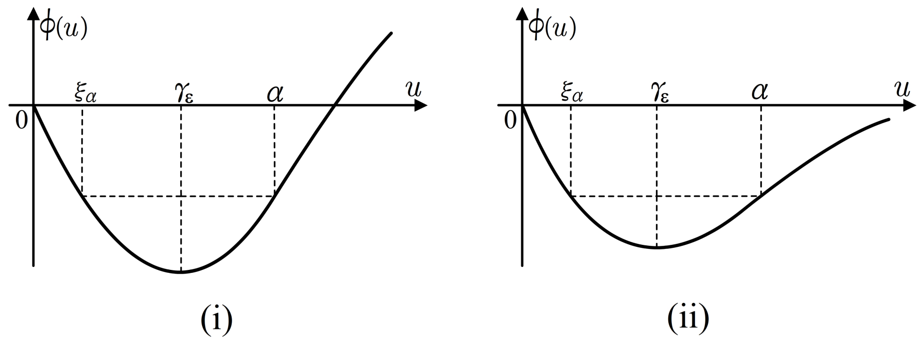

Figure 3.3: Graphs of $\phi(u)$ on $[0, \infty)$. (i) $\phi(u) > 0$ for some $u > 0$. (ii) $\phi(u) \le 0$ for all $u \ge 0$.

(II) We prove assertion (ii). We compute and observe that

$$\begin{aligned} T_\varepsilon''(\alpha) + \frac{2}{\alpha}T_\varepsilon'(\alpha) &= \frac{1}{\sqrt{2}\alpha^2}\int_0^\alpha \frac{\frac{3}{2}\left[\theta(\alpha) - \theta(u)\right]^2 + [F_\varepsilon(\alpha) - F_\varepsilon(u)][\phi(\alpha) - \phi(u)]}{[F(\alpha) - F(u)]^{5/2}}du \\ &\ge \frac{1}{\sqrt{2}\alpha^2}\int_0^\alpha \frac{\phi(\alpha) - \phi(u)}{[F_\varepsilon(\alpha) - F_\varepsilon(u)]^{3/2}}du, \end{aligned} \tag{3.16}$$

where $\phi(u) \equiv u\theta'(u) - \theta(u)$, see [10, (3.12)]. We obtain that

$$\phi(0) = 0 \quad \text{and} \quad \phi'(u) = u\theta''(u) = -u^2 f_\varepsilon''(u) \begin{cases} < 0 & \text{for } 0 \le u < \gamma_\varepsilon, \\ = 0 & \text{for } u = \gamma_\varepsilon, \\ > 0 & \text{for } u > \gamma_\varepsilon. \end{cases} \tag{3.17}$$

Let $\alpha \in [\kappa_\varepsilon, \infty)$ be given. By Lemma 3.3, we see that $\alpha \ge \kappa_\varepsilon > \gamma_\varepsilon$ for $\sigma_1 < \varepsilon < \bar{\varepsilon}$. If $\phi(\alpha) \ge 0$, by (3.17), we see that $\phi(\alpha) - \phi(u) > 0$ for $0 < u < \alpha$, and hence (3.15) holds by (3.16). While if $\phi(\alpha) < 0$, there exists $\xi_\alpha \in (0, \gamma_\varepsilon)$ such that $\phi(\xi_\alpha) = \phi(\alpha)$. See Fig. 3.3. So by [10, (3.15)], (F3) (ii) and (3.4),

$$T_\varepsilon''(\alpha) + \frac{2}{\alpha}T_\varepsilon'(\alpha) > \frac{-1}{\sqrt{2}\alpha^2\left[F_\varepsilon(\alpha) - F_\varepsilon(\xi_\alpha)\right]^{3/2}}G_\varepsilon(\alpha) \ge 0,$$

and hence (3.15) holds. Assume that $T_\varepsilon(\alpha)$ has a critical point $\alpha_1 \in [\kappa_\varepsilon, \infty)$. By (3.15), $T_\varepsilon''(\alpha_1) > 0$. So $T_\varepsilon(\alpha)$ has at most one critical point, a local minimum, on $[\kappa_\varepsilon, \infty)$. Therefore, assertion (ii) holds.

(III) We prove assertion (iii). By (F3) (iii), we see that $\rho_\varepsilon \leq \kappa_\varepsilon$ for $\sigma_1 < \varepsilon \leq \tilde{\varepsilon}$. We fix $\varepsilon \in (\sigma_1, \tilde{\varepsilon}]$ and $\alpha \in [\rho_\varepsilon, \kappa_\varepsilon]$. Assume that $\theta(\alpha) \leq 0$. By assertion (ii) of Lemma 3.4, we see that $\theta(\alpha) - \theta(u) < 0$ for $0 < u < \alpha$, see Fig. 3.2 (ii). It follows that $T_\varepsilon'(\alpha) < 0$ by (3.3). Assume that $\theta(\alpha) > 0$. By integration by parts and (F3) (ii)–(iii), we observe that

$$0 \geq 2H_\varepsilon(\kappa_\varepsilon) = \kappa_\varepsilon^2 \theta'(\kappa_\varepsilon) + G_\varepsilon(\kappa_\varepsilon) = \kappa_\varepsilon^2 \theta'(\kappa_\varepsilon).$$

So by (3.10), we have that $p_1(\varepsilon) < \rho_\varepsilon \leq \alpha \leq \kappa_\varepsilon \leq p_2(\varepsilon)$. Assume that $\theta(\alpha) > 0$. By assertion (ii) of Lemma 3.4, there exists $\bar{\alpha} \in (0, p_1(\varepsilon))$ such that $\theta(\bar{\alpha}) = \theta(\alpha)$. It follows that

$$\theta(\alpha) - \theta(u) \begin{cases} > 0 & \text{for } u \in (0, \bar{\alpha}), \\ = 0 & \text{for } u = \bar{\alpha}, \\ < 0 & \text{for } u \in (\bar{\alpha}, \alpha). \end{cases}$$

So by (3.3) and (F3) (iii), we obtain that

$$\begin{aligned}
T_\varepsilon'(\alpha) &= \frac{1}{2\sqrt{2}\alpha} \int_0^\alpha \frac{\theta(\alpha) - \theta(u)}{[F_\varepsilon(\alpha) - F_\varepsilon(u)]^{3/2}} du \\
&= \frac{1}{2\sqrt{2}\alpha} \left\{ \int_0^{\bar{\alpha}} \frac{\theta(\alpha) - \theta(u)}{[F_\varepsilon(\alpha) - F_\varepsilon(u)]^{3/2}} du + \int_{\bar{\alpha}}^\alpha \frac{\theta(\alpha) - \theta(u)}{[F_\varepsilon(\alpha) - F_\varepsilon(u)]^{3/2}} du \right\} \\
&< \frac{1}{2\sqrt{2}\alpha \, [F_\varepsilon(\alpha) - F_\varepsilon(\bar{\alpha})]^{3/2}} \left\{ \int_0^{\bar{\alpha}} [\theta(\alpha) - \theta(u)] \, du + \int_{\bar{\alpha}}^\alpha [\theta(\alpha) - \theta(u)] \, du \right\} \\
&= \frac{1}{2\sqrt{2}\alpha \, [F_\varepsilon(\alpha) - F_\varepsilon(\bar{\alpha})]^{3/2}} \left[\alpha\theta(\alpha) - \int_0^\alpha \theta(u) du \right] \\
&= \frac{1}{2\sqrt{2}\alpha \, [F_\varepsilon(\alpha) - F_\varepsilon(\bar{\alpha})]^{3/2}} \int_0^\alpha u\theta'(u) du = \frac{1}{2\sqrt{2}\alpha \, [F_\varepsilon(\alpha) - F_\varepsilon(\bar{\alpha})]^{3/2}} H_\varepsilon(\alpha) \leq 0.
\end{aligned}$$

So assertion (iii) holds.

The proof of Lemma 3.6 is complete. □

Lemma 3.7. *Consider (1.1) with $\varepsilon \in \Theta = (\sigma_1, \sigma_2)$ where $0 \leq \sigma_1 < \sigma_2 \leq \infty$. For any fixed $\alpha > 0$, $T_\varepsilon'(\alpha)$ is a continuously differentiable function of $\varepsilon \in I_\alpha$. Furthermore, $\frac{\partial}{\partial \varepsilon} T_\varepsilon'(\alpha) > 0$ for $0 < \alpha < \omega_\varepsilon$ and $\sigma_1 < \varepsilon < \bar{\varepsilon}$.*

Proof. First, for any fixed $\alpha > 0$, it can be proved that $T_\varepsilon'(\alpha)$ is a continuously differentiable function of $\varepsilon \in I_\alpha$. The proof is easy but tedious and consequently we omit it. Secondly, by (1.3), (1.4), (3.3) and (F5), we compute and obtain that, for $0 < \alpha < \omega_\varepsilon$,

$$\frac{\partial}{\partial \varepsilon} T_\varepsilon'(\alpha) = \frac{1}{4\sqrt{2}\alpha} \int_0^\alpha \frac{3\left(\frac{\partial}{\partial \varepsilon} I_1\right)(I_2) - 2\left(\frac{\partial}{\partial \varepsilon} I_1\right)(I_1) - 2\left(\frac{\partial}{\partial \varepsilon} I_2\right)(I_1)}{[F_\varepsilon(\alpha) - F_\varepsilon(u)]^{5/2}} du > 0.$$

The proof of Lemma 3.7 is complete. □

Lemma 3.8. *Consider (1.1) with $\tilde{\varepsilon} < \varepsilon < \bar{\varepsilon}$. Assume that $\gamma_\varepsilon < \eta_\varepsilon$. Then $[\alpha T_\varepsilon''(\alpha)]' > 0$ for $\gamma_\varepsilon \leq \alpha \leq \eta_\varepsilon$ and one of the following assertions (i)–(iii) holds:*

(i) $T'_\varepsilon(\alpha)$ *is a strictly increasing function of* α *on* $[\gamma_\varepsilon, \eta_\varepsilon]$.

(ii) $T'_\varepsilon(\alpha)$ *is a strictly decreasing function of* α *on* $[\gamma_\varepsilon, \eta_\varepsilon]$.

(iii) $T'_\varepsilon(\alpha)$ *is a strictly decreasing and then strictly increasing function of* α *on* $[\gamma_\varepsilon, \eta_\varepsilon]$.

Proof. By (F4), we compute and observe that

$$\left[\alpha T''_\varepsilon(\alpha)\right]' = \frac{1}{8\sqrt{2}\alpha^2}\int_0^\alpha \frac{K(\varepsilon,\alpha,u)}{[F_\varepsilon(\alpha)-F_\varepsilon(u)]^{7/2}}du > 0 \quad \text{for } \gamma_\varepsilon \le \alpha \le \eta_\varepsilon.$$

It follows that $\alpha T''_\varepsilon(\alpha)$ is a strictly increasing function of $\alpha \in [\gamma_\varepsilon, \eta_\varepsilon]$. So we observe that there are three cases:

Case 1. $T''_\varepsilon(\alpha) > 0$ for $\alpha \in [\gamma_\varepsilon, \eta_\varepsilon]$.

Case 2. $T''_\varepsilon(\alpha) < 0$ for $\alpha \in [\gamma_\varepsilon, \eta_\varepsilon)$.

Case 3. $T''_\varepsilon(\alpha) < 0$ for $\alpha \in [\gamma_\varepsilon, \breve{\alpha})$, $T''_\varepsilon(\alpha) > 0$ for $\alpha \in (\breve{\alpha}, \eta_\varepsilon]$, and $T''_\varepsilon(\breve{\alpha}) = 0$ for some $\breve{\alpha} \in (\gamma_\varepsilon, \eta_\varepsilon)$.

So by Cases 1–3, assertions (i)–(iii) hold.

The proof of Lemma 3.8 is complete. □

Lemma 3.9. *Consider* (1.1) *with* $\sigma_1 < \varepsilon < \bar{\varepsilon}$. *Either one of the following assertions (i)–(ii) holds:*

(i) $T_\varepsilon(\alpha)$ *is a strictly increasing function on* $(0,\infty)$.

(ii) $T_\varepsilon(\alpha)$ *has exactly one local maximum and exactly one local minimum on* $(0,\infty)$.

Proof. We fix $\varepsilon \in (\sigma_1, \bar{\varepsilon})$. Assume that assertion (i) does not hold. By Lemma 3.1 (i), $T_\varepsilon(\alpha)$ has a local maximum and a local minimum on $(0,\infty)$. Assume that $T_\varepsilon(\alpha)$ has two local maximum at some positive numbers $\alpha_{M_1} < \alpha_{M_2}$. Then there exists $\alpha_m \in (\alpha_{M_1}, \alpha_{M_2})$ such that $T_\varepsilon(\alpha_m)$ is the local minimum value. We consider four cases:

Case 1. $\tilde{\varepsilon} < \varepsilon < \bar{\varepsilon}$ and $\gamma_\varepsilon < \eta_\varepsilon$.

Case 2. $\tilde{\varepsilon} < \varepsilon < \bar{\varepsilon}$ and $\gamma_\varepsilon \ge \eta_\varepsilon$.

Case 3. $\sigma_1 < \varepsilon \le \tilde{\varepsilon}$ and $\gamma_\varepsilon < \eta_\varepsilon$.

Case 4. $\sigma_1 < \varepsilon \le \tilde{\varepsilon}$ and $\gamma_\varepsilon \ge \eta_\varepsilon$.

If Case 1 holds, by Lemmas 3.1 (ii) and 3.6 (ii), we observe that $\gamma_\varepsilon \le \alpha_m < \alpha_{M_2} < \kappa_\varepsilon = \eta_\varepsilon$. It is a contradiction by Lemma 3.8. If Case 2 holds, by Lemma 3.6 (ii), we observe that $0 < \alpha_{M_1} < \alpha_{M_2} < \kappa_\varepsilon = \eta_\varepsilon \le \gamma_\varepsilon$. It is a contradiction by Lemma 3.1 (ii). If Case 3 holds, by Lemmas 3.1 (ii) and 3.6 (ii)–(iii), we observe that $\gamma_\varepsilon \le \alpha_m < \alpha_{M_2} < \rho_\varepsilon = \eta_\varepsilon$. It is a contradiction by Lemma 3.8. If Case 4 holds, by Lemma 3.6 (ii)–(iii), we observe that $0 < \alpha_{M_1} < \alpha_{M_2} < \rho_\varepsilon = \eta_\varepsilon \le \gamma_\varepsilon$. It is a contradiction by Lemma 3.1 (ii). So $T_\varepsilon(\alpha)$ has exactly one local maximum.

Assume that $T_\varepsilon(\alpha)$ has two local minimum at some positive numbers $\alpha_{m_1} < \alpha_{m_2}$. By Lemma 3.1 (i), then there exist $\alpha_{M_1} \in (0, \alpha_{m_1})$ and $\alpha_{M_2} \in (\alpha_{m_1}, \alpha_{m_2})$ such that $T_\varepsilon(\alpha_{M_1})$ and $T_\varepsilon(\alpha_{M_2})$ are the local maximum values. By previous discussion, we obtain a contradiction. So $T_\varepsilon(\alpha)$ has exactly one local minimum.

By above, $T_\varepsilon(\alpha)$ has exactly one local maximum and exactly one local minimum on $(0,\infty)$.

The proof of Lemma 3.9 is complete. □

Lemma 3.10. *Consider (1.1) with $\sigma_1 < \varepsilon < \bar{\varepsilon}$. Either one of the following two assertions holds:*

(i) $T_\varepsilon(\alpha)$ is a strictly increasing function on $(0,\infty)$ and $T_\varepsilon(\alpha)$ has at most one critical point on $(0,\infty)$.

(ii) $T_\varepsilon(\alpha)$ has exactly two critical points, a local maximum at some α_M and a local minimum at some $\alpha_m > \alpha_M$ on $(0,\infty)$.

Proof. We fix $\varepsilon_* \in (\sigma_1, \bar{\varepsilon})$. By Lemma 3.9, either one of the following two cases holds:

Case 1. $T_{\varepsilon_*}(\alpha)$ is a strictly increasing function on $(0,\infty)$.

Case 2. $T_{\varepsilon_*}(\alpha)$ has exactly one local maximum at some $\alpha_M(\varepsilon_*)$ and exactly one local minimum at some $\alpha_m(\varepsilon_*)$ on $(0,\infty)$.

(I) We prove assertion (i) under Case 1. Case 1 implies that $T'_{\varepsilon_*}(\alpha) \geq 0$ for $\alpha > 0$. Assume that $T_{\varepsilon_*}(\alpha)$ has two critical points $\alpha_1(\varepsilon_*) < \alpha_2(\varepsilon_*)$ on $(0,\infty)$. We obtain that

$$T'_{\varepsilon_*}(\alpha_1(\varepsilon_*)) = T'_{\varepsilon_*}(\alpha_2(\varepsilon_*)) = T''_{\varepsilon_*}(\alpha_1(\varepsilon_*)) = T''_{\varepsilon_*}(\alpha_2(\varepsilon_*)) = 0.$$

So by (F5) and Lemma 3.6 (ii)–(iii), we observe that $0 < \alpha_1(\varepsilon_*) < \alpha_2(\varepsilon_*) < \eta_{\varepsilon_*} < \omega_{\varepsilon_*}$. We assert that there exists $\delta > 0$ such that

$$0 < \alpha_1(\varepsilon_*) < \alpha_2(\varepsilon_*) < \omega_\varepsilon \quad \text{for } \varepsilon_* - \delta \leq \varepsilon \leq \varepsilon_*. \tag{3.18}$$

Let $\hat{\varepsilon} \in (\varepsilon_* - \delta, \varepsilon_*)$ be given. By Lemma 3.7 and (3.18), we observe that

$$T'_{\hat{\varepsilon}}(\alpha_1(\varepsilon_*)) < T'_{\varepsilon_*}(\alpha_1(\varepsilon_*)) = 0 \quad \text{and} \quad T'_{\hat{\varepsilon}}(\alpha_2(\varepsilon_*)) < T'_{\varepsilon_*}(\alpha_2(\varepsilon_*)) = 0. \tag{3.19}$$

By Lemmas 3.1 (ii), 3.6 (ii)–(iii), and 3.8, we observe that, for $\sigma_1 < \varepsilon < \bar{\varepsilon}$, there are no open intervals $I \subset \mathbb{R}^+$ such that $T'_\varepsilon(\alpha) = 0$ on I. It implies that $T'_{\varepsilon_*}(\hat{\alpha}) > 0$ for some $\hat{\alpha} \in (\alpha_1(\varepsilon_*), \alpha_2(\varepsilon_*))$. So by continuity of $T'_\varepsilon(\alpha)$ of ε and (3.19), we choose $\hat{\varepsilon}$ sufficiently close to ε_* such that $T'_{\hat{\varepsilon}}(\alpha)$ has four roots $\alpha_{1,1}$, $\alpha_{1,2}$, $\alpha_{2,1}$, and $\alpha_{2,2}$ such that

$$\alpha_{1,1} < \alpha_1(\varepsilon_*) < \alpha_{1,2} < \alpha_{2,1} < \alpha_2(\varepsilon_*) < \alpha_{2,2}.$$

Furthermore, $T_{\hat{\varepsilon}}(\alpha_{1,1})$, $T_{\hat{\varepsilon}}(\alpha_{2,1})$ are local maximum values, and $T_{\hat{\varepsilon}}(\alpha_{1,2})$, $T_{\hat{\varepsilon}}(\alpha_{2,2})$ are local minimum values. It is a contradiction by Lemma 3.9. Therefore, assertion (i) holds.

Next, we prove assertion (3.18). Let $d_{\varepsilon_*} \equiv [\eta_{\varepsilon_*} + \alpha_2(\varepsilon_*)]/2$. Clearly, $\alpha_2(\varepsilon_*) < d_{\varepsilon_*} < \eta_{\varepsilon_*}$. If $\sigma_1 < \varepsilon_* \leq \tilde{\varepsilon}$, since $\eta_{\varepsilon_*} = \rho_{\varepsilon_*}$ and by Lemma 3.5, we observe that there exists $\delta_1 > 0$ such that

$$0 < \alpha_1(\varepsilon_*) < \alpha_2(\varepsilon_*) < d_{\varepsilon_*} < \rho_\varepsilon = \eta_\varepsilon < \omega_\varepsilon \quad \text{for } \varepsilon \in [\varepsilon_* - \delta_1, \varepsilon_*], \tag{3.20}$$

and hence assertion (3.18) holds. If $\tilde{\varepsilon} < \varepsilon^* < \bar{\varepsilon}$, since $\eta_{\varepsilon_*} = \kappa_{\varepsilon_*}$ and by Lemma 3.3, we observe that there exists $\delta_2 > 0$ such that

$$0 < \alpha_1(\varepsilon_*) < \alpha_2(\varepsilon_*) < d_{\varepsilon_*} < \kappa_\varepsilon = \eta_\varepsilon < \omega_\varepsilon \quad \text{for } \varepsilon \in [\varepsilon_* - \delta_2, \varepsilon_*], \tag{3.21}$$

and hence assertion (3.18) holds. Thus assertion (3.18) holds by (3.20) and (3.21).

(II) We prove assertion (ii) under Case 2. Case 2 implies that $T_{\varepsilon_*}(\alpha)$ has a critical point $\alpha_3(\varepsilon_*)$ on $(0,\infty)$, distinct from $\alpha_M(\varepsilon_*)$ and $\alpha_m(\varepsilon_*)$. It follows that $T'_{\varepsilon_*}(\alpha_3(\varepsilon_*)) = T''_{\varepsilon_*}(\alpha_3(\varepsilon_*)) = 0$.

So by (F5) and Lemma 3.6 (ii)–(iii), $0 < \alpha_3(\varepsilon_*) < \eta_{\varepsilon_*} < \omega_{\varepsilon_*}$. We assert that there exists $\delta > 0$ such that

$$0 < \alpha_3(\varepsilon_*) < \omega_\varepsilon \quad \text{for } \varepsilon_* - \delta \le \varepsilon \le \varepsilon_* + \delta. \tag{3.22}$$

By Lemmas 3.1 (ii), 3.6 (ii)–(iii), and 3.8, we observe that, for $\sigma_1 < \varepsilon < \bar{\varepsilon}$, there are no open intervals I such that $T'_\varepsilon(\alpha) = 0$ on I. By Lemma 3.7 and (3.22), we observe that if $T'_{\varepsilon_*}(\alpha)$ has a local minimum value at $\alpha = \alpha_3(\varepsilon_*)$, then

$$T'_\varepsilon(\alpha_3(\varepsilon_*)) < T'_{\varepsilon_*}(\alpha_3(\varepsilon_*)) = 0 \quad \text{for } \varepsilon_* - \delta < \varepsilon < \varepsilon_*;$$

if $T'_{\varepsilon_*}(\alpha)$ has a local maximum value at $\alpha = \alpha_3(\varepsilon_*)$, then

$$T'_\varepsilon(\alpha_3(\varepsilon_*)) > T'_{\varepsilon_*}(\alpha_3(\varepsilon_*)) = 0 \quad \text{for } \varepsilon_* < \varepsilon < \varepsilon_* + \delta.$$

So by continuity of $T'_\varepsilon(\alpha)$ of ε, there exists $\check{\varepsilon} \in (\sigma_1, \bar{\varepsilon})$ sufficiently close to ε_* such that $T_{\check{\varepsilon}}(\alpha)$ has four local extreme α_{31}, α_{32}, α_{33} and α_{34} such that α_{31} and α_{32} are in the neighborhood of $\alpha_M(\varepsilon_*)$ and $\alpha_m(\varepsilon_*)$ respectively, and $\alpha_{33} \in (0, \alpha_3(\varepsilon_*))$ and $\alpha_{34} \in (\alpha_3(\varepsilon_*), \infty)$, distinct from α_{31} and α_{32}, see Fig. 3.4. It is a contradiction by Lemma 3.9. Thus, assertion (ii) holds.

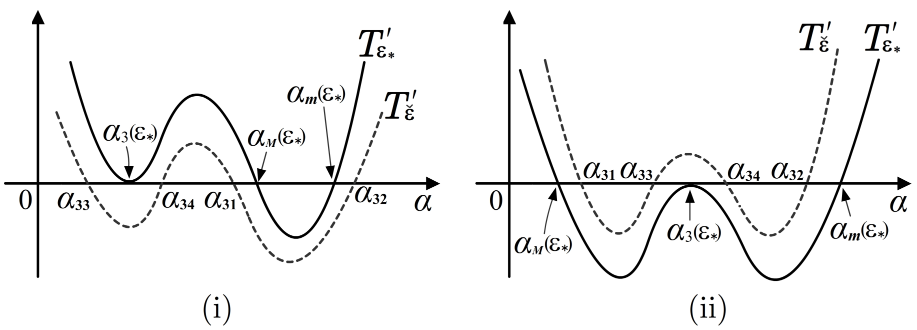

Figure 3.4: Local graphs of $T'_{\check{\varepsilon}}(\alpha)$ and $T'_{\varepsilon_*}(\alpha)$. (i) $T'_{\varepsilon_*}(\alpha_3(\varepsilon_*))$ is a local minimum value. (ii) $T'_{\varepsilon_*}(\alpha_3(\varepsilon_*))$ is a local maximum value.

Next, we prove assertion (3.22). If $\varepsilon_* \ne \tilde{\varepsilon}$, by continuities of ρ_ε and κ_ε, we observe that assertion (3.22) holds. If $\varepsilon_* = \tilde{\varepsilon}$, we let

$$\tilde{\eta}_\varepsilon \equiv \begin{cases} \rho_\varepsilon & \text{if } \sigma_1 < \varepsilon \le \tilde{\varepsilon}, \\ \kappa_\varepsilon - (\kappa_{\tilde{\varepsilon}} - \rho_{\tilde{\varepsilon}}) & \text{if } \tilde{\varepsilon} < \varepsilon < \bar{\varepsilon}. \end{cases}$$

Clearly, $\tilde{\eta}_\varepsilon$ is a continuous function of ε and $\tilde{\eta}_\varepsilon < \omega_\varepsilon$ on $(\sigma_1, \bar{\varepsilon})$ by Lemmas 3.3 and 3.5. Since $\alpha_3(\varepsilon_*) < \rho_{\varepsilon_*} = \tilde{\eta}_{\varepsilon_*} < \omega_{\varepsilon_*}$, assertion (3.22) holds.

The proof of Lemma 3.10 is complete. □

Let

$$\Omega = \left\{ \begin{array}{l} \varepsilon \in \Theta : T_\varepsilon(\alpha) \text{ has exactly two critical points,} \\ \text{a local maximum and a local minimum, on } (0, \infty_\varepsilon) \end{array} \right\}.$$

We then prove, in the next lemma, that the set Ω is open and connected.

Lemma 3.11. *Consider (1.1) with $\varepsilon \in \Theta = (\sigma_1, \sigma_2)$ where $0 \le \sigma_1 < \sigma_2 \le \infty$. The set Ω is nonempty, open and connected. Moreover, $\Omega = (\sigma_1, \varepsilon_0)$ for some $\varepsilon_0 \in (\tilde{\varepsilon}, \bar{\varepsilon})$.*

Proof. By Lemmas 3.6 (i) and 3.10, we have that

$$\Omega = \left\{ \begin{array}{l} \varepsilon \in (\sigma_1, \bar{\varepsilon}) : T_\varepsilon(\alpha) \text{ has exactly two critical points,} \\ \text{a local maximum and a local minimum, on } (0,\infty) \end{array} \right\}$$
$$= \left\{ \varepsilon \in (\sigma_1, \bar{\varepsilon}) : T'_\varepsilon(\alpha) < 0 \text{ for some } \alpha \in (0,\infty) \right\}. \tag{3.23}$$

(I) We show that Ω is open. Let $\varepsilon \in \Omega$. Then $T'_\varepsilon(\alpha_4) < 0$ for some $\alpha_4 \in (0,\infty)$. By Lemma 3.7, we observe that $T'_\zeta(\alpha_4) < 0$ for ζ belonging to some open neighborhood of ε. So Ω is open.

(II) We then show that Ω is nonempty and connected. First, we see that $(\sigma_1, \tilde{\varepsilon}] \subset \Omega$ by Lemma 3.6 (iii). It implies that Ω is nonempty. Suppose to the contrary that the set Ω is not connected, then there exist two numbers $\varepsilon_1 \notin \Omega$ and $\varepsilon_2 \in \Omega$ such that $\tilde{\varepsilon} < \varepsilon_1 < \varepsilon_2 < \bar{\varepsilon}$. So by (3.23), $T'_{\varepsilon_1}(\alpha) \geq 0$ on $(0,\infty)$. So by (F5) and Lemma 3.7, then

$$T'_{\varepsilon_2}(\alpha) > T'_{\varepsilon_1}(\alpha) \geq 0 \quad \text{for } 0 < \alpha < \omega_{\varepsilon_2} \leq \omega_{\varepsilon_1}. \tag{3.24}$$

Since $\varepsilon_2 \in \Omega$, we see that $T_{\varepsilon_2}(\alpha)$ has a local maximum at $\alpha_M(\varepsilon_2)$. So by Lemma 3.6 (ii), we further see that $T'_{\varepsilon_2}(\alpha_M(\varepsilon_2)) = 0$ and $\alpha_M(\varepsilon_2) < \kappa_{\varepsilon_2} < \omega_{\varepsilon_2}$. It is a contradiction by (3.24). So Ω is connect.

(III) Since Ω is open, connect and $(\sigma_1, \tilde{\varepsilon}] \subset \Omega$ and by Lemma 3.6 (i), there exists $\varepsilon_0 \in (\tilde{\varepsilon}, \bar{\varepsilon})$ such that $\Omega = (\sigma_1, \varepsilon_0)$.

The proof of Lemma 3.11 is complete. □

By Lemma 3.11, we see that, for $\varepsilon \in \Omega = (\sigma_1, \varepsilon_0)$, $T_\varepsilon(\alpha)$ has exactly two critical points, a local maximum at some $\alpha_M(\varepsilon)$ and a local minimum at some $\alpha_m(\varepsilon) > \alpha_M(\varepsilon)$ on Ω. So we have the following lemma.

Lemma 3.12. *Consider (1.1) with $\varepsilon \in \Omega$. Then the following assertions (i)–(ii) hold.*

(i) $\alpha_m(\varepsilon)$ is a continuous function on $(\sigma_1, \varepsilon_0)$. Furthermore,

$$\lim_{\varepsilon \to \varepsilon_0^-} \alpha_M(\varepsilon) = \lim_{\varepsilon \to \varepsilon_0^-} \alpha_m(\varepsilon) \equiv \alpha_0 \quad \textit{and} \quad T'_{\varepsilon_0}(\alpha_0) = 0. \tag{3.25}$$

(ii) Assume that ω_ε is a increasing function on $(\sigma_1, \tilde{\varepsilon}]$, and there exists a function $\beta_\varepsilon \in [\rho_\varepsilon, \kappa_\varepsilon]$ on $(\sigma_1, \tilde{\varepsilon})$ such that β_ε is decreasing on (σ_1, ε') and $(\varepsilon', \tilde{\varepsilon})$ for some $\varepsilon' \in (\sigma_1, \tilde{\varepsilon})$ respectively. Then $\alpha_M(\varepsilon)$ is a continuous function on $(\sigma_1, \varepsilon_0)$.

Proof. We divide this proof into next five steps.

Step 1. We prove that $\alpha_M(\varepsilon)$ is a increasing function on $[\tilde{\varepsilon}, \varepsilon_0)$, $\alpha_m(\varepsilon)$ is a decreasing function on $[\tilde{\varepsilon}, \varepsilon_0)$, and (3.25) holds. We first assert that

$$\theta(\alpha) - \theta(u) > 0 \quad \text{for } \alpha \geq \omega_\varepsilon \text{ and } \tilde{\varepsilon} \leq \varepsilon < \bar{\varepsilon}. \tag{3.26}$$

Assume that assertion (i) of Lemma 3.4 holds. It follows that (3.26) holds. Assume that assertion (ii) of Lemma 3.4 holds. Clearly, $\theta(\alpha) - \theta(u) > 0$ for $0 < u < \alpha \leq p_1(\varepsilon)$ and $\tilde{\varepsilon} \leq \varepsilon < \bar{\varepsilon}$. So by (3.3), we see that $T'_\varepsilon(\alpha) > 0$ for $0 < \alpha \leq p_1(\varepsilon)$ and $\tilde{\varepsilon} \leq \varepsilon < \bar{\varepsilon}$. So by (F3), (F5) and Lemma 3.6 (iii),

$$\omega_\varepsilon > \eta_\varepsilon = \begin{cases} \rho_\varepsilon > p_1(\varepsilon) & \text{for } \varepsilon = \tilde{\varepsilon}, \\ \kappa_\varepsilon > \gamma_\varepsilon > p_1(\varepsilon) & \text{for } \tilde{\varepsilon} < \varepsilon < \bar{\varepsilon}. \end{cases} \tag{3.27}$$

In addition, by (F6), we see that, for $\tilde{\varepsilon} \le \varepsilon < \bar{\varepsilon}$ and $0 < u < \omega_\varepsilon$,

$$\theta(\omega_\varepsilon) - \theta(u) = 2I_1(\varepsilon, \omega_\varepsilon, u) - I_2(\varepsilon, \omega_\varepsilon, u) > 0.$$

So by assertion (ii) of Lemma 3.4 and (3.27), we further see that (3.26) holds. By (3.3) and (3.26), $T'_\varepsilon(\alpha) > 0$ for $\alpha \ge \omega_\varepsilon$ and $\tilde{\varepsilon} \le \varepsilon < \bar{\varepsilon}$. It follows that

$$\alpha_M(\varepsilon) < \alpha_m(\varepsilon) < \omega_\varepsilon \quad \text{for } \tilde{\varepsilon} \le \varepsilon < \varepsilon_0. \tag{3.28}$$

Let $\varepsilon_1 < \varepsilon_2$ be given in $[\tilde{\varepsilon}, \varepsilon_0)$. By (F5) and (3.28), we see that

$$\alpha_M(\varepsilon_2) < \alpha_m(\varepsilon_2) < \omega_{\varepsilon_2} \le \omega_{\varepsilon_1}.$$

So by Lemma 3.7, we observe that

$$0 = T'_{\varepsilon_2}(\alpha_M(\varepsilon_2)) > T'_{\varepsilon_1}(\alpha_M(\varepsilon_2)) \;\text{ and }\; 0 = T'_{\varepsilon_2}(\alpha_m(\varepsilon_2)) > T'_{\varepsilon_1}(\alpha_m(\varepsilon_2)).$$

Then we obtain that

$$\alpha_M(\varepsilon_1) < \alpha_M(\varepsilon_2) < \alpha_m(\varepsilon_2) < \alpha_m(\varepsilon_1).$$

So $\alpha_M(\varepsilon)$ is a increasing function on $[\tilde{\varepsilon}, \varepsilon_0)$ and $\alpha_m(\varepsilon)$ is a decreasing function on $[\tilde{\varepsilon}, \varepsilon_0)$. Moreover, for $\tilde{\varepsilon} \le \varepsilon < \varepsilon_0$,

$$\alpha_M(\varepsilon) < \alpha^+ \equiv \lim_{\varepsilon \to \varepsilon_0^-} \alpha_M(\varepsilon) \le \alpha^- \equiv \lim_{\varepsilon \to \varepsilon_0^-} \alpha_m(\varepsilon) < \alpha_m(\varepsilon).$$

So $T'_\varepsilon(\alpha^+) < 0$ and $T'_\varepsilon(\alpha^-) < 0$ for $\tilde{\varepsilon} < \varepsilon < \varepsilon_0$. Then by Lemma 3.7, we further see that

$$0 \le T'_{\varepsilon_0}(\alpha^+) = \lim_{\varepsilon \to \varepsilon_0^-} T'_\varepsilon(\alpha^+) \le 0 \;\text{ and }\; 0 \le T'_{\varepsilon_0}(\alpha^-) = \lim_{\varepsilon \to \varepsilon_0^-} T'_\varepsilon(\alpha^-) \le 0.$$

So $T'_{\varepsilon_0}(\alpha^+) = T'_{\varepsilon_0}(\alpha^-) = 0$. By Lemmas 3.10 and 3.11, we have that $\alpha_0 \equiv \alpha^+ = \alpha^-$ and $T'_{\varepsilon_0}(\alpha_0) = 0$. It implies that (3.25) holds.

Step 2. We prove that

$$\alpha_m(\varepsilon) : [\tilde{\varepsilon}, \varepsilon_0) \longrightarrow (\alpha_0, \alpha_m(\tilde{\varepsilon})] \quad \text{is surjective,} \tag{3.29}$$

where α_0 is defined in Step 1. Let $\alpha_1 \in (\alpha_0, \alpha_m(\tilde{\varepsilon}))$. By Step 1, we see that

$$\alpha_M(\varepsilon_1) < \alpha_M(\varepsilon_2) < \alpha_m(\varepsilon_2) < \alpha_1 < \alpha_m(\varepsilon_1) \quad \text{for some } \varepsilon_1 < \varepsilon_2 \text{ in } (\tilde{\varepsilon}, \varepsilon_0).$$

It follows that $T'_{\varepsilon_1}(\alpha_1) < 0 < T'_{\varepsilon_2}(\alpha_1)$. So by Lemma 3.7, there exists $\varepsilon_3 \in (\varepsilon_1, \varepsilon_2) \subset (\tilde{\varepsilon}, \varepsilon_0)$ such that $T'_{\varepsilon_3}(\alpha_1) = 0$. By Lemma 3.11 and Step 1, we have $\alpha_m(\varepsilon_3) = \alpha_1$. It implies that (3.29) holds.

Step 3. We prove assertion (i). By Lemma 3.6 (iii), we see that $\alpha_m(\varepsilon) > \kappa_\varepsilon$ for $\sigma_1 < \varepsilon \le \tilde{\varepsilon}$. Since $T'_\varepsilon(\alpha_m(\varepsilon)) = 0$ for $\sigma_1 < \varepsilon \le \tilde{\varepsilon}$ and by (3.15), we observe that

$$T''_\varepsilon(\alpha_m(\varepsilon)) = T''_\varepsilon(\alpha_m(\varepsilon)) + \frac{2}{\alpha} T'_\varepsilon(\alpha_m(\varepsilon)) > 0 \quad \text{for } \sigma_1 < \varepsilon \le \tilde{\varepsilon}.$$

So by the Implicit Function Theorem and Lemma 3.11, we observe that

$$\alpha_m(\varepsilon) \text{ is a continuous function on } (\sigma_1, \tilde{\varepsilon}]. \tag{3.30}$$

In addition, by Step 1 and (3.29), we observe that

$$\alpha_m(\varepsilon) \text{ is a continuous function on } [\tilde{\varepsilon}, \varepsilon_0). \tag{3.31}$$

By (3.30) and (3.31), we obtain that $\alpha_m(\varepsilon)$ is a continuous function on $(\sigma_1, \varepsilon_0)$. So assertion (i) holds by Step 1.

Step 4. If ω_ε is a increasing function on $(\sigma_1, \tilde{\varepsilon}]$, we assert that

$$\alpha_M(\varepsilon) \text{ is a strictly increasing function on } (\sigma_1, \varepsilon_0). \tag{3.32}$$

By (F4) and Lemma 3.6 (iii), we see that $\alpha_M(\varepsilon) < \rho_\varepsilon = \eta_\varepsilon < \omega_\varepsilon$ for $\sigma_1 < \varepsilon \leq \tilde{\varepsilon}$. Let $\varepsilon_1 < \varepsilon_2$ be given in $(\sigma_1, \tilde{\varepsilon}]$. Then we have that $\alpha_M(\varepsilon_1) < \omega_{\varepsilon_1} \leq \omega_{\varepsilon_2}$. So by Lemma 3.7, we have that

$$T'_{\varepsilon_2}(\alpha_M(\varepsilon_1)) > T'_{\varepsilon_1}(\alpha_M(\varepsilon_1)) = 0,$$

which implies that $\alpha_M(\varepsilon_1) < \alpha_M(\varepsilon_2)$ or $\alpha_M(\varepsilon_1) > \alpha_m(\varepsilon_2)$. Assume that $\alpha_M(\varepsilon_1) > \alpha_m(\varepsilon_2)$. Since $\alpha_m(\varepsilon_2) > \alpha_M(\varepsilon_2)$, we observe that

$$\alpha_M(\varepsilon_2) < \alpha_M(\varepsilon_1) < \omega_{\varepsilon_1} \leq \omega_{\varepsilon_2}.$$

So by Lemma 3.7, we find that

$$T'_{\varepsilon_1}(\alpha_M(\varepsilon_2)) < T'_{\varepsilon_2}(\alpha_M(\varepsilon_2)) = 0 < T'_{\varepsilon_1}(\alpha_M(\varepsilon_2)),$$

which is a contradiction. Thus $\alpha_M(\varepsilon_1) < \alpha_M(\varepsilon_2)$. It implies that $\alpha_M(\varepsilon)$ is a strictly increasing function on $(\sigma_1, \tilde{\varepsilon}]$. By Step 1, we see that (3.32) holds.

Step 5. We prove that assertion (ii). We assert that

$$\alpha_M(\varepsilon) : (\sigma_1, \varepsilon_0) \longrightarrow \left(\lim_{\varepsilon \to \sigma_1^+} \alpha_M(\varepsilon), \alpha_0 \right) \text{ is surjective.} \tag{3.33}$$

So by (3.32), assertion (ii) holds. Next, we prove (3.33). Let $\alpha_2 \in \left(\lim_{\varepsilon \to \sigma_1^+} \alpha_M(\varepsilon), \alpha_0 \right)$ be given. We consider next three cases.

Case 1. $\alpha_2 = \alpha_M(\tilde{\varepsilon})$. Under Case 1, (3.33) holds immediately.

Case 2. $\alpha_M(\tilde{\varepsilon}) < \alpha_2 < \alpha_0$. Under Case 2, by Step 1 and (3.32), there exist $\varepsilon_- < \varepsilon_+$ in $(\tilde{\varepsilon}, \varepsilon_0)$ such that

$$\alpha_M(\varepsilon_-) < \alpha_2 < \alpha_M(\varepsilon_+) < \alpha_m(\varepsilon_+) < \alpha_m(\varepsilon_-).$$

It follows that $T'_{\varepsilon_-}(\alpha_2) < 0 < T'_{\varepsilon_+}(\alpha_2)$. So by Lemma 3.7, there exists $\varepsilon_1 \in (\varepsilon_-, \varepsilon_+) \subset (\tilde{\varepsilon}, \varepsilon_0)$ such that $T'_{\varepsilon_1}(\alpha_2) = 0$. Moreover, $\alpha_M(\varepsilon_1) = \alpha_2$ by Lemma 3.11 and Step 1. So (3.33) holds.

Case 3. $\lim_{\varepsilon \to \sigma_1^+} \alpha_M(\varepsilon) < \alpha_2 < \alpha_M(\tilde{\varepsilon})$. Under Case 3, we further consider next three subcases:

Case 3-1. $\alpha_2 = \alpha_M(\varepsilon')$. Under Case 3-1, clearly, (3.33) holds.

Case 3-2. $\alpha_2 < \alpha_M(\varepsilon')$. Under Case 3-2, by (3.32), there exists $\varepsilon_- \in (\sigma_1, \varepsilon')$ such that

$$\alpha_M(\varepsilon_-) < \alpha_2 < \alpha_M(\varepsilon') < \alpha_m(\varepsilon'). \tag{3.34}$$

Let ε be given in $[\varepsilon_-, \varepsilon')$. Since $\alpha_M(\varepsilon') < \rho_{\varepsilon'}$ by Lemmas 3.6 (iii) and 3.5, there exists $\delta > 0$ such that $\alpha_M(\varepsilon') < \rho_{\varepsilon'-\delta}$ and $\varepsilon < \varepsilon' - \delta$. So we further observe that

$$\alpha_M(\varepsilon') < \rho_{\varepsilon'-\delta} \leq \beta_{\varepsilon'-\delta} \leq \beta_\varepsilon \leq \kappa_\varepsilon < \alpha_m(\varepsilon).$$

So by (3.34), we see that $\alpha_M(\varepsilon_-) < \alpha_2 < \alpha_M(\varepsilon') < \alpha_m(\varepsilon)$ for $\varepsilon_- \leq \varepsilon \leq \varepsilon'$. Then we have that $T'_{\varepsilon_-}(\alpha_2) < 0 < T'_{\varepsilon'}(\alpha_2)$. It follows that $T'_{\varepsilon''}(\alpha_2) = 0$ for some $\varepsilon_1 \in (\varepsilon_-, \varepsilon')$. Furthermore, $\alpha_M(\varepsilon_1) = \alpha_2$. So (3.33) holds.

Case 3-3. $\alpha_2 > \alpha_M(\varepsilon')$. Under Case 3-1, similarly, there exists $\varepsilon_+ \in (\varepsilon', \tilde{\varepsilon})$ such that

$$\alpha_M(\varepsilon') < \alpha_2 < \alpha_M(\varepsilon_+) < \alpha_m(\varepsilon) \quad \text{for } \varepsilon' \leq \varepsilon \leq \varepsilon_+.$$

So by Lemma 3.7, there exists $\varepsilon_2 \in (\varepsilon', \varepsilon_+)$ such that $\alpha_M(\varepsilon_2) = \alpha_2$. It follows that (3.33) holds.

Thus by Cases 1–3, assertion (ii) holds.

The proof of Lemma 3.12 is complete. □

4 Proofs of main results

Proof of Theorem 2.1. To prove Theorem 2.1, by (3.2) and Lemma 3.1 (i), it suffices to prove assertions (M1)–(M3) in Section 3; see Fig. 3.1. Recall that:

(M1) For $\sigma_1 < \varepsilon < \varepsilon_0$, $T_\varepsilon(\alpha)$ has exactly two critical points, a local maximum at some α_M and a local minimum at some α_m ($> \alpha_M$), on $(0, \infty)$.

(M2) For $\varepsilon = \varepsilon_0$, $T'_{\varepsilon_0}(\alpha) > 0$ for $\alpha \in (0, \infty) \setminus \{\alpha_0\}$, and $T'_{\varepsilon_0}(\alpha_0) = 0$. In addition, $T''_{\varepsilon_0}(\alpha_0) = 0$ and $T'''_{\varepsilon_0}(\alpha_0) \neq 0$ if, for any fixed $u > 0$, $f'_\varepsilon(u)$ is continuously differentiable at $\varepsilon = \varepsilon_0$.

(M3) For $\varepsilon_0 < \varepsilon < \sigma_2$, $T'_\varepsilon(\alpha) > 0$ for $\alpha \in (0, \infty)$.

First, assertion (M1) immediately follows by Lemmas 3.11 and 3.1 (i).

Secondly, we prove assertion (M3). Obviously, assertion (M3) holds for $\bar{\varepsilon} \leq \varepsilon < \sigma_2$ by Lemma 3.6 (i). Assume that there exists $\varepsilon \in (\varepsilon_0, \bar{\varepsilon})$ such that $T_\varepsilon(\alpha)$ has a critical point α^* on $(0, \infty)$. Since $T'_\varepsilon(\alpha) \geq 0$ for $\alpha > 0$ by Lemma 3.11, we observe that $T'_\varepsilon(\alpha^*) = T''_\varepsilon(\alpha^*) = 0$. Since $\tilde{\varepsilon} < \varepsilon_0 < \bar{\varepsilon}$ and by Lemma 3.6 (ii) and (F5), we have that

$$0 < \alpha^* < \kappa_\varepsilon = \eta_\varepsilon < \omega_\varepsilon \leq \omega_{\varepsilon_0}.$$

In addition, by Lemma 3.7, we observe that $0 = T'_\varepsilon(\alpha^*) > T'_{\varepsilon_0}(\alpha^*) \geq 0$, which is a contradiction. So $T'_\varepsilon(\alpha) > 0$ for $\alpha \in (0, \infty)$ and $\varepsilon_0 < \varepsilon < \bar{\varepsilon}$. Thus assertion (M3) holds.

Finally, we prove assertion (M2). We have that $\lim_{\alpha \to 0^+} T_\varepsilon(\alpha) = 0$ and $\lim_{\alpha \to \infty} T_\varepsilon(\alpha) = \infty$ by Lemma 3.1 (i). By Lemmas 3.10–3.12, we see that

$$T'_{\varepsilon_0}(\alpha_0) = 0 \;\text{ and }\; T'_{\varepsilon_0}(\alpha) > 0 \;\text{ for } \alpha \in (0, \infty) \setminus \{\alpha_0\}. \tag{4.1}$$

Next, we assume that $f'_\varepsilon(u)$ is continuously differentiable at $\varepsilon = \varepsilon_0$ for any fixed $u > 0$. By (4.1), we obtain that $T''_{\varepsilon_0}(\alpha_0) = 0$. We then prove that $T'''_{\varepsilon_0}(\alpha_0) \neq 0$; we divide this proof into two steps.

Step 1. We prove that γ_ε is a continuous function at $\varepsilon = \varepsilon_0$. By (F1), there exist two sequences $\{\varepsilon_{1,n}\}_{n\in\mathbb{N}}$ and $\{\varepsilon_{2,n}\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \varepsilon_{1,n} = \lim_{n\to\infty} \varepsilon_{2,n} = \varepsilon_0$,

$$\liminf_{\varepsilon\to\varepsilon_0} \gamma_\varepsilon = \lim_{n\to\infty} \gamma_{\varepsilon_{1,n}} \quad \text{and} \quad \limsup_{\varepsilon\to\varepsilon_0} \gamma_\varepsilon = \lim_{n\to\infty} \gamma_{\varepsilon_{2,n}}.$$

Thus we observe that

$$f''_{\varepsilon_0}(\liminf_{\varepsilon\to\varepsilon_0} \gamma_\varepsilon) = f''_{\varepsilon_0}(\lim_{n\to\infty} \gamma_{\varepsilon_{1,n}}) = \lim_{n\to\infty} f''_{\varepsilon_{1,n}}(\gamma_{\varepsilon_{1,n}}) = 0,$$

$$f''_{\varepsilon_0}(\limsup_{\varepsilon\to\varepsilon_0} \gamma_\varepsilon) = f''_{\varepsilon_0}(\lim_{n\to\infty} \gamma_{\varepsilon_{2,n}}) = \lim_{n\to\infty} f''_{\varepsilon_{2,n}}(\gamma_{\varepsilon_{2,n}}) = 0.$$

So $f''_{\varepsilon_0}(\liminf_{\varepsilon\to\varepsilon_0} \gamma_\varepsilon) = f''_{\varepsilon_0}(\limsup_{\varepsilon\to\varepsilon_0} \gamma_\varepsilon) = 0$. By (F1), we see that

$$\gamma_{\varepsilon_0} = \liminf_{\varepsilon\to\varepsilon_0} \gamma_\varepsilon = \limsup_{\varepsilon\to\varepsilon_0} \gamma_\varepsilon = \lim_{\varepsilon\to\varepsilon_0} \gamma_\varepsilon,$$

which implies that γ_ε is a continuous function at $\varepsilon = \varepsilon_0$.

Step 2. We prove that $T'''_{\varepsilon_0}(\alpha_0) \neq 0$. Since $T'_{\varepsilon_0}(\alpha_0) = T''_{\varepsilon_0}(\alpha_0) = 0$ and by Lemma 3.6 (ii), we observe that $\alpha_0 < \kappa_{\varepsilon_0} = \eta_{\varepsilon_0}$. Assume that $\alpha_0 < \gamma_{\varepsilon_0}$. By continuity of γ_ε at $\varepsilon = \varepsilon_0$ and Step 1 in the proof of Lemma 3.12, we observe that

$$\alpha_M(\varepsilon) < \alpha_0 < \alpha_m(\varepsilon) < \gamma_\varepsilon \quad \text{for } \varepsilon \in (\tilde{\varepsilon}, \varepsilon_0) \text{ sufficiently close to } \varepsilon_0,$$

which is a contradiction by Lemma 3.1 (ii). Then we have that $\gamma_{\varepsilon_0} \leq \alpha_0 \leq \kappa_{\varepsilon_0}$. So by Lemma 3.8, we see that

$$\alpha_0 T'''_{\varepsilon_0}(\alpha_0) = T''_{\varepsilon_0}(\alpha_0) + \alpha_0 T'''_{\varepsilon_0}(\alpha_0) = \left[\alpha T''_{\varepsilon_0}(\alpha)\right]'\Big|_{\alpha=\alpha_0} > 0.$$

Thus $T'''_{\varepsilon_0}(\alpha_0) > 0$.

So by above, assertion (M2) holds.

The proof of Theorem 2.1 is complete. □

Proof of Theorem 2.2. For $\sigma_1 < \varepsilon < \varepsilon_0$, by Theorem 2.1 (i), we obtain that (1.1) has exactly one positive solution for $0 < \lambda < \lambda_*(\varepsilon)$ or $\lambda > \lambda^*(\varepsilon)$, exactly two positive solutions for $\lambda = \lambda_*(\varepsilon)$ or $\lambda = \lambda^*(\varepsilon)$, exactly three solutions for $\lambda_*(\varepsilon) < \lambda < \lambda^*(\varepsilon)$. While for $\varepsilon_0 \leq \varepsilon < \sigma_2$, by Theorem 2.1 (ii)–(iii), we obtain that (1.1) has exactly one positive solution for $\lambda > 0$. So by (3.1), we see that $\lambda^*(\varepsilon) \equiv T^2_\varepsilon(\alpha_M(\varepsilon))$ and $\lambda_*(\varepsilon) \equiv T^2_\varepsilon(\alpha_m(\varepsilon))$. By Lemma 3.12, we further see that $\lambda^*(\varepsilon)$ and $\lambda_*(\varepsilon)$ are continuous functions on $(\sigma_1, \varepsilon_0]$, and

$$\lim_{\varepsilon\to\varepsilon_0^-} \lambda^*(\varepsilon) = \lim_{\varepsilon\to\varepsilon_0^-} \lambda_*(\varepsilon) = [T_{\varepsilon_0}(\alpha_0)]^2 = \lambda_0.$$

Let $\varepsilon_1 < \varepsilon_2$ be two given numbers in $(\sigma_1, \varepsilon_0)$. By (F2), (3.32) and Lemma 3.11, we observe that $\alpha_M(\varepsilon_1) < \alpha_M(\varepsilon_2)$ and

$$\sqrt{\lambda^*(\varepsilon_1)} = T_{\varepsilon_1}(\alpha_M(\varepsilon_1)) < T_{\varepsilon_2}(\alpha_M(\varepsilon_1)) < T_{\varepsilon_2}(\alpha_M(\varepsilon_2)) = \sqrt{\lambda^*(\varepsilon_2)}.$$

So $\lambda^*(\varepsilon)$ is a strictly increasing function on $(\sigma_1, \varepsilon_0]$. Suppose to the contrary that $\lambda_*(\varepsilon_1) \geq \lambda_*(\varepsilon_2)$. Then by (F2) and (3.1),

$$T_{\varepsilon_1}(\alpha_m(\varepsilon_1)) = \sqrt{\lambda_*(\varepsilon_1)} \geq \sqrt{\lambda_*(\varepsilon_2)} = T_{\varepsilon_2}(\alpha_m(\varepsilon_2)) > T_{\varepsilon_1}(\alpha_m(\varepsilon_2)).$$

It follows that

$$\alpha_M(\varepsilon_2) < \alpha_m(\varepsilon_2) < \alpha_M(\varepsilon_1) < \alpha_m(\varepsilon_1),$$

which is a contradiction by (3.32). Thus, $\lambda_*(\varepsilon_2) > \lambda_*(\varepsilon_1)$. So $\lambda_*(\varepsilon)$ is a strictly decreasing function on $(\sigma_1, \varepsilon_0]$. Moreover,

$$0 \leq \lim_{\varepsilon \to \sigma_1^+} \lambda_*(\varepsilon) = \lim_{\varepsilon \to \sigma_1^+} [T_\varepsilon(\alpha_m(\varepsilon))]^2 \leq \lim_{\varepsilon \to \sigma_1^+} [T_\varepsilon(\alpha_M(\varepsilon))]^2 = \lim_{\varepsilon \to \sigma_1^+} \lambda^*(\varepsilon) < \lambda_0.$$

Finally, let us assume that $\lim_{\varepsilon \to \sigma_1^+} \rho_\varepsilon < \lim_{\varepsilon \to \sigma_1^+} \omega_\varepsilon$. We suppose to the contrary that $\lim_{\varepsilon \to \sigma_1^+} \lambda_*(\varepsilon) = \lim_{\varepsilon \to \sigma_1^+} \lambda^*(\varepsilon)$. Let $\varepsilon_3 \in (\sigma_1, \varepsilon')$ be fixed. By Lemma 3.6 (iii) and (3.32), we have that, for $\sigma_1 < \varepsilon < \varepsilon_3$,

$$\alpha^+ \equiv \lim_{\varepsilon \to \sigma_1^+} \alpha_M(\varepsilon) < \alpha_M(\varepsilon) < \alpha_M(\varepsilon_3) < \rho_{\varepsilon_3} \leq \beta_{\varepsilon_3} \leq \beta_\varepsilon \leq \kappa_\varepsilon < \alpha_m(\varepsilon). \tag{4.2}$$

In addition, we have that

$$\alpha^+ \leq \lim_{\varepsilon \to \sigma_1^+} \rho_\varepsilon < \lim_{\varepsilon \to \sigma_1^+} \omega_\varepsilon. \tag{4.3}$$

Let $\alpha \in I \equiv \left(\alpha^+, \min\left\{\lim_{\varepsilon \to \sigma_1^+} \omega_\varepsilon, \beta_{\varepsilon_3}\right\}\right)$. So by (3.32), (4.2) and (4.3), there exists $\varepsilon_4 \in (\sigma_1, \varepsilon_3)$ such that

$$\alpha_M(\varepsilon) < \alpha < \beta_{\varepsilon_3} < \alpha_m(\varepsilon) \quad \text{for } \sigma_1 < \varepsilon < \varepsilon_4. \tag{4.4}$$

So we see that, for $\sigma_1 < \varepsilon < \varepsilon_4$,

$$\lambda_*(\varepsilon) = [T_\varepsilon(\alpha_m(\varepsilon))]^2 < [T_\varepsilon(\alpha)]^2 < [T_\varepsilon(\alpha_M(\varepsilon))]^2 = \lambda^*(\varepsilon).$$

It follows that

$$\lim_{\varepsilon \to \sigma_1^+} \lambda_*(\varepsilon) = \lim_{\varepsilon \to \sigma_1^+} \lambda^*(\varepsilon) = \lim_{\varepsilon \to \sigma_1^+} [T_\varepsilon(\alpha)]^2.$$

Since α is arbitrary, we observe that $\lim_{\varepsilon \to \sigma_1^+} [T_\varepsilon(\alpha)]^2$ is constant for $\alpha \in I$, which implies that $\lim_{\varepsilon \to \sigma_1^+} T'_\varepsilon(\alpha) = 0$ for $\alpha \in I$. Furthermore, by Lemma 3.7 and (4.4),

$$0 = \lim_{\varepsilon \to \sigma_1^+} T'_\varepsilon(\alpha) < T'_\varepsilon(\alpha) < 0 \quad \text{for } \alpha \in I \text{ and } \sigma_1 < \varepsilon < \varepsilon_4,$$

which is a contradiction. Thus, $\lim_{\varepsilon \to \sigma_1^+} \lambda_*(\varepsilon) < \lim_{\varepsilon \to \sigma_1^+} \lambda^*(\varepsilon)$ if $\lim_{\varepsilon \to \sigma_1^+} \rho_\varepsilon < \lim_{\varepsilon \to \sigma_1^+} \omega_\varepsilon$.

The proof of Theorem 2.2 is complete. □

Proof of Theorem 2.3. First, for $\varepsilon \geq \bar{\varepsilon} = 0.25$, it is easy to show that the bifurcation curve S_ε of (1.5) is monotone increasing on the $(\lambda, \|u\|_\infty)$-plane and all positive solutions of (1.5) are nondegenerate, see [3, p. 482] and Fig. 1.1 (iii). Hence Theorem 2.3 holds for $\varepsilon \geq 0.25$.

Next, we prove Theorem 2.3 for $0 < \varepsilon \leq \bar{\varepsilon} = 0.25$ by applying Theorem 2.1. That is, we prove that

$$f_\varepsilon(u) = \exp\left(\frac{u}{1 + \varepsilon u}\right) \in C^3[0, \infty)$$

satisfies (F1)–(F6), and for any fixed $u > 0$, $f'_\varepsilon(u)$ is a continuously differentiable function of $\varepsilon \in (\tilde{\varepsilon}, \bar{\varepsilon})$. In this case, for (1.5) with $0 < \varepsilon \leq \bar{\varepsilon} = 0.25$, we take that

$$\varepsilon \in \Theta = (\sigma_1, \sigma_2) = (0, 0.3), \qquad 0 = \sigma_1 < \tilde{\varepsilon} = \frac{1}{\tilde{a}}\ (\approx 0.243) < \bar{\varepsilon} = 0.25 < 0.3 = \sigma_2,$$

where

$$\tilde{a} \equiv \inf\left\{ a > 4 : \int_0^{\frac{a(a-2)+a\sqrt{a(a-4)}}{2}} u g_a(u) - u^2 g'_a(u) du < 0 \right\} \approx 4.107,$$

and $g_a(u) \equiv f_{\varepsilon=1/a}(u) = \exp\left(\frac{au}{a+u}\right)$, cf. [7, (1.4)]. Clearly, for any fixed $\varepsilon \in \Theta = (0, 0.3)$, $f_\varepsilon(u) = \exp\left(\frac{u}{1+\varepsilon u}\right) \in C^3[0, \infty)$, $f_\varepsilon(0) = 1 > 0$, $f_\varepsilon(u) > 0$ on $(0, \infty)$, and $f'_\varepsilon(u)$ is a continuously differentiable, strictly decreasing function of $\varepsilon \in \Theta = (0, 0.3)$. We then compute and find that, for $\varepsilon \in \Theta = (0, 0.3)$,

$$f''_\varepsilon(u) = -\frac{2\varepsilon^2 (u - \gamma_\varepsilon)}{(1+\varepsilon u)^4} \exp\left(\frac{u}{1+\varepsilon u}\right) \begin{cases} > 0 & \text{for } 0 < u < \gamma_\varepsilon, \\ = 0 & \text{for } u = \gamma_\varepsilon = \frac{1-2\varepsilon}{2\varepsilon^2} > 0, \\ < 0 & \text{for } u > \gamma_\varepsilon, \end{cases}$$

$$\lim_{u \to \infty} \frac{f_\varepsilon(u)}{u} = \lim_{u \to \infty} \frac{\exp\left(\frac{u}{1+\varepsilon u}\right)}{u} = 0.$$

So $f_\varepsilon(u)$ satisfies (F1) and (F2) with any fixed $\varepsilon \in \Theta = (0, 0.3)$.

We then prove that $f_\varepsilon(u)$ satisfies (F3)–(F6) with $0 = \sigma_1 < \tilde{\varepsilon}\ (\approx 0.243) < \bar{\varepsilon} = 0.25 < 0.3 = \sigma_2$. It is easy to see that, for fixed $a = 1/\varepsilon$, $g_a(u) = \exp(\frac{au}{a+u}) \in C^3[0, \infty)$ and

$$g''_a(u) = f''_{\frac{1}{a}}(u) \begin{cases} > 0 & \text{for } 0 < u < \hat{\gamma}_a, \\ = 0 & \text{for } u = \hat{\gamma}_a \equiv \frac{a(a-2)}{2} > 0, \\ < 0 & \text{for } u > \hat{\gamma}_a. \end{cases}$$

Huang and Wang [7,8] proved the following assertions (I)–(VII):

(I) $g_a(\hat{\gamma}_a) - \hat{\gamma}_a g'_a(\hat{\gamma}_a) \geq 0$ for $2 < a \leq 4$. (So (F3) (i) holds with $0.25 = \bar{\varepsilon} \leq \varepsilon < \sigma_2 = 0.3$.)

(II) For $a > 4$, the function $\int_0^u t^3 g''_a(t) dt$ has a positive zero $\hat{\kappa}_a$ in $(0, \infty)$. (So (F3) (ii) holds with $0 = \sigma_1 < \varepsilon \leq \bar{\varepsilon} = 0.25$.)

(III) For $a \geq \tilde{a} \approx 4.107$, there exists $\hat{\rho}_a \in (0, \hat{\kappa}_a]$ such that

$$\int_0^u t g_a(t) - t^2 g'_a(t) dt \begin{cases} = 0 & \text{if } u = \hat{\rho}_a, \\ < 0 & \text{if } \hat{\rho}_a < u \leq \hat{\kappa}_a. \end{cases}$$

(So (F3) (iii) holds with $0 = \sigma_1 < \varepsilon \leq \tilde{\varepsilon} = 1/\tilde{a} \approx 0.243$.)

(IV) There exists $a^*\ (\approx 4.166) \in (\tilde{a}, \infty)$ such that

$$\hat{\eta}_a \begin{cases} > \hat{\gamma}_a & \text{for } 4 < a < a^*, \\ \leq \hat{\gamma}_a & \text{for } a \geq a^*, \end{cases} \qquad \text{where } \hat{\eta}_a \equiv \begin{cases} \hat{\rho}_a & \text{if } a \geq \tilde{a}, \\ \hat{\kappa}_a & \text{if } 4 < a < \tilde{a}. \end{cases}$$

(V) $K(\frac{1}{a}, u, v) > 0$ for $u \in [\hat{\gamma}_a, \hat{\eta}_a]$, $0 < v < u$ and $4 < a < a^* \approx 4.166$.

(VI) For $a > 4$, we have that $\hat{\omega}_a > \hat{\eta}_a$ and

$$N(v,u) \equiv 3\left[\frac{\partial}{\partial\varepsilon} I_1\left(\frac{1}{a},u,v\right)\right] I_2\left(\frac{1}{a},u,v\right) - 2\left[\frac{\partial}{\partial\varepsilon} I_1\left(\frac{1}{a},u,v\right)\right] I_1\left(\frac{1}{a},u,v\right)$$
$$-2\left[\frac{\partial}{\partial\varepsilon} I_2\left(\frac{1}{a},u,v\right)\right] I_1\left(\frac{1}{a},u,v\right) > 0 \quad \text{for } 0 < v < u < \hat{\omega}_a,$$

where

$$\hat{\omega}_a \equiv \begin{cases} 12 & \text{if } 4 < a < 6, \\ 3 & \text{if } a \geq 6. \end{cases}$$

(VII) $\hat{\theta}(12) - \hat{\theta}(u) > 0$ for $0 < u < 12$ and $4 < a \leq \tilde{a} \approx 4.107$, where

$$\hat{\theta}(u) \equiv 2\int_0^u g_a(t)dt - ug_a(u) \quad \text{for } u \geq 0.$$

Notice that assertions (I)–(III) follow by [8, p. 771 and Lemma 13], assertion (IV) follows by [8, (4) and (28)–(31)], assertion (V) follows by [7, Lemma 2.6 and (2.32)] and [8, (28)–(31)], assertion (VI) follows by [8, Lemma 21 and its proof], and assertion (VII) follows by [8, Lemma 12(i)].

By assertions (I)–(III), we observe that $f_\varepsilon(u)$ satisfies (F3).

By assertions (IV) and (V), we observe that, if $a > 4$ and $\hat{\eta}_a > \hat{\gamma}_a$, then $K(\frac{1}{a},u,v) > 0$ for $u \in [\hat{\gamma}_a, \hat{\eta}_a]$, $0 < v < u$. It follows that $f_\varepsilon(u)$ satisfies (F4) with $m = 0$ for $0 = \sigma_1 < \varepsilon < \bar{\varepsilon} = 0.25$.

By assertion (VI), we see that $\hat{\omega}_a$ is a monotone decreasing function of a on $(4, \tilde{a})$. Let

$$\omega_\varepsilon \equiv \hat{\omega}_{\frac{1}{\varepsilon}} = \begin{cases} 3 & \text{if } 0 < \varepsilon \leq \frac{1}{6}, \\ 12 & \text{if } \frac{1}{6} < \varepsilon < \frac{1}{4} = 0.25 = \bar{\varepsilon}. \end{cases} \tag{4.5}$$

So by assertion (VI) again, $f_\varepsilon(u)$ satisfies (F5) for $0 = \sigma_1 < \varepsilon \leq \bar{\varepsilon} = 0.25$.

Since $\tilde{a}\,(\approx 4.107) < 6$ and by assertions (VI) and (VII), we see that, for $0 < u < \hat{\omega}_a$ and $4 < a \leq \tilde{a}$,

$$2I_1\left(\frac{1}{a},\hat{\omega}_a,u\right) - I_2\left(\frac{1}{a},\hat{\omega}_a,u\right) = 2I_1\left(\frac{1}{a},12,u\right) - I_2\left(\frac{1}{a},12,u\right) = \hat{\theta}(12) - \hat{\theta}(u) > 0,$$

which implies that $f_\varepsilon(u)$ satisfies (F6) for $0.243 \approx \tilde{\varepsilon} \leq \varepsilon < \bar{\varepsilon} = 0.25$.

By above and Theorem 2.1, we obtain that Theorem 2.3 holds for $0 < \varepsilon \leq \bar{\varepsilon} = 0.25$.

The proof of Theorem 2.3 is complete. □

Proof of Theorem 2.4. In the proof of Theorem 2.3, we have verified that $f_\varepsilon(u) = \exp\left(\frac{u}{1+\varepsilon u}\right)$ satisfies (F1)–(F6) for $0 < \varepsilon \leq 0.25$. By (4.5), ω_ε is monotone increasing for $0 = \sigma_1 < \varepsilon \leq \tilde{\varepsilon}$. Let

$$\hat{\beta}_a \equiv \begin{cases} \hat{\kappa}_a & \text{for } \tilde{a} < a \leq a^*, \\ \hat{\gamma}_a = \frac{a(a-2)}{2} & \text{for } a > a^*, \end{cases}$$

where $a^*\,(\approx 4.166)$ is defined by [8, (4)]. By [8, Lemma 13(i)], we see that $\hat{\beta}_a$ is a strictly increasing function on $(\tilde{a}, a^*)$ and (a^*, ∞), respectively. By [8, (30) and (31)], we find that $\hat{\rho}_a \leq \hat{\beta}_a \leq \hat{\kappa}_a$ for $a > \tilde{a}$. Let $\beta_\varepsilon = \hat{\beta}_{1/\varepsilon}$ and $\varepsilon' = 1/a^*$. Then $\beta_\varepsilon \in [\rho_\varepsilon, \kappa_\varepsilon]$ is a strictly decreasing function on $(0, \varepsilon')$ and $(\varepsilon', \tilde{\varepsilon})$, respectively. Clearly, we compute that

$$\lim_{\varepsilon \to 0^+} H_\varepsilon(u) = \left(-u^2 + 3u - 3\right) e^u + 3 \quad \text{for } u > 0.$$

We observe that $\lim_{\varepsilon\to 0^+} H_\varepsilon(0) = 0$, $\lim_{\varepsilon\to 0^+} H_\varepsilon(2) = -e^2 + 3\,(\approx -4.38) < 0$, and $\lim_{\varepsilon\to 0^+} H_\varepsilon(u)$ is strictly increasing on $(0,1)$ and then strictly decreasing on $(1,\infty)$. Thus $\lim_{\varepsilon\to 0^+} H_\varepsilon(u)$ has a unique positive zero which is less than 2. So by (4.5), we obtain that

$$\lim_{\varepsilon\to 0^+} \rho_\varepsilon < 2 < 3 = \lim_{\varepsilon\to 0^+} \omega_\varepsilon.$$

So by Theorem 2.2 and [8, (10)], we see that all results of Theorem 2.4 hold. □

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