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On the existence, uniqueness and regularity of solutions for a class of micropolar fluids with shear dependent viscosities

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Abstract. In this paper we consider a model describing the motion of a class of micropolar fluids with shear-dependent viscosities in a smooth domain $\Omega \subset \mathbb{R}^2$. Under the conditions that the external force and vortex viscosity μ_r are small in a suitable sense, we proved the existence and uniqueness of regularized solutions for the problem by using the iterative method.

Keywords: existence and uniqueness, regularity, shear dependent viscosity, micropolar fluids.

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1 Introduction and main result

The objective of the present work is to study the existence and uniqueness of strong solutions of a system associated to the steady equations for the motion of incompressible micropolar fluids with shear dependent viscosities in a bounded domain $\Omega \subset \mathbb{R}^2$ having a smooth boundary. More precisely, we will study the following system

$$\begin{cases} (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \operatorname{div}\,[(1+|\mathcal{D}\boldsymbol{u}|)^{p-2}\mathcal{D}\boldsymbol{u}] + \nabla\eta = \mu_r \operatorname{rot}\boldsymbol{w} + \boldsymbol{f}, & \text{in } \Omega,\\ \operatorname{div}\,\boldsymbol{u} = 0, & \text{in } \Omega,\\ (\boldsymbol{u}\cdot\nabla)\boldsymbol{w} - \mu_1\Delta\boldsymbol{w} - \mu_2\nabla\operatorname{div}\boldsymbol{w} + \mu_r\boldsymbol{w} = \mu_r\operatorname{rot}\boldsymbol{u} + \boldsymbol{g}, & \text{in } \Omega, \end{cases} \tag{1.1}$$

together with the boundary conditions

$$\boldsymbol{u}|_{\partial\Omega} = 0, \qquad \boldsymbol{w}|_{\partial\Omega} = 0, \tag{1.2}$$

where $\mathcal{D}\boldsymbol{u} = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)$, $p \in (1,2)$. The vector-valued functions $\boldsymbol{u} = (u_1, u_2, u_3), \boldsymbol{w} = (w_1, w_2, w_3)$ and the scalar function η denote respectively, the velocity, the angular velocity of rotation of particles and the pressure of the fluid. The vector-valued functions $\boldsymbol{f}$ and $\boldsymbol{g}$ denote

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respectively, the external sources of linear and angular momentum. The positive constants μ_1 and μ_2 are the spin viscosities, μ_r is the vortex viscosity. For simplicity, in this paper, we take $\mu_1 = \mu_2 = 1$.

The micropolar fluid model, firstly introduced by Eringen in [7], is a substantial generalization of the classic Navier–Stokes equations in the sense that the microstructure of the fluid particles is taken into account. Physically, micropolar fluids represent fluids consisting of rigid randomly oriented (or spherical) particles suspended in a viscous medium, see e.g. [4,7,17]. We note that micropolar fluids enables us to consider some physical phenomena that can not be treated by the classical Navier–Stokes equations for the viscous incompressible fluids such as suspensions, lubricants, blood motion in animals and liquid crystals.

If the exponent index $p = 2$, then (1.1)–(1.2) reduces the the classical micropolar fluid system and there are many results on the existence and uniqueness of solutions for it. For example, the existence of weak solutions in any connected open set $S \subseteq \mathbb{R}^d$ (cf. [24], in any bounded domain $\Omega \subset \mathbb{R}^d$) and strong solutions in any bounded domain $\Omega \subset \mathbb{R}^d$ established by Galdi and Rionero [13], and Yamaguchi [27], respectively. For the same problem, Łukaszewicz [16] proved the existence and uniqueness of strong solutions in 1989, and, in 1990, established the global existence of weak solutions for arbitrary initial data $(\boldsymbol{u}_0, \boldsymbol{w}_0) \in L^2_\sigma \times L^2$ (see [17]). Using a spectral Galerkin method, Rojas-Medar [23] proved the local existence and uniqueness of strong solutions. Ortega-Torres and Rojas-Medar proved the global existence of a strong solution by assuming small initial data, (see [20]). Linearization and successive approximations have been considered in [3,21] to give sufficient conditions on the kinematics pressure in order to obtain regularity and uniqueness of the weak solutions to the micropolar fluid equations. Recently, Loayza and Rojas-Medar [18] investigated regularity criteria for weak solutions of the micropolar fluid equations in a bounded three-dimensional domain. For more details, one can also refer [9–11,15,17,26] and the reference cited therein.

The case of the exponent $p \neq 2$ (i.e. the non-Newtonian micropolar fluid or called the micropolar fluid with shear dependent viscosities) is less studied. Araújo et al. [1] proved the existence of weak solutions by using Galerkin and compactness arguments. Uniqueness and periodicity of solutions are also considered. In [2], the author studied the long time behavior of the two-dimensional flow for non-Newtonian micropolar fluids in bounded smooth domains, in the sense of pullback attractors. They proved the existence and upper semicontinuity of the pullback attractors with respect to the viscosity coefficient of the model.

In the present work, as we said previously, we are interested in the flow of micropolar fluids with shear-dependent viscosities in a smooth domain $\Omega \subseteq R^2$. Under the conditions that the external sources $\boldsymbol{f}, \boldsymbol{g}$ and the vortex viscosity μ_r are small in a suitable sense, we proved the existence and uniqueness of regularized solutions for the problem by using the iterative method.

Throughout the paper, as usual, we denote by $\mathcal{V} = \{v \in C_0^\infty(\Omega);\ \operatorname{div} v = 0\}$ and the spaces

$$V_q(\Omega) := \text{the completion of } \mathcal{V} \text{ in the } W^{1,q}\text{-norm},$$

for $q = 2$ we simply write $V(\Omega)$. We also denote by $(C^{m,\gamma}(\bar{\Omega}),\ \|\cdot\|_{C^{m,\gamma}}), m$ nonnegative integer and $\gamma \in (0,1)$, the Hölder space with order m. By $W^{-1,q'}(\Omega),\ q' = \frac{q}{q-1}$, the strong dual of $W_0^{1,q}(\Omega)$ with norm $\|\cdot\|_{-1,q}$.

Next, we introduce the notions of solutions to (1.1)–(1.2).

Definition 1.1. Assume that $\boldsymbol{f} \in L^2(\Omega)$, $\boldsymbol{g} \in L^2(\Omega)$. We say that $(\boldsymbol{u}, \boldsymbol{w})$ is a pair of $C^{1,\gamma}(\bar{\Omega}) \times W^{1,2}(\Omega)$-solution of problem (1.1)–(1.2). If $\boldsymbol{u} \in C^{1,\gamma}(\bar{\Omega})$, for some $\gamma \in (0,1)$, $\operatorname{div} \boldsymbol{u} = 0$,

$\boldsymbol{u}|_{\partial\Omega} = 0$, $\boldsymbol{w} \in W_0^{1,2}(\Omega)$ and it satisfies the following integral identity for $\forall\varphi \in V_q(\Omega)$ and $\forall\psi \in C_0^\infty(\Omega)$

$$\int_\Omega (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\varphi dx + \int_\Omega (1+|\mathcal{D}\boldsymbol{u}|)^{p-2}\mathcal{D}\boldsymbol{u}\mathcal{D}\varphi dx = \int_\Omega \mu_r \operatorname{rot}\boldsymbol{w}\varphi dx + \int_\Omega \boldsymbol{f}\varphi dx, \tag{1.3}$$

$$\int_\Omega (\boldsymbol{u}\cdot\nabla)\boldsymbol{w}\psi dx + \int_\Omega \nabla\boldsymbol{w}\nabla\psi dx + \int_\Omega \operatorname{div}\boldsymbol{w}\operatorname{div}\psi dx + \int_\Omega \mu_r\boldsymbol{w}\psi dx = \int_\Omega \mu_r \operatorname{rot}\boldsymbol{u}\psi dx + \int_\Omega \boldsymbol{g}\psi dx. \tag{1.4}$$

Remark 1.2. We observe that if $\boldsymbol{u}$ satisfies (1.3) then we can apply the theorem of de Rham (see [25, Lemma 2.2.1]) to find a pressure η at least in $L^2(\Omega)$ such that the pair $(\boldsymbol{u},\eta)$ satisfies the following integral identity for $\forall\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\varphi dx + \int_\Omega (1+|\mathcal{D}\boldsymbol{u}|)^{p-2}\mathcal{D}\boldsymbol{u}\mathcal{D}\varphi dx - \int_\Omega \eta\nabla\cdot\varphi dx = \int_\Omega \mu_r rot\boldsymbol{w}\varphi dx + \int_\Omega \boldsymbol{f}\varphi dx.$$

The validity of the reverse implication is obvious.In the sequel we shall refer to $(\boldsymbol{u},\boldsymbol{w})$ or $(\boldsymbol{u},\eta,\boldsymbol{w})$ as solution of system (1.1)–(1.2) without distinction.

Our aim is to prove the following theorems.

Theorem 1.3. *Assume that $p \in (1,2), q > 2$,and let $\gamma_0 = 1 - \frac{2}{q}$. Let Ω be a domain of class C^2, and let be $\boldsymbol{f} \in L^q(\Omega), \boldsymbol{g} \in L^2(\Omega)$. If $\|\boldsymbol{f}\|_q \le \delta_1$, $\|\boldsymbol{g}\|_2 \le \delta_2$, $\mu_r < \delta_3$ where δ_1, δ_2, δ_3 are positive constants small in a suitable sense (see (3.2), (3.13), (3.16)), then there exist a unique $C^{1,\gamma}(\bar\Omega) \times W^{1,2}(\Omega)$-solution $(\boldsymbol{u},\eta,\boldsymbol{w})$ of problem (1.1)–(1.2) such that*

$$\boldsymbol{u} \in C^{1,\gamma}(\bar\Omega), \;\; \eta \in C^{0,\gamma}(\bar\Omega), \;\; \boldsymbol{w} \in W^{2,2}(\Omega), \qquad \forall\gamma < \gamma_0,$$

and

$$\|\boldsymbol{u}\|_{C^{1,\gamma}} + \|\eta\|_{C^{0,\gamma}} + \|\boldsymbol{w}\|_{2,2} \le 2(\tilde{c_0}\|\boldsymbol{f}\|_q + c_0\|\boldsymbol{g}\|_2),$$

where $\tilde{c_0}, c_0$ are positive constants.

Theorem 1.4. *In addition to the assumptions of Theorem 1.3, if $q > 4$ and $\|\boldsymbol{f}\|_q$, $\|\boldsymbol{g}\|_2$, μ_r are sufficiently small (see (4.4)), then there exists a solution $(\boldsymbol{u},\eta,\boldsymbol{w})$ of problem (1.1)–(1.2) such that*

$$\boldsymbol{u} \in W^{2,2}(\Omega) \cap C^{1,\gamma}(\bar\Omega), \quad \eta \in W^{1,2} \cap C^{0,\gamma}(\bar\Omega), \quad \boldsymbol{w} \in W^{2,2}(\Omega), \qquad \forall\gamma < \gamma_0.$$

The present work is organized as follows: in Section 2 we state preliminaries results that will be used later in the paper. Section 3 is dedicated to give the proof of Theorem 1.3. More precisely, in Section 3.1 we construct approximate solutions to the original nonlinear problem by iterate scheme, then derive the uniform estimate for such approximate solutions. The results are used in Section 3.2 to prove the convergence of the solutions. The existence and uniqueness results are proved in Section 3.3 and Section 3.4 respectively. Finally, in Section 4, we prove the regularity result (Theorem 1.4).

2 Preliminary lemmas

In this section, we recall the following useful results.

Lemma 2.1 ([22]). *For any $q \geq 1$, there exists a constant c_1 such that*

$$\|v\|_q + \|\nabla v\|_q \leq c_1 \|\mathcal{D}v\|_q, \quad \text{for each } v \in V_q(\Omega).$$

Hence the two quantities above are equivalent norms in $V_q(\Omega)$.

Lemma 2.2 ([19]). *If a distribution g is such that $\nabla g \in W^{-1,q}(\Omega)$, then $g \in L^q(\Omega)$ and*

$$\|g\|_{L^q_\sharp} \leq C\|\nabla g\|_{-1,q},$$

where $L^q_\sharp = L^q/\mathbb{R}$.

Lemma 2.3 ([5]). *For any given real numbers $\xi, \eta \geq 0$ and $1 < p < 2$ the following inequality holds true:*

$$\left| \frac{1}{(1+\xi)^{2-p}} - \frac{1}{(1+\eta)^{2-p}} \right| \leq (2-p)|\xi - \eta|.$$

Lemma 2.4 ([8]). *For an arbitrary tensor D, define $S(D) \equiv (1+|D|)^{p-2}D$, $1 < p < 2$. Then there exist a constant C such that, for any pair of tensors D_1 and D_2,*

$$(S(D_1) - S(D_2)) \cdot (D_1 - D_2) \geq C \frac{|D_1 - D_2|^2}{(1+|D_1|+|D_2|)^{2-p}}.$$

3 The proof of Theorem 1.3

As already stated, in order to prove Theorem 1.3 we use the method of successive approximations.

3.1 Approximating linear problems

We construct approximate solutions, inductively, as follows:

(i) first define $\boldsymbol{u}^{-1} = \boldsymbol{w}^{-1} = 0$, and

(ii) assuming that $(\boldsymbol{u}^{m-1}, \boldsymbol{w}^{m-1})$ was defined for $m \geq 1$, let $\boldsymbol{u}^m, \boldsymbol{w}^m$ be the unique solution to the following boundary problems:

$$\begin{cases} -\text{div}\,[(1+|\mathcal{D}\boldsymbol{u}^{m-1}|)^{p-2}\mathcal{D}\boldsymbol{u}^m] + \nabla\eta^m = \mu_r \,\text{rot}\, \boldsymbol{w}^{m-1} + \boldsymbol{f} - (\boldsymbol{u}^{m-1}\cdot\nabla)\boldsymbol{u}^{m-1}, & \text{in } \Omega, \\ \text{div}\, \boldsymbol{u}^m = 0, & \text{in } \Omega, \\ -\Delta \boldsymbol{w}^m - \nabla\,\text{div}\,\boldsymbol{w}^m = \mu_r \,\text{rot}\, \boldsymbol{u}^{m-1} - \mu_r \boldsymbol{w}^{m-1} + \boldsymbol{g} - (\boldsymbol{u}^{m-1}\cdot\nabla)\boldsymbol{w}^{m-1}, & \text{in } \Omega, \\ \boldsymbol{u}^m|_{\partial\Omega} = 0, \quad \boldsymbol{w}^m|_{\partial\Omega} = 0. \end{cases} \tag{3.1}$$

The following result holds true.

Proposition 3.1. *Assume that $p \in (1,2)$, $q > 2$ and let $\gamma_0 = 1 - \frac{2}{q}$. Let Ω be a domain of class C^2, and let be $\boldsymbol{f} \in L^q(\Omega)$, $\boldsymbol{g} \in L^2(\Omega)$. Then, for any $m \in \mathbb{N}$ there exists a weak solution $(\boldsymbol{u}^m, \eta^m, \boldsymbol{w}^m)$ of problem (3.1) such that*

$$\boldsymbol{u}^m \in C^{1,\gamma_0}(\bar{\Omega}), \quad \eta^m \in C^{0,\gamma_0}(\bar{\Omega}), \quad \boldsymbol{w}^m \in W^{2,2}(\Omega).$$

Moreover, if $\boldsymbol{f}$ and $\boldsymbol{g}$ satisfy the assumption

$$\tilde{c_0}\|\boldsymbol{f}\|_q + c_0\|\boldsymbol{g}\|_2 < \frac{\left[C\tilde{c_0}\mu_r + (C+1)c_0\mu_r - 1\right]^2}{4(\tilde{c_0}+c_0)}, \tag{3.2}$$

where μ_r is properly small satisfying (3.13), then

$$\|\boldsymbol{u}^m\|_{C^{1,\gamma_0}} + \|\eta^m\|_{C^{0,\gamma_0}} + \|\boldsymbol{w}^m\|_{2,2} \le 2(\tilde{c_0}\|\boldsymbol{f}\|_q + c_0\|\boldsymbol{g}\|_2), \quad \textit{uniformly in } m \in \mathbb{N}. \tag{3.3}$$

Proof. Setting $I_m = \|\boldsymbol{u}^m\|_{C^{1,\gamma_0}} + \|\eta^m\|_{C^{0,\gamma_0}} + \|\boldsymbol{w}^m\|_{2,2}$. Let be $m = 0$, first of all, we consider the following boundary-value problem

$$\begin{cases} -\Delta \boldsymbol{w}^0 - \nabla \operatorname{div} \boldsymbol{w}^0 = \boldsymbol{g}, & \text{in } \Omega, \\ \boldsymbol{w}^0|_{\partial\Omega} = 0, \end{cases}$$

where $\boldsymbol{g} \in L^2(\Omega)$. According to the theory of elliptic equation, we can find a solution $\boldsymbol{w}^0 \in W^{2,2}(\Omega)$ and get

$$\|\boldsymbol{w}^0\|_{2,2} \le c_0\|\boldsymbol{g}\|_2. \tag{3.4}$$

Then we consider the following boundary-value problem

$$\begin{cases} -\frac{1}{2}\triangle \boldsymbol{u}^0 + \nabla\eta^0 = \boldsymbol{f}, & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u}^0 = 0, & \text{in } \Omega, \\ \boldsymbol{u}^0|_{\partial\Omega} = 0, \end{cases} \tag{3.5}$$

where $\boldsymbol{f} \in L^2(\Omega)$. We can find a solution $(\boldsymbol{u}^0, \eta^0) \in C^{1,\gamma_0}(\bar{\Omega}) \times C^{0,\gamma_0}(\bar{\Omega})$ (see [6, Theorem 3.2]) and

$$\|\boldsymbol{u}^0\|_{C^{1,\gamma_0}} + \|\eta^0\|_{C^{0,\gamma_0}} \le \tilde{c}(\|\boldsymbol{u}^0\|_{1,2} + \|\boldsymbol{f}\|_q) = c(\|\boldsymbol{u}^0\|_{1,2} + \|\boldsymbol{f}\|_q), \tag{3.6}$$

where $c > 1$. By writing the definition of weak solution of $(3.5)_1$ with the test function φ replaced by $\boldsymbol{u}^0$ we get

$$\int_\Omega |\mathcal{D}\boldsymbol{u}^0|^2 dx = \int_\Omega \boldsymbol{f}\boldsymbol{u}^0 dx.$$

By Lemma 2.1 and the Hölder inequality, we have

$$\int_\Omega |\mathcal{D}\boldsymbol{u}^0|^2 dx = \|\mathcal{D}\boldsymbol{u}^0\|_2^2 \ge \frac{1}{c_1^2}\|\boldsymbol{u}^0\|_{1,2}^2,$$

$$\int_\Omega \boldsymbol{f}\boldsymbol{u}^0 dx \le \|\boldsymbol{f}\|_2\|\boldsymbol{u}^0\|_2 \le \|\boldsymbol{f}\|_q\|\boldsymbol{u}^0\|_{1,2},$$

which obviously implies $\|\boldsymbol{u}^0\|_{1,2} \le c_1^2\|\boldsymbol{f}\|_q$. So we can get from (3.4) and (3.6) that

$$I_0 = \|\boldsymbol{u}^0\|_{C^{1,\gamma_0}} + \|\eta^0\|_{C^{0,\gamma_0}} + \|\boldsymbol{w}^0\|_{2,2} \le c(1+c_1^2)\|\boldsymbol{f}\|_q + c_0\|\boldsymbol{g}\|_2. \tag{3.7}$$

Let be $m \ge 1$, assuming that $(\boldsymbol{u}^m, \eta^m, \boldsymbol{w}^m) \in C^{1,\gamma_0}(\bar{\Omega}) \times C^{0,\gamma_0}(\bar{\Omega}) \times W^{2,2}(\bar{\Omega})$ is a solution of (3.1). Firstly, we consider the following boundary-value problem

$$\begin{cases} -\Delta \boldsymbol{w}^{m+1} - \nabla \operatorname{div} \boldsymbol{w}^{m+1} = \mu_r \operatorname{rot} \boldsymbol{u}^m - \mu_r \boldsymbol{w}^m + \boldsymbol{g} - (\boldsymbol{u}^m \cdot \nabla)\boldsymbol{w}^m, & \text{in } \Omega, \\ \boldsymbol{w}^{m+1}|_{\partial\Omega} = 0, \end{cases}$$

where $g \in L^2(\Omega)$. Since the assumption implies that

$$\|(\boldsymbol{u}^m \cdot \nabla)\boldsymbol{w}^m\|_2 \le \|\boldsymbol{u}^m\|_2 \|\nabla \boldsymbol{w}^m\|_2 \le \|\boldsymbol{u}^m\|_{C^{1,\gamma_0}} \|\boldsymbol{w}^m\|_{2,2},$$

then $\mu_r \operatorname{rot} \boldsymbol{u}^m - \mu_r \boldsymbol{w}^m + \boldsymbol{g} - (\boldsymbol{u}^m \cdot \nabla)\boldsymbol{w}^m$ belongs to $L^2(\Omega)$. According to the theory of elliptic equation, we can find a solution $\boldsymbol{w}^{m+1} \in W^{2,2}(\Omega)$ and

$$\begin{aligned} \|\boldsymbol{w}^{m+1}\|_{2,2} &\le c_0(\|\mu_r \operatorname{rot} \boldsymbol{u}^m\|_2 + \|\mu_r \boldsymbol{w}^m\|_2 + \|\boldsymbol{g}\|_2 + \|(\boldsymbol{u}^m \cdot \nabla)\boldsymbol{w}^m\|_2 \\ &\le c_0(C\mu_r \|\boldsymbol{u}^m\|_{C^{1,\gamma_0}} + \mu_r \|\boldsymbol{w}^m\|_{2,2} + \|\boldsymbol{g}\|_2 + \|\boldsymbol{u}^m\|_{C^{1,\gamma_0}} \|\boldsymbol{w}^m\|_{2,2}). \end{aligned} \tag{3.8}$$

Secondly, we consider the boundary-value problem

$$\begin{cases} -\operatorname{div}[(1+|\mathcal{D}\boldsymbol{u}^m|)^{p-2}\mathcal{D}\boldsymbol{u}^{m+1}] + \nabla \eta^{m+1} = \mu_r \operatorname{rot} \boldsymbol{w}^m + \boldsymbol{f} - (\boldsymbol{u}^m \cdot \nabla)\boldsymbol{u}^m, & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u}^{m+1} = 0, & \text{in } \Omega, \\ \boldsymbol{u}^{m+1}|_{\partial\Omega} = 0, \end{cases} \tag{3.9}$$

where $\boldsymbol{f} \in L^q(\Omega)$. Since the assumption implies that

$$\begin{aligned} \|\mu_r \operatorname{rot} \boldsymbol{w}^m\|_q &\le C\mu_r \|\nabla \boldsymbol{w}^m\|_q \le C\mu_r \|\boldsymbol{w}^m\|_{2,2}, \\ \|(\boldsymbol{u}^m \cdot \nabla)\boldsymbol{u}^m\|_q &\le \|\boldsymbol{u}^m\|_\infty \|\nabla \boldsymbol{u}^m\|_q \le \|\boldsymbol{u}^m\|^2_{C^{1,\gamma_0}}, \end{aligned}$$

then $\mu_r \operatorname{rot} \boldsymbol{w}^m + \boldsymbol{f} - (\boldsymbol{u}^m \cdot \nabla)\boldsymbol{u}^m$ belongs to $L^q(\Omega)$. Then we can get a solution $(\boldsymbol{u}^{m+1}, \eta^{m+1}) \in C^{1,\gamma_0}(\bar{\Omega}) \times C^{0,\gamma_0}(\bar{\Omega})$ (see [6, Theorem 3.2]) such that

$$\begin{aligned} &\|\boldsymbol{u}^{m+1}\|_{C^{1,\gamma_0}} + \|\eta^{m+1}\|_{C^{0,\gamma_0}} \\ &\quad \le \tilde{c}\big(\|\boldsymbol{u}^{m+1}\|_{1,2} + \|\mu_r \operatorname{rot} \boldsymbol{w}^m\|_q + \|\boldsymbol{f}\|_q + \|(\boldsymbol{u}^m \cdot \nabla)\boldsymbol{u}^m\|_q\big) \\ &\quad \le c(1 + \|\boldsymbol{u}^m\|_{C^{1,\gamma_0}})^r \cdot (\|\boldsymbol{u}^{m+1}\|_{1,2} + C\mu_r \|\boldsymbol{w}^m\|_{2,2} + \|\boldsymbol{f}\|_q + \|\boldsymbol{u}^m\|^2_{C^{1,\gamma_0}}), \end{aligned} \tag{3.10}$$

where the exponent r is a real number greater than 2.

By writing the definition of weak solution of $(3.9)_1$ with the test function φ replaced by $\boldsymbol{u}^{m+1}$ we get

$$\int_\Omega (1+|\mathcal{D}\boldsymbol{u}^m|)^{p-2}|\mathcal{D}\boldsymbol{u}^{m+1}|^2 dx = \int_\Omega \mu_r \operatorname{rot} \boldsymbol{w}^m \boldsymbol{u}^{m+1} dx + \int_\Omega \boldsymbol{f}\boldsymbol{u}^{m+1} dx - \int_\Omega (\boldsymbol{u}^m \cdot \nabla)\boldsymbol{u}^m \boldsymbol{u}^{m+1} dx.$$

Since $1 < p < 2$, by Lemma 2.1 and the Hölder inequality, there follows

$$\begin{aligned} \int_\Omega (1+|\mathcal{D}\boldsymbol{u}^m|)^{p-2}|\mathcal{D}\boldsymbol{u}^{m+1}|^2 dx &\ge (1+\|\mathcal{D}\boldsymbol{u}^m\|_\infty)^{p-2}\|\mathcal{D}\boldsymbol{u}^{m+1}\|_2^2 \\ &\ge \frac{1}{c_1^2}(1+\|\boldsymbol{u}^m\|_{C^{1,\gamma_0}})^{p-2}\|\boldsymbol{u}^{m+1}\|_{1,2}^2, \end{aligned}$$

$$\begin{aligned} &\left| \int_\Omega \mu_r \operatorname{rot} \boldsymbol{w}^m \boldsymbol{u}^{m+1} dx + \int_\Omega \boldsymbol{f}\boldsymbol{u}^{m+1} dx - \int_\Omega (\boldsymbol{u}^m \cdot \nabla)\boldsymbol{u}^m \boldsymbol{u}^{m+1} dx \right| \\ &\quad \le \|\mu_r \operatorname{rot} \boldsymbol{w}^m\|_2 \|\boldsymbol{u}^{m+1}\|_2 + \|\boldsymbol{f}\|_2 \|\boldsymbol{u}^{m+1}\|_2 + \|(\boldsymbol{u}^m \cdot \nabla)\boldsymbol{u}^m\|_2 \|\boldsymbol{u}^{m+1}\|_2 \\ &\quad \le C\mu_r \|\boldsymbol{w}^m\|_{2,2} \|\boldsymbol{u}^{m+1}\|_{1,2} + \|\boldsymbol{f}\|_q \|\boldsymbol{u}^{m+1}\|_{1,2} + \|\boldsymbol{u}^m\|^2_{C^{1,\gamma_0}} \|\boldsymbol{u}^{m+1}\|_{1,2}, \end{aligned}$$

which implies

$$\|\boldsymbol{u}^{m+1}\|_{1,2} \le c_1^2 (1+\|\boldsymbol{u}^m\|_{C^{1,\gamma_0}})^{2-p} (C\mu_r \|\boldsymbol{w}^m\|_{2,2} + \|\boldsymbol{f}\|_q + \|\boldsymbol{u}^m\|^2_{C^{1,\gamma_0}}). \tag{3.11}$$

Combining (3.8), (3.10) and (3.11), we obtain

$$\begin{aligned} I_{m+1} &= \|\boldsymbol{u}^{m+1}\|_{C^{1,\gamma_0}} + \|\eta^{m+1}\|_{C^{0,\gamma_0}} + \|\boldsymbol{w}^{m+1}\|_{2,2} \\ &\le c(1+c_1^2)(1+\|\boldsymbol{u}^m\|_{C^{1,\gamma_0}})^{r+2-p}(C\mu_r\|\boldsymbol{w}^m\|_{2,2} + \|\boldsymbol{f}\|_q + \|\boldsymbol{u}^m\|^2_{C^{1,\gamma_0}}) \\ &\quad + c_0(C\mu_r\|\boldsymbol{u}^m\|_{C^{1,\gamma_0}} + \mu_r\|\boldsymbol{w}^m\|_{2,2} + \|\boldsymbol{g}\|_2 + \|\boldsymbol{u}^m\|_{C^{1,\gamma_0}}\|\boldsymbol{w}^m\|_{2,2}) \\ &\le c(1+c_1^2)(1+I_m)^{r+2-p}(C\mu_r I_m + \|\boldsymbol{f}\|_q + I_m^2) + c_0\big[(C+1)\mu_r I_m + \|\boldsymbol{g}\|_2 + I_m^2\big]. \end{aligned} \tag{3.12}$$

We shall prove the boundedness of the sequence $\{I_m\}$ by a fixed point argument. Setting, for any $t \ge 0$

$$\psi(t) = c(1+c_1^2)(1+t)^{r+2-p}(C\mu_r t + \|\boldsymbol{f}\|_q + t^2) + c_0[(C+1)\mu_r t + \|\boldsymbol{g}\|_2 + t^2] - t.$$

We look for a root of $\psi(t)$. Let us observe that if $0 \le t \le 1$, then

$$\begin{aligned} \psi(t) &\le c(1+c_1^2)2^{r+2-p}(C\mu_r t + \|\boldsymbol{f}\|_q + t^2) + c_0[(C+1)\mu_r t + \|\boldsymbol{g}\|_2 + t^2] - t \\ &= \big[c(1+c_1^2)2^{r+2-p} + c_0\big]t^2 + \big[Cc(1+c_1^2)2^{r+2-p}\mu_r + (C+1)c_0\mu_r - 1\big]t \\ &\quad + c(1+c_1^2)2^{r+2-p}\|\boldsymbol{f}\|_q + c_0\|\boldsymbol{g}\|_2. \end{aligned}$$

Define

$$\begin{aligned} h(t) &= \big[c(1+c_1^2)2^{r+2-p} + c_0\big]t^2 + \big[Cc(1+c_1^2)2^{r+2-p}\mu_r + (C+1)c_0\mu_r - 1\big]t \\ &\quad + c(1+c_1^2)2^{r+2-p}\|\boldsymbol{f}\|_q + c_0\|\boldsymbol{g}\|_2, \end{aligned}$$

we note that if

$$\mu_r \le \frac{1}{Cc(1+c_1^2)2^{r+2-p} + (C+1)c_0}, \tag{3.13}$$

then the function $h(t)$ has two positive roots $s_1 < s_2$, if and only if the discriminant $\Delta > 0$, namely

$$c(1+c_1^2)2^{r+2-p}\|\boldsymbol{f}\|_q + c_0\|\boldsymbol{g}\|_2 < \frac{[Cc(1+c_1^2)2^{r+2-p}\mu_r + (C+1)c_0\mu_r - 1]^2}{4[c(1+c_1^2)2^{r+2-p} + c_0]},$$

and we have that

$$0 < s_1 = \frac{1 - Cc(1+c_1^2)2^{r+2-p}\mu_r - (C+1)c_0\mu_r - \sqrt{\Delta}}{2[c(1+c_1^2)2^{r+2-p} + c_0]} < 1.$$

Since $c > 1$ and consequently $2[c(1+c_1^2)2^{r+2-p} + c_0] > 1$. Since $\psi(0) > 0$ and $\psi(t) \le h(t)$, when $t \in [0,1]$, there exists t_1, with $0 < t_1 < s_1$ such that $\psi(t_1) = 0$, i.e.

$$c(1+c_1^2)(1+t_1)^{r+2-p}(C\mu_r t_1 + \|\boldsymbol{f}\|_q + t_1^2) + c_0[(C+1)\mu_r t_1 + \|\boldsymbol{g}\|_2 + t_1^2] - t_1 = 0.$$

Since $t_1 > 0$, it follows that $c(1+c_1^2)\|\boldsymbol{f}\|_q + c_0\|\boldsymbol{g}\|_2 - t_1 \le 0$, recalling (3.7), we get $t_1 \ge c(1+c_1^2)\|\boldsymbol{f}\|_q + c_0\|\boldsymbol{g}\|_2 \ge I_0$. If we suppose that $I_m \le t_1$, by inequality (3.12) and the fact that $\psi(t_1) = 0$ we obtain

$$\begin{aligned} I_{m+1} &\le c(1+c_1^2)(1+I_m)^{r+2-p}[C\mu_r I_m + \|\boldsymbol{f}\|_q + I_m^2] + c_0\big[(C+1)\mu_r I_m + \|\boldsymbol{g}\|_2 + I_m^2\big] \\ &\le c(1+c_1^2)(1+t_1)^{r+2-p}(C\mu_r t_1 + \|\boldsymbol{f}\|_q + t_1^2) + c_0\big[(C+1)\mu_r t_1 + \|\boldsymbol{g}\|_2 + t_1^2\big] \\ &= \psi(t_1) + t_1 \\ &= t_1, \end{aligned}$$

which proves our claim. Therefore

$$I_m \le t_1 < s_1 < c(1+c_1^2)2^{r+3-p}\|\boldsymbol{f}\|_q + 2c_0\|\boldsymbol{g}\|_2 \le 1, \quad \forall m \in \mathbb{N}.$$

Let $\tilde{c_0} = c(1+c_1^2)2^{r+2-p}$, we can get $I_m \le 2(\tilde{c_0}\|\boldsymbol{f}\|_q + c_0\|\boldsymbol{g}\|_2)$. □

3.2 Convergence of approximate solutions

For any $j \in \mathbb{N}$, set $\boldsymbol{P}^{j+1} = \boldsymbol{u}^{j+1} - \boldsymbol{u}^j$, $Q^{j+1} = \eta^{j+1} - \eta^j$, $\boldsymbol{R}^{j+1} = \boldsymbol{w}^{j+1} - \boldsymbol{w}^j$. Taking $m = j$ and $j+1$, respectively, in the weak formula of (3.1)$_1$, then subtracting one from the other, we can get for $\forall \varphi \in C_0^\infty(\Omega)$

$$\begin{aligned}\int_\Omega (1 + |\mathcal{D}\boldsymbol{u}^j|)^{p-2}\mathcal{D}\boldsymbol{u}^{j+1}\mathcal{D}\varphi dx = &\int_\Omega (1 + |\mathcal{D}\boldsymbol{u}^{j-1}|)^{p-2}\mathcal{D}\boldsymbol{u}^j\mathcal{D}\varphi dx \\ &+ \int_\Omega \mu_r \operatorname{rot} \boldsymbol{R}^j \varphi dx - \int_\Omega (\boldsymbol{P}^j \cdot \nabla)\boldsymbol{u}^j \varphi dx \\ &- \int_\Omega (\boldsymbol{u}^{j-1} \cdot \nabla \boldsymbol{P}\varphi dx + \int_\Omega Q^{j+1} \operatorname{div} \varphi dx.\end{aligned}$$

Next, by subtracting $\int_\Omega (1 + |\mathcal{D}\boldsymbol{u}^j|)^{p-2}\mathcal{D}\boldsymbol{u}^j\mathcal{D}\varphi dx$ from both sides of the above equality, we could obtain

$$\begin{aligned}\int_\Omega (1 + |\mathcal{D}\boldsymbol{u}^j|)^{p-2}\mathcal{D}\boldsymbol{P}^{j+1}\mathcal{D}\varphi dx = &\int_\Omega [(1 + |\mathcal{D}\boldsymbol{u}^{j-1}|)^{p-2} - (1 + |\mathcal{D}\boldsymbol{u}^j|)^{p-2}]\mathcal{D}\boldsymbol{u}^j\mathcal{D}\varphi dx \\ &+ \int_\Omega \mu_r \operatorname{rot} \boldsymbol{R}^j \varphi dx - \int_\Omega (\boldsymbol{P}^j \cdot \nabla)\boldsymbol{u}^j \varphi dx \\ &- \int_\Omega (\boldsymbol{u}^{j-1} \cdot \nabla)\boldsymbol{P}^j \varphi dx + \int_\Omega Q^{j+1} \operatorname{div} \varphi dx.\end{aligned} \tag{3.14}$$

This identity, by a continuity argument, still holds with $\varphi \in V(\Omega)$, in which case the last term of (3.14) vanishes. Here, we recall that $\boldsymbol{u}^{j-1} = \boldsymbol{w}^{j-1} = 0$ for $j = 0$ and then $\boldsymbol{P}^0 = \boldsymbol{u}^0, \boldsymbol{R}^0 = \boldsymbol{w}^0$.

Similarly, by taking $m = j$ and $j+1$, respectively, in the weak formula of (3.1)$_3$, then subtracting one from the other, we can get for $\forall \psi \in H_0^1(\Omega)$

$$\begin{aligned}\int_\Omega \operatorname{div} \boldsymbol{R}^{j+1} \operatorname{div} \psi dx + \int_\Omega \nabla \boldsymbol{R}^{j+1}\nabla\psi dx = &\int_\Omega \mu_r \operatorname{rot} \boldsymbol{P}^j \psi dx - \int_\Omega \mu_r \boldsymbol{R}^j \psi dx \\ &- \int_\Omega (\boldsymbol{P}^j \cdot \nabla)\boldsymbol{w}^j \psi dx - \int_\Omega (\boldsymbol{u}^{j-1} \cdot \nabla)\boldsymbol{R}^j \psi dx.\end{aligned} \tag{3.15}$$

Proposition 3.2. *Assume that all the assumptions of Proposition 3.1 are satisfied and let $\{\boldsymbol{u}^m\}, \{\eta^m\}$ and $\{\boldsymbol{w}^m\}$ be the corresponding sequence. Then, if*

$$\begin{aligned}&(1 + 2\tilde{c_0}\|\boldsymbol{f}\|_q + 2c_0\|\boldsymbol{g}\|_2)^{2-p} \\ &\cdot \Big[(2 - p + 2c_1^2 + Cc_1 + C) \cdot (2\tilde{c_0}\|\boldsymbol{f}\|_q + 2c_0\|\boldsymbol{g}\|_2) + C(c_1 + 1)\mu_r\Big] < 1,\end{aligned} \tag{3.16}$$

with $\tilde{c_0}$ and c_0 given by Proposition 3.1, the series $\sum_m \boldsymbol{P}^m$ converges to a function $\boldsymbol{P}$ in $W^{1,2}(\Omega)$, the series $\sum_m Q^m$ converges to a function Q in $L^2(\Omega)$, the series $\sum_m \boldsymbol{R}^m$ converges to a function $\boldsymbol{R}$ in $W^{1,2}(\Omega)$.

Proof. First, let us verify that the following estimates hold:

(a)

$$\begin{aligned}\|D\boldsymbol{P}^1\|_2 + \|\boldsymbol{R}^1\|_{1,2} \le &(2\tilde{c_0}\|\boldsymbol{f}\|_q + 2c_0\|\boldsymbol{g}\|_2) \cdot (1 + 2\tilde{c_0}\|\boldsymbol{f}\|_q + 2c_0\|\boldsymbol{g}\|_2)^{2-p} \\ &\cdot \big[(2 - p + c_1 + C)(2\tilde{c_0}\|\boldsymbol{f}\|_q + 2c_0\|\boldsymbol{g}\|_2) + C\mu_r(1 + c_1)\big];\end{aligned}$$

(b) if, for $j \ge 1$, it holds that

$$\begin{aligned}&\|D\boldsymbol{P}^j\|_2 + \|\boldsymbol{R}^j\|_{1,2} \\ &\le (2\tilde{c_0}\|\boldsymbol{f}\|_q + 2c_0\|\boldsymbol{g}\|_2) \cdot \frac{(2 - p + c_1 + C)(2\tilde{c_0}\|\boldsymbol{f}\|_q + 2c_0\|\boldsymbol{g}\|_2) + C\mu_r(1 + c_1)}{(2 - p + 2c_1^2 + Cc_1 + C)(2\tilde{c_0}\|\boldsymbol{f}\|_q + 2c_0\|\boldsymbol{g}\|_2) + C(c_1 + 1)\mu_r} \\ &\quad \cdot \Big\{(1+2\tilde{c_0}\|\boldsymbol{f}\|_q+2c_0\|\boldsymbol{g}\|_2)^{2-p}\big[(2-p+2c_1^2+Cc_1+C)(2\tilde{c_0}\|\boldsymbol{f}\|_q+2c_0\|\boldsymbol{g}\|_2)+C(c_1+1)\mu_r\big]\Big\}^j,\end{aligned}$$

then

$$\|D\boldsymbol{P}^{j+1}\|_2 + \|\boldsymbol{R}^{j+1}\|_{1,2} \tag{3.17}$$
$$\leq (2\tilde{c_0}\|\boldsymbol{f}\|_q + 2c_0\|\boldsymbol{g}\|_2)\frac{(2-p+c_1+C)(2\tilde{c_0}\|\boldsymbol{f}\|_q + 2c_0\|\boldsymbol{g}\|_2) + C\mu_r(1+c_1)}{(2-p+2c_1^2+Cc_1+C)(2\tilde{c_0}\|\boldsymbol{f}\|_q + 2c_0\|\boldsymbol{g}\|_2) + C(c_1+1)\mu_r}$$
$$\cdot \Big\{(1+2\tilde{c_0}\|\boldsymbol{f}\|_q+2c_0\|\boldsymbol{g}\|_2)^{2-p}\big[(2-p+2c_1^2+Cc_1+C)(2\tilde{c_0}\|\boldsymbol{f}\|_q+2c_0\|\boldsymbol{g}\|_2)+C(c_1+1)\mu_r\big]\Big\}^{j+1}.$$

By the above arguments, setting $j=0$ and testing with $\boldsymbol{P}^1$ in (3.14), we get

$$\int_\Omega (1+|\mathcal{D}\boldsymbol{u}^0|)^{p-2}|\mathcal{D}\boldsymbol{P}^1|^2 dx$$
$$= \int_\Omega [1-(1+|\mathcal{D}\boldsymbol{u}^0|)^{p-2}]\mathcal{D}\boldsymbol{u}^0\mathcal{D}\boldsymbol{P}^1 dx + \int_\Omega \mu_r \operatorname{rot} \boldsymbol{w}^0\boldsymbol{P}^1 dx - \int_\Omega (\boldsymbol{u}^0\cdot\nabla)\boldsymbol{u}^0\boldsymbol{P}^1 dx.$$

Since $p<2$, then

$$\int_\Omega (1+|\mathcal{D}\boldsymbol{u}^0|)^{p-2}|\mathcal{D}\boldsymbol{P}^1|^2 dx \geq (1+\|\mathcal{D}\boldsymbol{u}^0\|_\infty)^{p-2}\|\mathcal{D}\boldsymbol{P}^1\|_2^2 \geq (1+\|\boldsymbol{u}^0\|_{C^{1,\gamma_0}})^{p-2}\|\mathcal{D}\boldsymbol{P}^1\|_2^2,$$

by using the Hölder inequality, Lemma 2.3 and Lemma 2.1 we get

$$\left|\int_\Omega [1-(1+|\mathcal{D}\boldsymbol{u}^0|)^{p-2}]\mathcal{D}\boldsymbol{u}^0\mathcal{D}\boldsymbol{P}^1 dx + \int_\Omega \mu_r \operatorname{rot} \boldsymbol{w}^0\boldsymbol{P}^1 dx - \int_\Omega (\boldsymbol{u}^0\cdot\nabla)\boldsymbol{u}^0\boldsymbol{P}^1 dx\right|$$
$$\leq (2-p)\|\mathcal{D}\boldsymbol{u}^0\|_2\|\mathcal{D}\boldsymbol{u}^0\|_\infty\|\mathcal{D}\boldsymbol{P}^1\|_2 + \|\mu_r \operatorname{rot} \boldsymbol{w}^0\|_2\|\boldsymbol{P}^1\|_2 + \|(\boldsymbol{u}^0\cdot\nabla)\boldsymbol{u}^0\|_2\|\boldsymbol{P}^1\|_2$$
$$\leq (2-p)\|\boldsymbol{u}^0\|_{C^{1,\gamma_0}}^2\|\mathcal{D}\boldsymbol{P}^1\|_2 + Cc_1\mu_r\|\boldsymbol{w}^0\|_{1,2}\|\mathcal{D}\boldsymbol{P}^1\|_2 + c_1\|\boldsymbol{u}^0\|_{C^{1,\gamma_0}}^2\|\mathcal{D}\boldsymbol{P}^1\|_2,$$

hence

$$\|\mathcal{D}\boldsymbol{P}^1\|_2 \leq (1+\|\boldsymbol{u}^0\|_{C^{1,\gamma_0}})^{2-p}\cdot\Big[(2-p)\|\boldsymbol{u}^0\|_{C^{1,\gamma_0}}^2 + Cc_1\mu_r\|\boldsymbol{w}^0\|_{1,2} + c_1\|\boldsymbol{u}^0\|_{C^{1,\gamma_0}}^2\Big]. \tag{3.18}$$

Nextly, setting $j=0$ and testing with $\boldsymbol{R}^1$ in (3.15), we get

$$\int_\Omega |\operatorname{div} \boldsymbol{R}^1|^2 dx + \int_\Omega |\nabla\boldsymbol{R}^1|^2 dx = \int_\Omega \mu_r \operatorname{rot} \boldsymbol{u}^0\boldsymbol{R}^1 dx - \int_\Omega \mu_r\boldsymbol{w}^0\boldsymbol{R}^1 dx - \int_\Omega (\boldsymbol{u}^0\cdot\nabla)\boldsymbol{w}^0\boldsymbol{R}^1 dx,$$

by the Hölder inequality and Lemma 2.1 we get

$$\int_\Omega |\operatorname{div} \boldsymbol{R}^1|^2 dx + \int_\Omega |\nabla\boldsymbol{R}^1|^2 dx \geq C\|\boldsymbol{R}^1\|_{1,2}^2,$$

and

$$\left|\int_\Omega \mu_r \operatorname{rot} \boldsymbol{u}^0\boldsymbol{R}^1 dx - \int_\Omega \mu_r\boldsymbol{w}^0\boldsymbol{R}^1 dx - \int_\Omega (\boldsymbol{u}^0\cdot\nabla)\boldsymbol{w}^0\boldsymbol{R}^1 dx\right|$$
$$\leq \|\mu_r \operatorname{rot} \boldsymbol{u}^0\|_2\|\boldsymbol{R}^1\|_2 + \|\mu_r\boldsymbol{w}^0\|_2\|\boldsymbol{R}^1\|_2 + \|(\boldsymbol{u}^0\cdot\nabla)\boldsymbol{w}^0\|_2\|\boldsymbol{R}^1\|_2$$
$$\leq C\mu_r\|\boldsymbol{u}^0\|_{C^{1,\gamma_0}}\|\boldsymbol{R}^1\|_{1,2} + \mu_r\|\boldsymbol{w}^0\|_{2,2}\|\boldsymbol{R}^1\|_{1,2} + \|\boldsymbol{u}^0\|_{C^{1,\gamma_0}}\|\boldsymbol{w}^0\|_{2,2}\|\boldsymbol{R}^1\|_{1,2},$$

hence

$$\|\boldsymbol{R}^1\|_{1,2} \leq C\mu_r\|\boldsymbol{u}^0\|_{C^{1,\gamma_0}} + C\mu_r\|\boldsymbol{w}^0\|_{2,2} + C\|\boldsymbol{u}^0\|_{C^{1,\gamma_0}}\|\boldsymbol{w}^0\|_{2,2}. \tag{3.19}$$

Combining (3.18) (3.19) and by using estimate (3.3), we obtain

$$\begin{aligned}\|\mathcal{D}\boldsymbol{P}^1\|_2+\|\boldsymbol{R}^1\|_{1,2}\le{}&(1+\|\boldsymbol{u}^0\|_{C^{1,\gamma_0}})^{2-p}\cdot\big[(2-p+c_1)\|\boldsymbol{u}^0\|^2_{C^{1,\gamma_0}}+Cc_1\mu_r\|\boldsymbol{w}^0\|_{1,2}\big]\\&+C\mu_r\|\boldsymbol{u}^0\|_{C^{1,\gamma_0}}+C\mu_r\|\boldsymbol{w}^0\|_{2,2}+C\|\boldsymbol{u}^0\|_{C^{1,\gamma_0}}\|\boldsymbol{w}^0\|_{2,2}\\\le{}&(1+2\tilde{c_0}\|\boldsymbol{f}\|_q+2c_0\|\boldsymbol{g}\|_2)^{2-p}\cdot\big[(2-p+c_1+C)(2\tilde{c_0}\|\boldsymbol{f}\|_q+2c_0\|\boldsymbol{g}\|_2)\\&+C\mu_r(1+c_1)\big](2\tilde{c_0}\|\boldsymbol{f}\|_q+2c_0\|\boldsymbol{g}\|_2).\end{aligned}$$

We arrive at (a).

Let us pass to estimate (b). Assume that the hypothesis in (b) holds. As for (a), by setting $j\ge1$ and $\varphi=\boldsymbol{P}^{j+1}\in V(\Omega)$ in (3.14), we get

$$\begin{aligned}&\int_\Omega(1+|\mathcal{D}\boldsymbol{u}^j|)^{p-2}|\mathcal{D}\boldsymbol{P}^{j+1}|^2dx\\&\quad=\int_\Omega\big[(1+|\mathcal{D}\boldsymbol{u}^{j-1}|)^{p-2}-(1+|\mathcal{D}\boldsymbol{u}^j|)^{p-2}\big]\mathcal{D}\boldsymbol{u}^j\mathcal{D}\boldsymbol{P}^{j+1}dx\\&\qquad+\int_\Omega\mu_r\operatorname{rot}\boldsymbol{R}^j\boldsymbol{P}^{j+1}dx-\int_\Omega(\boldsymbol{P}^j\cdot\nabla)\boldsymbol{u}^j\boldsymbol{P}^{j+1}dx-\int_\Omega(\boldsymbol{u}^{j-1}\cdot\nabla)\boldsymbol{P}^j\boldsymbol{P}^{j+1}dx.\end{aligned}$$

Since $p<2$, we get

$$\begin{aligned}\int_\Omega(1+|\mathcal{D}\boldsymbol{u}^j|)^{p-2}|\mathcal{D}\boldsymbol{P}^{j+1}|^2dx&\ge(1+\|\mathcal{D}\boldsymbol{u}^j\|_\infty)^{p-2}\|\mathcal{D}\boldsymbol{P}^{j+1}\|_2^2\\&\ge(1+\|\boldsymbol{u}^j\|_{C^{1,\gamma_0}})^{p-2}\|\mathcal{D}\boldsymbol{P}^{j+1}\|_2^2,\end{aligned}$$

then the Hölder inequality , Lemma 2.3 and Lemma 2.1 yield that

$$\begin{aligned}&\Big|\int_\Omega\big[(1+|\mathcal{D}\boldsymbol{u}^{j-1}|)^{p-2}-(1+|\mathcal{D}\boldsymbol{u}^j|)^{p-2}\big]\mathcal{D}\boldsymbol{u}^j\mathcal{D}\boldsymbol{P}^{j+1}dx\Big|\\&\qquad\le(2-p)\|\mathcal{D}\boldsymbol{P}^j\|_2\|\mathcal{D}\boldsymbol{u}^j\|_\infty\|\mathcal{D}\boldsymbol{P}^{j+1}\|_2\\&\qquad\le(2-p)\|\mathcal{D}\boldsymbol{P}^j\|_2\|\boldsymbol{u}^j\|_{C^{1,\gamma_0}}\|\mathcal{D}\boldsymbol{P}^{j+1}\|_2,\end{aligned}$$

$$\begin{aligned}&\Big|\int_\Omega\mu_r\operatorname{rot}\boldsymbol{R}^j\boldsymbol{P}^{j+1}dx-\int_\Omega(\boldsymbol{P}^j\cdot\nabla)\boldsymbol{u}^j\boldsymbol{P}^{j+1}dx-\int_\Omega(\boldsymbol{u}^{j-1}\cdot\nabla)\boldsymbol{P}^j\boldsymbol{P}^{j+1}dx\Big|\\&\quad\le\|\mu_r\operatorname{rot}\boldsymbol{R}^j\|_2\|\boldsymbol{P}^{j+1}\|_2+\|(\boldsymbol{P}^j\cdot\nabla)\boldsymbol{u}^j\|_2\|\boldsymbol{P}^{j+1}\|_2+\|(\boldsymbol{u}^{j-1}\cdot\nabla)\boldsymbol{P}^j\|_2\|\boldsymbol{P}^{j+1}\|_2\\&\quad\le Cc_1\mu_r\|\boldsymbol{R}^j\|_{1,2}\|\mathcal{D}\boldsymbol{P}^{j+1}\|_2+c_1^2\|\mathcal{D}\boldsymbol{P}^j\|_2\|\boldsymbol{u}^j\|_{C^{1,\gamma_0}}\|\mathcal{D}\boldsymbol{P}^{j+1}\|_2\\&\qquad+c_1^2\|\boldsymbol{u}^{j-1}\|_{C^{1,\gamma_0}}\|\mathcal{D}\boldsymbol{P}^j\|_2\|\mathcal{D}\boldsymbol{P}^{j+1}\|_2,\end{aligned}$$

hence

$$\begin{aligned}\|\mathcal{D}\boldsymbol{P}^{j+1}\|_2\le{}&(1+\|\boldsymbol{u}^j\|_{C^{1,\gamma_0}})^{2-p}\cdot\Big[(2-p)\|\mathcal{D}\boldsymbol{P}^j\|_2\|\boldsymbol{u}^j\|_{C^{1,\gamma_0}}\\&+Cc_1\mu_r\|\boldsymbol{R}^j\|_{1,2}+c_1^2\|\mathcal{D}\boldsymbol{P}^j\|_2\|\boldsymbol{u}^j\|_{C^{1,\gamma_0}}+c_1^2\|\boldsymbol{u}^{j-1}\|_{C^{1,\gamma_0}}\|\mathcal{D}\boldsymbol{P}^j\|_2\Big].\end{aligned}\tag{3.20}$$

Then setting $j\ge1$ and testing with $\boldsymbol{R}^{j+1}$ in (3.15), we get

$$\begin{aligned}\int_\Omega|\operatorname{div}\boldsymbol{R}^{j+1}|^2dx+\int_\Omega|\nabla\boldsymbol{R}^{j+1}|^2dx={}&\int_\Omega\mu_r\operatorname{rot}\boldsymbol{P}^j\boldsymbol{R}^{j+1}dx-\int_\Omega\mu_r\boldsymbol{R}^j\boldsymbol{R}^{j+1}dx\\&-\int_\Omega(\boldsymbol{P}^j\cdot\nabla)\boldsymbol{w}^j\boldsymbol{R}^{j+1}dx-\int_\Omega(\boldsymbol{u}^{j-1}\cdot\nabla)\boldsymbol{R}^j\boldsymbol{R}^{j+1}dx,\end{aligned}$$

by using the Hölder inequality and Lemma 2.1 we get

$$\int_\Omega |\operatorname{div} \boldsymbol{R}^{j+1}|^2 dx + \int_\Omega |\nabla \boldsymbol{R}^{j+1}|^2 dx \geq C\|\boldsymbol{R}^{j+1}\|_{1,2}^2,$$

and

$$\begin{aligned}
&\left| \int_\Omega \mu_r \operatorname{rot} \boldsymbol{P}^j \boldsymbol{R}^{j+1} dx - \int_\Omega \mu_r \boldsymbol{R}^j \boldsymbol{R}^{j+1} dx - \int_\Omega (\boldsymbol{P}^j \cdot \nabla) \boldsymbol{w}^j \boldsymbol{R}^{j+1} dx - \int_\Omega (\boldsymbol{u}^{j-1} \cdot \nabla) \boldsymbol{R}^j \boldsymbol{R}^{j+1} dx \right| \\
&\quad \leq \|\mu_r \operatorname{rot} \boldsymbol{P}^j\|_2 \|\boldsymbol{R}^{j+1}\|_2 + \|\mu_r \boldsymbol{R}^j\|_2 \|\boldsymbol{R}^{j+1}\|_2 + \|\boldsymbol{P}^j\|_2 \|\nabla \boldsymbol{w}^j\|_\infty \|\boldsymbol{R}^{j+1}\|_2 \\
&\qquad + \|(\boldsymbol{u}^{j-1} \cdot \nabla) \boldsymbol{R}^j\|_2 \|\boldsymbol{R}^{j+1}\|_2 \\
&\quad \leq C c_1 \mu_r \|\mathcal{D}\boldsymbol{P}^j\|_2 \|\boldsymbol{R}^{j+1}\|_{1,2} + \mu_r \|\boldsymbol{R}^j\|_{1,2} \|\boldsymbol{R}^{j+1}\|_{1,2} + C c_1 \|\mathcal{D}\boldsymbol{P}^j\|_2 \|\boldsymbol{w}^j\|_{2,2} \|\boldsymbol{R}^{j+1}\|_{1,2} \\
&\qquad + \|\boldsymbol{u}^{j-1}\|_{C^{1,\gamma_0}} \|\boldsymbol{R}^j\|_{1,2} \|\boldsymbol{R}^{j+1}\|_{1,2},
\end{aligned}$$

hence

$$\|\boldsymbol{R}^{j+1}\|_{1,2} \leq C c_1 \mu_r \|\mathcal{D}\boldsymbol{P}^j\|_2 + C\mu_r \|\boldsymbol{R}^j\|_{1,2} + C c_1 \|\mathcal{D}\boldsymbol{P}^j\|_2 \|\boldsymbol{w}^j\|_{2,2} + C\|\boldsymbol{u}^{j-1}\|_{C^{1,\gamma_0}} \|\boldsymbol{R}^j\|_{1,2}. \tag{3.21}$$

Combining (3.20) and (3.21) and appealing to (3.3), we get

$$\begin{aligned}
&\|\mathcal{D}\boldsymbol{P}^{j+1}\|_2 + \|\boldsymbol{R}^{j+1}\|_{1,2} \\
&\quad \leq (1 + \|\boldsymbol{u}^j\|_{C^{1,\gamma_0}})^{2-p} \cdot \big[(2-p)\|\mathcal{D}\boldsymbol{P}^j\|_2 \|\boldsymbol{u}^j\|_{C^{1,\gamma_0}} + C c_1 \mu_r \|\boldsymbol{R}^j\|_{1,2} + c_1^2 \|\mathcal{D}\boldsymbol{P}^j\|_2 \|\boldsymbol{u}^j\|_{C^{1,\gamma_0}} \\
&\qquad + c_1^2 \|\boldsymbol{u}^{j-1}\|_{C^{1,\gamma_0}} \|\mathcal{D}\boldsymbol{P}^j\|_2\big] + C c_1 \mu_r \|\mathcal{D}\boldsymbol{P}^j\|_2 + C\mu_r \|\boldsymbol{R}^j\|_{1,2} + C c_1 \|\mathcal{D}\boldsymbol{P}^j\|_2 \|\boldsymbol{w}^j\|_{2,2} \\
&\qquad + C\|\boldsymbol{u}^{j-1}\|_{C^{1,\gamma_0}} \|\boldsymbol{R}^j\|_{1,2} \\
&\quad \leq \big[(1 + \|\boldsymbol{u}^j\|_{C^{1,\gamma_0}})^{2-p}((2-p)\|\boldsymbol{u}^j\|_{C^{1,\gamma_0}} + c_1^2 \|\boldsymbol{u}^j\|_{C^{1,\gamma_0}} + c_1^2 \|\boldsymbol{u}^{j-1}\|_{C^{1,\gamma_0}}) + C c_1 \mu_r \\
&\qquad + C c_1 \|\boldsymbol{w}^j\|_{2,2}\big] \cdot \|\mathcal{D}\boldsymbol{P}^j\|_2 + \big[C c_1 \mu_r (1 + \|\boldsymbol{u}^j\|_{C^{1,\gamma_0}})^{2-p} + C\mu_r + C\|\boldsymbol{u}^{j-1}\|_{C^{1,\gamma_0}}\big] \cdot \|\boldsymbol{R}^j\|_{1,2} \\
&\quad \leq (1 + 2\tilde{c_0}\|\boldsymbol{f}\|_q + 2c_0\|\boldsymbol{g}\|_2)^{2-p}\big[(2 - p + 2c_1^2 + C c_1 + C)(2\tilde{c_0}\|\boldsymbol{f}\|_q + 2c_0\|\boldsymbol{g}\|_2) + C(c_1+1)\mu_r\big] \\
&\qquad \cdot (\|\mathcal{D}\boldsymbol{P}^j\|_2 + \|\boldsymbol{R}^j\|_{1,2}),
\end{aligned}$$

which gives (3.17) via the hypothesis in (b). Therefore, by induction, (3.17) holds for any given $j \in \mathbb{N}$.

By the assumption (3.16), the series $\sum_j (\|\mathcal{D}\boldsymbol{P}^j\|_2 + \|\boldsymbol{R}^j\|_{1,2})$ converges. Since $\sum_j \|\mathcal{D}\boldsymbol{P}^j\|_2$ and $\sum_j \|\boldsymbol{R}^j\|_{1,2}$ are positive series, both $\sum_j \|\mathcal{D}\boldsymbol{P}^j\|_2$ and $\sum_j \|\boldsymbol{R}^j\|_{1,2}$ converge. Therefore, by the completeness of $W^{1,2}(\Omega)$ there follows the convergence of the series $\sum_j \boldsymbol{P}^j(x)$ and $\sum_j \boldsymbol{R}^j(x)$ in the norm $\|\cdot\|_{1,2}$ to a function $\boldsymbol{P}(x) \in W^{1,2}(\Omega)$ and $\boldsymbol{R}(x) \in W^{1,2}(\Omega)$ respectively.

By (3.14) the following identity holds in the distributional sense

$$\begin{aligned}
\nabla Q^{j+1} &= \nabla \cdot [(1 + |\mathcal{D}\boldsymbol{u}^j|)^{p-2} \mathcal{D}\boldsymbol{P}^{j+1}] - \nabla \cdot \{[(1 + |\mathcal{D}\boldsymbol{u}^{j-1}|)^{p-2} - (1 + |\mathcal{D}\boldsymbol{u}^j|)^{p-2}]\mathcal{D}\boldsymbol{u}^j\} \\
&\quad + \mu_r \operatorname{rot} \boldsymbol{R}^j - (\boldsymbol{P}^j \cdot \nabla)\boldsymbol{u}^j - (\boldsymbol{u}^{j-1} \cdot \nabla)\boldsymbol{P}^j.
\end{aligned}$$

In order to get estimates on the L^2-norm of Q^{j+1}, by Lemma 2.2 it is sufficient to estimate the $W^{-1,2}$-norm of the right-hand side of the previous equations. The first term can be estimated as follows

$$\|\nabla \cdot [(1 + |\mathcal{D}\boldsymbol{u}^j|)^{p-2} \mathcal{D}\boldsymbol{P}^{j+1}]\|_{-1,2} \leq \|(1 + |\mathcal{D}\boldsymbol{u}^j|)^{p-2} \mathcal{D}\boldsymbol{P}^{j+1}\|_2 \leq \|\mathcal{D}\boldsymbol{P}^{j+1}\|_2.$$

For the second one, we have

$$\begin{aligned}
&\big\|\nabla\cdot\{[(1+|\mathcal{D}\boldsymbol{u}^{j-1}|)^{p-2}-(1+|\mathcal{D}\boldsymbol{u}^{j}|)^{p-2}]\mathcal{D}\boldsymbol{u}^{j}\}\big\|_{-1,2}\\
&\quad\le \big\|\big[(1+|\mathcal{D}\boldsymbol{u}^{j-1}|)^{p-2}-(1+|\mathcal{D}\boldsymbol{u}^{j}|)^{p-2}\big]\mathcal{D}\boldsymbol{u}^{j}\big\|_{2}\\
&\quad\le (2-p)\|\mathcal{D}\boldsymbol{P}^{j}\|_2\|\mathcal{D}\boldsymbol{u}^{j}\|_2\\
&\quad\le (2-p)\|\mathcal{D}\boldsymbol{P}^{j}\|_2\|\boldsymbol{u}^{j}\|_{C^{1,\gamma_0}}.
\end{aligned}$$

Finally, using Lemma 2.1

$$\begin{aligned}
&\|\mu_r\,\mathrm{rot}\,\boldsymbol{R}^j-(\boldsymbol{P}^j\cdot\nabla)\boldsymbol{u}^j-(\boldsymbol{u}^{j-1}\cdot\nabla)\boldsymbol{P}^j\|_{-1,2}\\
&\quad\le \|\mu_r\,\mathrm{rot}\,\boldsymbol{R}^j-(\boldsymbol{P}^j\cdot\nabla)\boldsymbol{u}^j-(\boldsymbol{u}^{j-1}\cdot\nabla)\boldsymbol{P}^j\|_{2}\\
&\quad\le \|\mu_r\,\mathrm{rot}\,\boldsymbol{R}^j\|_2+\|(\boldsymbol{P}^j\cdot\nabla)\boldsymbol{u}^j\|_2+\|(\boldsymbol{u}^{j-1}\cdot\nabla)\boldsymbol{P}^j\|_{2}\\
&\quad\le C\mu_r\|\boldsymbol{R}^j\|_{1,2}+c_1\|\mathcal{D}\boldsymbol{P}^j\|_2\|\boldsymbol{u}^j\|_{C^{1,\gamma_0}}+c_1\|\boldsymbol{u}^{j-1}\|_{C^{1,\gamma_0}}\|\mathcal{D}\boldsymbol{P}^j\|_2.
\end{aligned}$$

Combining all these above and taking into account estimates (3.17) and (3.3), straightforward calculations lead to

$$\begin{aligned}
\|Q^{j+1}\|_2&\le C\|\nabla Q^{j+1}\|_{-1,2}\\
&\le C\big[\|\mathcal{D}\boldsymbol{P}^{j+1}\|_2+(2-p)\|\mathcal{D}\boldsymbol{P}^j\|_2\|\boldsymbol{u}^j\|_{C^{1,\gamma_0}}+C\mu_r\|\boldsymbol{R}^j\|_{1,2}+c_1\|\mathcal{D}\boldsymbol{P}^j\|_2\|\boldsymbol{u}^j\|_{C^{1,\gamma_0}}\\
&\qquad+c_1\|\boldsymbol{u}^{j-1}\|_{C^{1,\gamma_0}}\|\mathcal{D}\boldsymbol{P}^j\|_2\big]\\
&\le C\big[(2-p+2c_1+2c_1^2)(2\tilde{c_0}\|\boldsymbol{f}\|_q+2c_0\|\boldsymbol{g}\|_2)+C(c_1+1)\mu_r\big]\\
&\qquad\cdot(1+2\tilde{c_0}\|\boldsymbol{f}\|_q+2c_0\|\boldsymbol{g}\|_2)^{2-p}\cdot(\|\mathcal{D}\boldsymbol{P}^j\|_2+\|\boldsymbol{R}^j\|_{1,2}).
\end{aligned}$$

Hence, using again the bound (3.16), we can state that there exists a function $Q(x)\in L^2(\Omega)$ to which the series $\sum_j Q^j(x)$ converges in the L^2-norm. □

3.3 Existence results

Set $\|\boldsymbol{f}\|_q\le\delta_1, \|\boldsymbol{g}\|_2\le\delta_2, \mu_r<\delta_3$, where $\delta_1,\delta_2,\delta_3$ are small enough to meet the requirements of Proposition 3.1 and Proposition 3.2, where $\delta_3=\min\{\frac{1}{C\tilde{c_0}+c_0(C+1)},\frac{1}{C(1+c_1)}\}$. Since the sequences $\{\boldsymbol{u}^m\},\{\eta^m\},\{\boldsymbol{w}^m\}$ constructed in Proposition 3.1 satisfy the following relations

$$\boldsymbol{u}^m(x)=\sum_{j=1}^m\boldsymbol{P}^j(x)+\boldsymbol{u}^0(x),\quad \eta^m(x)=\sum_{j=1}^m Q^j(x)+\eta^0(x),\quad \boldsymbol{w}^m(x)=\sum_{j=1}^m\boldsymbol{R}^j(x)+\boldsymbol{w}^0(x),$$

setting $\boldsymbol{u}(x)=\boldsymbol{P}(x)+\boldsymbol{u}^0(x)$, $\eta(x)=Q(x)+\eta^0(x)$, $\boldsymbol{w}(x)=\boldsymbol{R}(x)+\boldsymbol{w}^0(x)$ with $\boldsymbol{P}(x),Q(x)$ and $\boldsymbol{R}(x)$ as in Proposition 3.2, the sequences $\{\boldsymbol{u}^m(x)\},\{\eta^m(x)\}$ and $\{\boldsymbol{w}^m(x)\}$ converge to the functions $\boldsymbol{u}(x),\eta(x)$ and $\boldsymbol{w}(x)$ respectively in the $W^{1,2},L^2$ and $W^{1,2}-$norms. On the other hand, recalling Proposition 3.1, by Arzelà–Ascoli theorem, there exists a subsequence $\{\boldsymbol{u}^{k_m}\}$ converging in $C^{1,\gamma}(\bar\Omega)$, hence in $W^{1,2}(\Omega)$, to a function $\tilde{\boldsymbol{u}}$. Since all the sequence $\{\boldsymbol{u}^m\}$ converges to $\boldsymbol{u}$ in $W^{1,2}(\Omega)$, then $\boldsymbol{u}=\tilde{\boldsymbol{u}}$. In the same way one can prove that $\eta\in C^{0,\gamma}(\bar\Omega)$ and $\boldsymbol{w}\in W^{2,2}(\Omega)$. Since $\nabla\cdot\boldsymbol{u}^{k_m}=0$, $\boldsymbol{u}^{k_m}|_{\partial\Omega}=0$ and $\boldsymbol{w}^{k_m}|_{\partial\Omega}=0$, for any $m\in\mathbb{N}$, it follows that $\nabla\cdot\boldsymbol{u}=0, \boldsymbol{u}|_{\partial\Omega}=0$ and $\boldsymbol{w}|_{\partial\Omega}=0$.

Let us prove that

$$\begin{aligned}&\int_\Omega (1+|\mathcal{D}\boldsymbol{u}|)^{p-2}\mathcal{D}\boldsymbol{u}\mathcal{D}\varphi dx+\int_\Omega(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\varphi dx-\int_\Omega \eta \operatorname{div}\varphi dx-\int_\Omega \mu_r \operatorname{rot}\boldsymbol{w}\varphi dx\\ &\quad=\lim_{m\to\infty}\Big\{\int_\Omega(1+|\mathcal{D}\boldsymbol{u}^{m-1}|)^{p-2}\mathcal{D}\boldsymbol{u}^m\mathcal{D}\varphi dx+\int_\Omega(\boldsymbol{u}^{m-1}\cdot\nabla)\boldsymbol{u}^{m-1}\varphi dx\\ &\qquad-\int_\Omega \eta^m\operatorname{div}\varphi dx-\int_\Omega\mu_r\operatorname{rot}\boldsymbol{w}^{m-1}\varphi dx\Big\},\quad \text{for all } \varphi\in C_0^\infty(\Omega),\end{aligned}\tag{3.22}$$

and

$$\begin{aligned}&\int_\Omega \operatorname{div}\boldsymbol{w}\operatorname{div}\psi dx+\int_\Omega\nabla\boldsymbol{w}\nabla\psi dx+\int_\Omega(\boldsymbol{u}\cdot\nabla)\boldsymbol{w}\psi dx-\int_\Omega\mu_r\operatorname{rot}\boldsymbol{u}\psi dx+\int_\Omega\mu_r\boldsymbol{w}\psi dx\\ &\quad=\lim_{m\to\infty}\Big\{\int_\Omega\operatorname{div}\boldsymbol{w}^m\operatorname{div}\psi dx+\int_\Omega\nabla\boldsymbol{w}^m\nabla\psi dx+\int_\Omega(\boldsymbol{u}^{m-1}\cdot\nabla)\boldsymbol{w}^{m-1}\psi dx\\ &\qquad-\int_\Omega\mu_r\operatorname{rot}\boldsymbol{u}^{m-1}\psi dx+\int_\Omega\mu_r\boldsymbol{w}^{m-1}\psi dx\Big\},\quad\text{for all } \psi\in C_0^\infty(\Omega).\end{aligned}\tag{3.23}$$

Firstly, by using the Hölder inequality we get

$$\begin{aligned}&\Big|\int_\Omega(1+|\mathcal{D}\boldsymbol{u}^{m-1}|)^{p-2}\mathcal{D}\boldsymbol{u}^m\mathcal{D}\varphi dx-\int_\Omega(1+|\mathcal{D}\boldsymbol{u}|)^{p-2}\mathcal{D}\boldsymbol{u}\mathcal{D}\varphi dx\Big|\\ &\quad=\Big|\int_\Omega(1+|\mathcal{D}\boldsymbol{u}^{m-1}|)^{p-2}(\mathcal{D}\boldsymbol{u}^m-\mathcal{D}\boldsymbol{u})\mathcal{D}\varphi dx\\ &\qquad+\int_\Omega[(1+|\mathcal{D}\boldsymbol{u}^{m-1}|)^{p-2}-(1+|\mathcal{D}\boldsymbol{u}|)^{p-2}]\mathcal{D}\boldsymbol{u}\mathcal{D}\varphi dx\Big|\\ &\quad\le\|\mathcal{D}\boldsymbol{u}^m-\mathcal{D}\boldsymbol{u}\|_2\|\mathcal{D}\varphi\|_2+(2-p)\|\mathcal{D}\boldsymbol{u}^{m-1}-\mathcal{D}\boldsymbol{u}\|_2\|\mathcal{D}\boldsymbol{u}\|_2\|\mathcal{D}\varphi\|_\infty,\end{aligned}$$

$$\begin{aligned}&\Big|\int_\Omega(\boldsymbol{u}^{m-1}\cdot\nabla)\boldsymbol{u}^{m-1}\varphi dx-\int_\Omega(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\varphi dx\Big|\\ &\quad=\Big|\int_\Omega[(\boldsymbol{u}^{m-1}-\boldsymbol{u})\cdot\nabla]\boldsymbol{u}^{m-1}\varphi dx+\int_\Omega(\boldsymbol{u}\cdot\nabla)(\boldsymbol{u}^{m-1}-\boldsymbol{u})\varphi dx\Big|\\ &\quad\le\|\boldsymbol{u}^{m-1}-\boldsymbol{u}\|_2\|\nabla\boldsymbol{u}^{m-1}\|_2\|\varphi\|_\infty+\|\boldsymbol{u}\|_2\|\nabla\boldsymbol{u}^{m-1}-\nabla\boldsymbol{u}\|_2\|\varphi\|_\infty,\end{aligned}$$

$$\Big|\int_\Omega\eta^m\operatorname{div}\varphi dx-\int_\Omega\eta\operatorname{div}\varphi dx\Big|=\Big|\int_\Omega(\eta^m-\eta)\operatorname{div}\varphi dx\Big|\le\|\eta^m-\eta\|_2\|\operatorname{div}\varphi\|_2,$$

$$\begin{aligned}\Big|\int_\Omega\mu_r\operatorname{rot}\boldsymbol{w}^{m-1}\varphi dx-\int_\Omega\mu_r\operatorname{rot}\boldsymbol{w}\varphi dx\Big|&\le\mu_r\|\operatorname{rot}\boldsymbol{w}^{m-1}-\operatorname{rot}\boldsymbol{w}\|_2\|\varphi\|_2\\ &\le C\mu_r\|\nabla\boldsymbol{w}^{m-1}-\nabla\boldsymbol{w}\|_2\|\varphi\|_2,\end{aligned}$$

and such quantities tend to zero as m goes to infinity, thanks to the $W^{1,2}$ convergence of $\boldsymbol{u}^m$, $\boldsymbol{w}^m$, the L^2 convergence of η^m and the boundedness of the norms $\|\mathcal{D}\boldsymbol{u}\|_2$, $\|\mathcal{D}\varphi\|_\infty$, $\|\mathcal{D}\varphi\|_2$, $\|\nabla\boldsymbol{u}^{m-1}\|_2$, $\|\varphi\|_\infty$, $\|\boldsymbol{u}\|_2$, $\|\varphi\|_2$. Observing that the right-hand side of (3.22) is equal to $\int_\Omega \boldsymbol{f}\varphi dx$, we have that

$$\int_\Omega(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\varphi dx+\int_\Omega(1+|\mathcal{D}\boldsymbol{u}|)^{p-2}\mathcal{D}\boldsymbol{u}\mathcal{D}\varphi dx-\int_\Omega\eta\nabla\cdot\varphi dx=\int_\Omega\mu_r\operatorname{rot}\boldsymbol{w}\varphi dx+\int_\Omega\boldsymbol{f}\varphi dx,$$

for any $\varphi \in C_0^\infty(\Omega)$.

Secondly, we have

$$\begin{aligned}
&\left| \int_\Omega \operatorname{div} \boldsymbol{w}^m \operatorname{div} \psi dx + \int_\Omega \nabla \boldsymbol{w}^m \nabla \psi dx - \int_\Omega \operatorname{div} \boldsymbol{w} \operatorname{div} \psi dx - \int_\Omega \nabla \boldsymbol{w} \nabla \psi dx \right| \\
&\quad = \left| \int_\Omega (\operatorname{div} \boldsymbol{w}^m - \operatorname{div} \boldsymbol{w}) \operatorname{div} \psi dx + \int_\Omega (\nabla \boldsymbol{w}^m - \nabla \boldsymbol{w}) \nabla \psi dx \right| \\
&\quad \le C \|\nabla \boldsymbol{w}^m - \nabla \boldsymbol{w}\|_2 \|\nabla \psi\|_2 + \|\nabla \boldsymbol{w}^m - \nabla \boldsymbol{w}\|_2 \|\nabla \psi\|_2, \\
&\left| \int_\Omega (\boldsymbol{u}^{m-1} \cdot \nabla) \boldsymbol{w}^{m-1} \psi dx - \int_\Omega (\boldsymbol{u} \cdot \nabla) \boldsymbol{w} \psi dx \right| \\
&\quad = \left| \int_\Omega [(\boldsymbol{u}^{m-1} - \boldsymbol{u}) \cdot \nabla] \boldsymbol{w}^{m-1} \psi dx + \int_\Omega (\boldsymbol{u} \cdot \nabla)(\boldsymbol{w}^{m-1} - \boldsymbol{w}) \psi dx \right| \\
&\quad \le \|\boldsymbol{u}^{m-1} - \boldsymbol{u}\|_2 \|\nabla \boldsymbol{w}^{m-1}\|_2 \|\psi\|_\infty + \|\boldsymbol{u}\|_2 \|\nabla \boldsymbol{w}^{m-1} - \nabla \boldsymbol{w}\|_2 \|\psi\|_\infty,
\end{aligned}$$

$$\begin{aligned}
\left| \int_\Omega \mu_r \operatorname{rot} \boldsymbol{u}^{m-1} \psi dx - \int_\Omega \mu_r \operatorname{rot} \boldsymbol{u} \psi dx \right| &\le \mu_r \| \operatorname{rot} \boldsymbol{u}^{m-1} - \operatorname{rot} \boldsymbol{u}\|_2 \|\varphi\|_2 \\
&\le C\mu_r \|\nabla \boldsymbol{u}^{m-1} - \nabla \boldsymbol{u}\|_2 \|\varphi\|_2, \\
\left| \int_\Omega \mu_r \boldsymbol{w}^{m-1} \psi dx - \int_\Omega \mu_r \boldsymbol{w} \psi dx \right| &\le \mu_r \|\boldsymbol{w}^{m-1} - \boldsymbol{w}\|_2 \|\varphi\|_2 \\
&\le \mu_r \|\boldsymbol{w}^{m-1} - \boldsymbol{w}\|_{2,2} \|\varphi\|_2.
\end{aligned}$$

Similarly as above, we get

$$\int_\Omega (\boldsymbol{u} \cdot \nabla) \boldsymbol{w} \psi dx + \int_\Omega \operatorname{div} \boldsymbol{w} \operatorname{div} \psi dx + \int_\Omega \nabla \boldsymbol{w} \nabla \psi dx + \int_\Omega \mu_r \boldsymbol{w} \psi dx = \int_\Omega \mu_r \operatorname{rot} \boldsymbol{u} \psi dx + \int_\Omega \boldsymbol{g} \psi dx,$$

for any $\psi \in C_0^\infty(\Omega)$. By Definition 1.1 and Remark 1.2, we know $(\boldsymbol{u}, \eta, \boldsymbol{w})$ is a solution of problem (1.1)–(1.2).

Finally, passing to the limit in the following estimate and by the lower semi-continuity of the norms, we get

$$\begin{aligned}
\|\boldsymbol{u}\|_{C^{1,\gamma}} + \|\eta\|_{C^{0,\gamma}} + \|\boldsymbol{w}\|_{2,2} &\le \|\boldsymbol{u} - \boldsymbol{u}^{k_m}\|_{C^{1,\gamma}} + \|\boldsymbol{u}^{k_m}\|_{C^{1,\gamma}} + \|\eta - \eta^{k_m}\|_{C^{0,\gamma}} + \|\eta^{k_m}\|_{C^{0,\gamma}} + \|\boldsymbol{w}^{k_m}\|_{2,2} \\
&\le 2(\tilde{c}_0 \|\boldsymbol{f}\|_q + c_0 \|\boldsymbol{g}\|_2).
\end{aligned}$$

3.4 Uniqueness results

Assume that $(\boldsymbol{u}_1, \boldsymbol{w}_1)$ and $(\boldsymbol{u}_2, \boldsymbol{w}_2)$ are two solutions of problem (1.1)–(1.2). Let $\bar{\boldsymbol{u}} = \boldsymbol{u}_1 - \boldsymbol{u}_2$, $\bar{\boldsymbol{w}} = \boldsymbol{w}_1 - \boldsymbol{w}_2$. Using Definition 1.1, we bring $(\boldsymbol{u}_1, \boldsymbol{w}_1), (\boldsymbol{u}_2, \boldsymbol{w}_2)$ into (1.3) and subtract one from the other, then test with $\boldsymbol{\varphi} = \bar{\boldsymbol{u}} = \boldsymbol{u}_1 - \boldsymbol{u}_2 \in V_q(\Omega)$, we get

$$\int_\Omega \left[S(\mathcal{D}\boldsymbol{u}_1) - S(\mathcal{D}\boldsymbol{u}_2) \right] \cdot (\mathcal{D}\boldsymbol{u}_1 - \mathcal{D}\boldsymbol{u}_2) dx = \int_\Omega \mu_r \operatorname{rot} \bar{\boldsymbol{w}} \bar{\boldsymbol{u}} dx + \int_\Omega (\bar{\boldsymbol{u}} \cdot \nabla) \bar{\boldsymbol{u}} \boldsymbol{u}_1 dx.$$

Using the Hölder inequality, we can write

$$\begin{aligned}
\|\mathcal{D}\bar{\boldsymbol{u}}\|_p^p &= \int_\Omega \left(\frac{|\mathcal{D}\bar{\boldsymbol{u}}|^2}{(1 + |\mathcal{D}\boldsymbol{u}_1| + |\mathcal{D}\boldsymbol{u}_2|)^{2-p}} \right)^{\frac{p}{2}} \cdot (1 + |\mathcal{D}\boldsymbol{u}_1| + |\mathcal{D}\boldsymbol{u}_2|)^{\frac{p(2-p)}{2}} dx \\
&\le \left(\int_\Omega \frac{|\mathcal{D}\bar{\boldsymbol{u}}|^2}{(1 + |\mathcal{D}\boldsymbol{u}_1| + |\mathcal{D}\boldsymbol{u}_2|)^{2-p}} dx \right)^{\frac{p}{2}} \cdot \left[\int_\Omega (1 + |\mathcal{D}\boldsymbol{u}_1| + |\mathcal{D}\boldsymbol{u}_2|)^p dx \right]^{\frac{2-p}{2}}.
\end{aligned}$$

Hence, recalling Lemma 2.4, we obtain

$$\begin{aligned}\|\mathcal{D}\bar{\boldsymbol{u}}\|_p^2 \leq& \int_\Omega \frac{|\mathcal{D}\bar{\boldsymbol{u}}|^2}{(1+|\mathcal{D}\boldsymbol{u}_1|+|\mathcal{D}\boldsymbol{u}_2|)^{2-p}}dx \cdot \left[\int_\Omega (1+|\mathcal{D}\boldsymbol{u}_1|+|\mathcal{D}\boldsymbol{u}_2|)^p dx\right]^{\frac{2-p}{p}} \\ \leq& C\Big(\int_\Omega \big(S(\mathcal{D}\boldsymbol{u}_1)-S(\mathcal{D}\boldsymbol{u}_2)\big)\cdot(\mathcal{D}\boldsymbol{u}_1-\mathcal{D}\boldsymbol{u}_2)dx\Big)\cdot\Big(1+\|\mathcal{D}\boldsymbol{u}_1\|_p^{2-p}+\|\mathcal{D}\boldsymbol{u}_2\|_p^{2-p}\Big) \\ \leq& C\Big(\int_\Omega \mu_r \operatorname{rot}\bar{\boldsymbol{w}}\bar{\boldsymbol{u}}dx+\int_\Omega(\bar{\boldsymbol{u}}\cdot\nabla)\bar{\boldsymbol{u}}\boldsymbol{u}_1dx\Big)\cdot(1+\|\mathcal{D}\boldsymbol{u}_1\|_p+\|\mathcal{D}\boldsymbol{u}_2\|_p).\end{aligned}$$

Using Hölder's and Sobolev's inequality, we have

$$\begin{aligned}\Big|\int_\Omega \mu_r \operatorname{rot}\bar{\boldsymbol{w}}\bar{\boldsymbol{u}}dx+\int_\Omega(\bar{\boldsymbol{u}}\cdot\nabla)\bar{\boldsymbol{u}}\boldsymbol{u}_1dx\Big| \leq& \|\mu_r\operatorname{rot}\bar{\boldsymbol{w}}\|_{\frac{2p}{3p-2}}\|\bar{\boldsymbol{u}}\|_{\frac{2p}{2-p}}+C\|\nabla\boldsymbol{u}_1\|_{\frac{p}{2p-2}}\|\bar{\boldsymbol{u}}\|^2_{\frac{2p}{2-p}} \\ \leq& C\mu_r\|\bar{\boldsymbol{w}}\|_{1,2}\|\mathcal{D}\bar{\boldsymbol{u}}\|_p+C\|\boldsymbol{u}_1\|_{C^{1,\gamma_0}}\|\mathcal{D}\bar{\boldsymbol{u}}\|_p^2.\end{aligned}$$

So we get

$$\|\mathcal{D}\bar{\boldsymbol{u}}\|_p \leq C\big(\mu_r\|\bar{\boldsymbol{w}}\|_{1,2}+\|\boldsymbol{u}_1\|_{C^{1,\gamma_0}}\|\mathcal{D}\bar{\boldsymbol{u}}\|_p\big)\cdot\big(1+\|\mathcal{D}\boldsymbol{u}_1\|_p+\|\mathcal{D}\boldsymbol{u}_2\|_p\big). \tag{3.24}$$

Inserting $(\boldsymbol{u}_1,\boldsymbol{w}_1),(\boldsymbol{u}_2,\boldsymbol{w}_2)$ into (1.4) and subtract one from the other, then test with $\boldsymbol{\psi}=\bar{\boldsymbol{w}}=\boldsymbol{w}_1-\boldsymbol{w}_2\in W^{1,2}(\Omega)$, we get

$$\begin{aligned}\int_\Omega|\operatorname{div}\bar{\boldsymbol{w}}|^2dx+\int_\Omega|\nabla\bar{\boldsymbol{w}}|^2dx =& \int_\Omega\mu_r\operatorname{rot}\bar{\boldsymbol{u}}\bar{\boldsymbol{w}}dx-\int_\Omega\mu_r|\bar{\boldsymbol{w}}|^2dx-\int_\Omega(\bar{\boldsymbol{u}}\cdot\nabla)\boldsymbol{w}_1\bar{\boldsymbol{w}}dx \\ &-\int_\Omega(\boldsymbol{u}_2\cdot\nabla)\bar{\boldsymbol{w}}\bar{\boldsymbol{w}}dx.\end{aligned}$$

Since

$$\int_\Omega|\operatorname{div}\bar{\boldsymbol{w}}|^2dx+\int_\Omega|\nabla\bar{\boldsymbol{w}}|^2dx\geq C\|\bar{\boldsymbol{w}}\|_{1,2}^2,$$

and

$$\begin{aligned}&\Big|\int_\Omega\mu_r\operatorname{rot}\bar{\boldsymbol{u}}\bar{\boldsymbol{w}}dx-\int_\Omega\mu_r|\bar{\boldsymbol{w}}|^2dx-\int_\Omega(\bar{\boldsymbol{u}}\cdot\nabla)\boldsymbol{w}_1\bar{\boldsymbol{w}}dx-\int_\Omega(\boldsymbol{u}_2\cdot\nabla)\bar{\boldsymbol{w}}\bar{\boldsymbol{w}}dx\Big| \\ &\leq\|\mu_r\operatorname{rot}\bar{\boldsymbol{u}}\|_p\|\bar{\boldsymbol{w}}\|_{\frac{p}{p-1}}+\mu_r\|\bar{\boldsymbol{w}}\|_2^2+\|\bar{\boldsymbol{u}}\|_{\frac{2p}{2-p}}\|\nabla\boldsymbol{w}_1\|_{\frac{2p}{3p-2}}\|\bar{\boldsymbol{w}}\|_\infty+\|\boldsymbol{u}_2\|_\infty\|\nabla\bar{\boldsymbol{w}}\|_2\|\bar{\boldsymbol{w}}\|_2 \\ &\leq C\mu_r\|\mathcal{D}\bar{\boldsymbol{u}}\|_p\|\bar{\boldsymbol{w}}\|_{1,2}+\mu_r\|\bar{\boldsymbol{w}}\|_{1,2}^2+C\|\mathcal{D}\bar{\boldsymbol{u}}\|_p\|\boldsymbol{w}_1\|_{2,2}\|\bar{\boldsymbol{w}}\|_{1,2}+\|\boldsymbol{u}_2\|_{C^{1,\gamma_0}}\|\bar{\boldsymbol{w}}\|_{1,2}^2,\end{aligned}$$

we get

$$\|\bar{\boldsymbol{w}}\|_{1,2}\leq C\mu_r\|\mathcal{D}\bar{\boldsymbol{u}}\|_p+C\mu_r\|\bar{\boldsymbol{w}}\|_{1,2}+C\|\mathcal{D}\bar{\boldsymbol{u}}\|_p\|\boldsymbol{w}_1\|_{2,2}+C\|\boldsymbol{u}_2\|_{C^{1,\gamma_0}}\|\bar{\boldsymbol{w}}\|_{1,2}. \tag{3.25}$$

Combining (3.24) and (3.25), we finally obtain

$$\begin{aligned}\|\mathcal{D}\bar{\boldsymbol{u}}\|_p+\|\bar{\boldsymbol{w}}\|_{1,2}\leq&\ \big[C\|\boldsymbol{u}_1\|_{C^{1,\gamma_0}}(1+\|\mathcal{D}\boldsymbol{u}_1\|_p+\|\mathcal{D}\boldsymbol{u}_2\|_p)+C\mu_r+C\|\boldsymbol{w}_1\|_{2,2}\big]\cdot\|\mathcal{D}\bar{\boldsymbol{u}}\|_p \\ &+\big[C\mu_r(1+\|\mathcal{D}\boldsymbol{u}_1\|_p+\|\mathcal{D}\boldsymbol{u}_2\|_p)+C\mu_r+C\|\boldsymbol{u}_2\|_{C^{1,\gamma_0}}\big]\cdot\|\bar{\boldsymbol{w}}\|_{1,2} \\ \leq&\ C(2\tilde{c_0}\|\boldsymbol{f}\|_q+2c_0\|\boldsymbol{g}\|_2+\mu_r)\cdot(1+4\tilde{c_0}\|\boldsymbol{f}\|_q+4c_0\|\boldsymbol{g}\|_2)\cdot(\|\mathcal{D}\bar{\boldsymbol{u}}\|_p+\|\bar{\boldsymbol{w}}\|_{1,2}).\end{aligned}$$

So if $2\tilde{c_0}\|\boldsymbol{f}\|_q+2c_0\|\boldsymbol{g}\|_2+\mu_r$ is sufficiently small, the uniqueness follows.

4 Proof of Theorem 1.4

Throughout the proof we assume that $\|f\|_q \le \delta_1$, $\|g\|_2 \le \delta_2$, $\mu_r < \delta_3$, in this way all the hypotheses of Theorem 3.1 are satisfied and we can find the sequences $\{u^m\}, \{\eta^m\}, \{w^m\}$, as in Proposition 3.2, converging to the solution (u, η, w). In order to get $D^2 u \in L^2(\Omega)$ we proceed by induction on m. Firstly, we have (see [5, Theorem 3.2])

$$\|u^0\|_{2,2} + \|\eta^0\|_{1,2} \le \tilde{c_1}(\|u^0\|_{1,2} + \|f\|_2).$$

Since $\|u^0\|_{1,2} \le c_1^2 \|f\|_2$, there follows $\|u^0\|_{2,2} + \|\eta^0\|_{1,2} \le \tilde{c_1}(1 + c_1^2)\|f\|_2$, it implies that $\|D^2 u^0\|_2 \le \tilde{c_1}(1 + c_1^2)\|f\|_2$. We assume that $D^2 u^m \in L^2(\Omega)$ and we go forward with the step $m + 1$. Let us consider the following boundary-value problem

$$\begin{cases} -\mathrm{div}\big[\big(1 + J_\varepsilon(|\mathcal{D}u^m|)\big)^{p-2}\mathcal{D}u_\varepsilon^{m+1}\big] + \nabla\eta_\varepsilon^{m+1} = \mu_r \,\mathrm{rot}\, w^m + f - (u^m \cdot \nabla)u^m, & \text{in } \Omega, \\ \mathrm{div}\, u_\varepsilon^{m+1} = 0, & \text{in } \Omega, \\ u_\varepsilon^{m+1}|_{\partial\Omega} = 0, \end{cases} \tag{4.1}$$

where J_ε denotes the Friedrichs mollifier. Since $\mu_r \,\mathrm{rot}\, w^m + f - (u^m \cdot \nabla)u^m \in L^q(\Omega)$, then, as in (3.10), there exists a solution $(u_\varepsilon^{m+1}, \eta_\varepsilon^{m+1}) \in C^{1,\gamma_0}(\bar{\Omega}) \times C^{0,\gamma_0}(\bar{\Omega})$ and satisfies

$$\begin{aligned} &\|u_\varepsilon^{m+1}\|_{C^{1,\gamma_0}} + \|\eta_\varepsilon^{m+1}\|_{C^{0,\gamma_0}} \\ &\quad \le c\big(1 + \|J_\varepsilon(|\mathcal{D}u^m|)\|_{C^{0,\gamma_0}}\big)^r \cdot \big(\|u_\varepsilon^{m+1}\|_{1,2} + \|\mu_r \,\mathrm{rot}\, w^m\|_q + \|f\|_q + \|(u^m \cdot \nabla)u^m\|_q\big). \end{aligned}$$

Since

$$\begin{aligned} \|u_\varepsilon^{m+1}\|_{1,2} &\le c_1^2\big(1 + \|J_\varepsilon(|\mathcal{D}u^m|)\|_{C^{0,\gamma_0}}\big)^{2-p} \cdot (C\mu_r\|w^m\|_{2,2} + \|f\|_q + \|u^m\|^2_{C^{1,\gamma_0}}), \\ \|J_\varepsilon(|\mathcal{D}u^m|)\|_{C^{0,\gamma_0}} &\le \|Du^m\|_{C^{0,\gamma_0}} \le \|u^m\|_{C^{1,\gamma_0}}, \end{aligned}$$

there follows

$$\begin{aligned} &\|u_\varepsilon^{m+1}\|_{C^{1,\gamma_0}} + \|\eta_\varepsilon^{m+1}\|_{C^{0,\gamma_0}} \\ &\quad \le c(1 + c_1^2)\big(1 + \|u^m\|_{C^{1,\gamma_0}}\big)^{r+2-p} \cdot (C\mu_r\|w^m\|_{2,2} + \|f\|_q + \|u^m\|^2_{C^{1,\gamma_0}}) \\ &\quad \le c(1 + c_1^2)\big(1 + 2\tilde{c_0}\|f\|_q + 2c_0\|g\|_2\big)^{r+2-p} \cdot \big[C\mu_r(2\tilde{c_0}\|f\|_q + 2c_0\|g\|_2) \\ &\qquad + \|f\|_q + (2\tilde{c_0}\|f\|_q + 2c_0\|g\|_2)^2\big]. \end{aligned} \tag{4.2}$$

Further, by Theorem 3.2 in [5], we have $(u_\varepsilon^{m+1}, \eta_\varepsilon^{m+1}) \in W^{2,2}(\Omega) \times W^{1,2}(\Omega)$. Next, we concentrate on deriving the corresponding estimates which are uniform in ε.

Multiply $(4.1)_1$ by $\Delta u_\varepsilon^{m+1}$ and integrate on $\Omega_\eta = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \eta\}$, for some $\varepsilon < \eta$, We get

$$\begin{aligned} &\frac{1}{2}\int_{\Omega_\eta} \big(1 + J_\varepsilon(|\mathcal{D}u^m|)\big)^{p-2}|\Delta u_\varepsilon^{m+1}|^2 dx \\ &\quad = \int_{\Omega_\eta} (2-p)\big(1 + J_\varepsilon(|\mathcal{D}u^m|)\big)^{p-3}\mathcal{D}u_\varepsilon^{m+1}\big(\Delta u_\varepsilon^{m+1} \otimes \nabla J_\varepsilon(|\mathcal{D}u^m|)\big)dx + \int_{\Omega_\eta} \nabla\eta_\varepsilon^{m+1}\Delta u_\varepsilon^{m+1} dx \\ &\qquad - \int_{\Omega_\eta} \mu_r \,\mathrm{rot}\, w^m \Delta u_\varepsilon^{m+1} dx - \int_{\Omega_\eta} f\Delta u_\varepsilon^{m+1} dx + \int_{\Omega_\eta} (u^m \cdot \nabla)u^m \Delta u_\varepsilon^{m+1} dx = \sum_{i=1}^5 H_i. \end{aligned}$$

Since

$$\|\nabla J_\varepsilon(|\mathcal{D}u^m|)\|_{2,\Omega_\eta} = \|J_\varepsilon(\nabla|\mathcal{D}u^m|)\|_{2,\Omega_\eta} \le \|\nabla|\mathcal{D}u^m|\|_{2,\Omega_\eta},$$

there follows

$$|H_1| \le \|\mathcal{D}\boldsymbol{u}_\varepsilon^{m+1}\|_{\infty,\Omega_\eta}\|\Delta\boldsymbol{u}_\varepsilon^{m+1}\|_{2,\Omega_\eta}\|\nabla J_\varepsilon(|\mathcal{D}\boldsymbol{u}^m|)\|_{2,\Omega_\eta}$$
$$\le \|\boldsymbol{u}_\varepsilon^{m+1}\|_{C^{1,\gamma_0}}\|\Delta\boldsymbol{u}_\varepsilon^{m+1}\|_{2,\Omega_\eta}\|\nabla|\mathcal{D}\boldsymbol{u}^m|\|_{2,\Omega_\eta}.$$

By using the divergence theorem

$$|H_2| = \Big|\int_{\partial\Omega_\eta}\eta_\varepsilon^{m+1}\Delta\boldsymbol{u}_\varepsilon^{m+1}\cdot\boldsymbol{n}d\sigma\Big| \le \|\eta_\varepsilon^{m+1}\|_{W^{\frac{1}{2},2}(\partial\Omega_\eta)}\|\Delta\boldsymbol{u}_\varepsilon^{m+1}\cdot\boldsymbol{n}\|_{W^{-\frac{1}{2},2}(\partial\Omega_\eta)}.$$

Since $\gamma_0 = 1-\frac{2}{q}$, $q>4$, there follows $\gamma_0 > \frac{1}{2}$, then $\|\eta_\varepsilon^{m+1}\|_{W^{\frac{1}{2},2}(\partial\Omega_\eta)} \le C\|\eta_\varepsilon^{m+1}\|_{C^{0,\gamma_0}}$ (see [14]). $\|\Delta\boldsymbol{u}_\varepsilon^{m+1}\cdot\boldsymbol{n}\|_{W^{-\frac{1}{2},2}(\partial\Omega_\eta)} \le \|\Delta\boldsymbol{u}_\varepsilon^{m+1}\|_{2,\Omega_\eta} + \|\nabla\cdot\Delta\boldsymbol{u}_\varepsilon^{m+1}\|_{2,\Omega_\eta} = \|\Delta\boldsymbol{u}_\varepsilon^{m+1}\|_{2,\Omega_\eta}$ (see [12, Chapter *III*]). Hence

$$|H_2| \le C\|\eta_\varepsilon^{m+1}\|_{C^{0,\gamma_0}}\|\Delta\boldsymbol{u}_\varepsilon^{m+1}\|_{2,\Omega_\eta}.$$

Moreover,

$$|H_3| \le \|\mu_r \operatorname{rot}\boldsymbol{w}^m\|_2\|\Delta\boldsymbol{u}_\varepsilon^{m+1}\|_{2,\Omega_\eta} \le C\mu_r\|\boldsymbol{w}^m\|_{2,2}\|\Delta\boldsymbol{u}_\varepsilon^{m+1}\|_{2,\Omega_\eta},$$
$$|H_4| \le \|\boldsymbol{f}\|_2\|\Delta\boldsymbol{u}_\varepsilon^{m+1}\|_{2,\Omega_\eta} \le \|\boldsymbol{f}\|_q\|\Delta\boldsymbol{u}_\varepsilon^{m+1}\|_{2,\Omega_\eta},$$
$$|H_5| \le \|(\boldsymbol{u}^m\cdot\nabla)\boldsymbol{u}^m\|_2\|\Delta\boldsymbol{u}_\varepsilon^{m+1}\|_{2,\Omega_\eta} \le \|\boldsymbol{u}^m\|^2_{C^{1,\gamma_0}}\|\Delta\boldsymbol{u}_\varepsilon^{m+1}\|_{2,\Omega_\eta},$$

and

$$\Big|\int_{\Omega_\eta}\big(1+J_\varepsilon(|\mathcal{D}\boldsymbol{u}^m|)\big)^{p-2}|\Delta\boldsymbol{u}_\varepsilon^{m+1}|^2dx\Big| \ge \big(1+\|J_\varepsilon(|\mathcal{D}\boldsymbol{u}^m|)\|_{C^{0,\gamma_0}}\big)^{p-2}\|\Delta\boldsymbol{u}_\varepsilon^{m+1}\|^2_{2,\Omega_\eta}$$
$$\ge (1+\|\boldsymbol{u}^m|\|_{C^{1,\gamma_0}})^{p-2}\|\Delta\boldsymbol{u}_\varepsilon^{m+1}\|^2_{2,\Omega_\eta}.$$

The above estimates, recalling (4.2), imply that $\Delta\boldsymbol{u}_\varepsilon^{m+1}\in L^2(\Omega_\eta)$ and

$$\|\Delta\boldsymbol{u}_\varepsilon^{m+1}\|_{2,\Omega_\eta} \le 2(1+\|\boldsymbol{u}^m|\|_{C^{1,\gamma_0}})^{2-p}\cdot\big[\|\boldsymbol{f}\|_q + \|\boldsymbol{u}_\varepsilon^{m+1}\|_{C^{1,\gamma_0}}\|\nabla|\mathcal{D}\boldsymbol{u}^m|\|_{2,\Omega_\eta} + C\|\eta_\varepsilon^{m+1}\|_{C^{0,\gamma_0}}$$
$$+ C\mu_r\|\boldsymbol{w}^m\|_{2,2} + \|\boldsymbol{u}^m\|^2_{C^{1,\gamma_0}}\big]$$
$$\le Cc(1+c_1^2)(1+2\tilde{c_0}\|\boldsymbol{f}\|_q + 2c_0\|\boldsymbol{g}\|_2)^{r+4-2p}\cdot(\|\nabla|\mathcal{D}\boldsymbol{u}^m|\|_{2,\Omega_\eta} + C)$$
$$\cdot\big[C\mu_r(2\tilde{c_0}\|\boldsymbol{f}\|_q + 2c_0\|\boldsymbol{g}\|_2) + \|\boldsymbol{f}\|_q + (2\tilde{c_0}\|\boldsymbol{f}\|_q + 2c_0\|\boldsymbol{g}\|_2)^2\big].$$

Since the previous estimate holds for any $\eta > 0$, we can replace $\|\Delta\boldsymbol{u}_\varepsilon^{m+1}\|_{2,\Omega_\eta}$ with $\|\Delta\boldsymbol{u}_\varepsilon^{m+1}\|_{2,\Omega}$. By the boundedness of $\Delta\boldsymbol{u}_\varepsilon^{m+1}$ in $L^2(\Omega)$, uniformly in ε, we deduce the existence of a subsequence weekly converging in $L^2(\Omega)$.

On the other hand, for any fixed $m\in\mathbb{N}$, $\boldsymbol{u}_\varepsilon^{m+1}$ tends to $\boldsymbol{u}^{m+1}$ in $W^{1,2}(\Omega)$ as $\varepsilon\to 0$. By using the definition of weak solution for $\boldsymbol{u}_\varepsilon^{m+1}$ and $\boldsymbol{u}^{m+1}$ and testing with $\boldsymbol{u}^{m+1}-\boldsymbol{u}_\varepsilon^{m+1}$, we get

$$\int_\Omega\big(1+J_\varepsilon(|\mathcal{D}\boldsymbol{u}^m|)\big)^{p-2}\mathcal{D}\boldsymbol{u}_\varepsilon^{m+1}(\mathcal{D}\boldsymbol{u}^{m+1}-\mathcal{D}\boldsymbol{u}_\varepsilon^{m+1})dx$$
$$= \int_\Omega\big(1+|\mathcal{D}\boldsymbol{u}^m|\big)^{p-2}\mathcal{D}\boldsymbol{u}^{m+1}(\mathcal{D}\boldsymbol{u}^{m+1}-\mathcal{D}\boldsymbol{u}_\varepsilon^{m+1})dx,$$

hence

$$\int_\Omega\big(1+|\mathcal{D}\boldsymbol{u}^m|\big)^{p-2}|\mathcal{D}\boldsymbol{u}^{m+1}-\mathcal{D}\boldsymbol{u}_\varepsilon^{m+1}|^2dx$$
$$= \int_\Omega\big|\big(1+J_\varepsilon(|\mathcal{D}\boldsymbol{u}^m|)\big)^{p-2} - \big(1+|\mathcal{D}\boldsymbol{u}^m|\big)^{p-2}\big|\mathcal{D}\boldsymbol{u}_\varepsilon^{m+1}(\mathcal{D}\boldsymbol{u}^{m+1}-\mathcal{D}\boldsymbol{u}_\varepsilon^{m+1})dx.$$

Since

$$\left|\int_\Omega \big(1+|\mathcal{D}\boldsymbol{u}^m|\big)^{p-2}|\mathcal{D}\boldsymbol{u}^{m+1}-\mathcal{D}\boldsymbol{u}_\varepsilon^{m+1}|^2dx\right| \geq (1+\|\mathcal{D}\boldsymbol{u}^m\|_\infty)^{p-2}\|(\mathcal{D}\boldsymbol{u}^{m+1}-\mathcal{D}\boldsymbol{u}_\varepsilon^{m+1})\|_2^2$$
$$\geq (1+\|\boldsymbol{u}^m\|_{C^{1,\gamma_0}})^{p-2}\|(\mathcal{D}\boldsymbol{u}^{m+1}-\mathcal{D}\boldsymbol{u}_\varepsilon^{m+1})\|_2^2,$$

$$\left|\int_\Omega \big|\big(1+J_\varepsilon(|\mathcal{D}\boldsymbol{u}^m|)\big)^{p-2}-\big(1+|\mathcal{D}\boldsymbol{u}^m|\big)^{p-2}\big|\mathcal{D}\boldsymbol{u}_\varepsilon^{m+1}(\mathcal{D}\boldsymbol{u}^{m+1}-\mathcal{D}\boldsymbol{u}_\varepsilon^{m+1})dx\right|$$
$$\leq (2-p)\|J_\varepsilon(|\mathcal{D}\boldsymbol{u}^m|)-\mathcal{D}\boldsymbol{u}^m\|_2\|\mathcal{D}\boldsymbol{u}_\varepsilon^{m+1}\|_\infty\|\mathcal{D}\boldsymbol{u}^{m+1}-\mathcal{D}\boldsymbol{u}_\varepsilon^{m+1}\|_2$$
$$\leq (2-p)\|J_\varepsilon(|\mathcal{D}\boldsymbol{u}^m|)-\mathcal{D}\boldsymbol{u}^m\|_2\|\boldsymbol{u}_\varepsilon^{m+1}\|_{C^{1,\gamma_0}}\|\mathcal{D}\boldsymbol{u}^{m+1}-\mathcal{D}\boldsymbol{u}_\varepsilon^{m+1}\|_2,$$

there follows

$$\|\mathcal{D}\boldsymbol{u}^{m+1}-\mathcal{D}\boldsymbol{u}_\varepsilon^{m+1}\|_2 \leq (2-p)\cdot(1+\|\boldsymbol{u}^m\|_{C^{1,\gamma_0}})^{2-p}\|J_\varepsilon(|\mathcal{D}\boldsymbol{u}^m|)-\mathcal{D}\boldsymbol{u}^m\|_2\|\boldsymbol{u}_\varepsilon^{m+1}\|_{C^{1,\gamma_0}},$$

we can get $\mathcal{D}\boldsymbol{u}^{m+1}\in L^2(\Omega)$ as $\varepsilon\to 0$.

By using the strong convergence of $\boldsymbol{u}_\varepsilon^{m+1}$ to $\boldsymbol{u}^{m+1}$ in $W^{1,2}(\Omega)$, we also deduce that the limit point of the subsequence of $\Delta\boldsymbol{u}_\varepsilon^{m+1}$ in $L^2(\Omega)$ is $\Delta\boldsymbol{u}^{m+1}$. Since $\|D^2\boldsymbol{u}^{m+1}\|_2\leq C\|\Delta\boldsymbol{u}^{m+1}\|_2$, setting $B=1+2\tilde{c_0}\|\boldsymbol{f}\|_q+2c_0\|\boldsymbol{g}\|_2$, we get

$$\begin{aligned}\|D^2\boldsymbol{u}^{m+1}\|_2 \leq CB^{r+4-2p}\big[C\mu_r(B-1)+\|\boldsymbol{f}\|_q+(B-1)^2\big]\|D^2\boldsymbol{u}^m\|_2\\ +CB^{r+4-2p}\big[C\mu_r(B-1)+\|\boldsymbol{f}\|_q+(B-1)^2\big].\end{aligned}\tag{4.3}$$

Set

$$\begin{aligned}\Phi(z)=CB^{r+4-2p}\big[C\mu_r(B-1)+\|\boldsymbol{f}\|_q+(B-1)^2\big]z\\ +CB^{r+4-2p}\big[C\mu_r(B-1)+\|\boldsymbol{f}\|_q+(B-1)^2\big].\end{aligned}$$

If

$$CB^{r+4-2p}[C\mu_r(B-1)+\|\boldsymbol{f}\|_q+(B-1)^2]<1,\tag{4.4}$$

then there exists $z_0>0$ such that $\Phi(z_0)=z_0$. Let $\bar{m}=\min\{m\in\mathbb{N}:\|D^2\boldsymbol{u}^m\|_2\leq z_0\}$. Assume that $\bar{m}=+\infty$. Since $\Phi(z)<z$ for any $z>z_0$, then, using (4.3) we get

$$\|D^2\boldsymbol{u}^{m+1}\|_2\leq\Phi(\|D^2\boldsymbol{u}^m\|_2)\leq\|D^2\boldsymbol{u}^m\|_2.$$

Therefore $\|D^2\boldsymbol{u}^m\|_2\leq\|D^2\boldsymbol{u}^0\|_2\leq\tilde{c_1}(1+c_1^2)\|\boldsymbol{f}\|_2$, for every $m\in\mathbb{N}$. On the other hand, if $\bar{m}<+\infty$, since $\Phi(z)$ is increasing, for any $m\geq\bar{m}$, we have

$$\|D^2\boldsymbol{u}^{m+1}\|_2\leq\Phi(\|D^2\boldsymbol{u}^m\|_2)\leq\Phi(z_0)=z_0.$$

Hence, by induction, $\|D^2\boldsymbol{u}^m\|_2\leq z_0$ for any $m\geq\bar{m}$. Finally

$$\|D^2\boldsymbol{u}^m\|_2\leq\max\{\tilde{c_1}(1+c_1^2)\|\boldsymbol{f}\|_2,\ z_0\}.$$

By the uniform boundedness of the L^2-norm of $D^2\boldsymbol{u}^m$, using the strong convergence in $W^{1,2}(\Omega)$ of $\boldsymbol{u}^m$ to the solution $\boldsymbol{u}$ of problem (1.1), we deduce that if the condition (4.4), $\|\boldsymbol{f}\|_q\leq\delta_1$, $\|\boldsymbol{g}\|_2\leq\delta_2$, $\mu_r<\delta_3$ are satisfied, then $\boldsymbol{u}\in W^{2,2}(\Omega)$, $\boldsymbol{w}\in W^{2,2}(\Omega)$. By $(1.1)_1$, we have that $\nabla\eta=\mu_r\operatorname{rot}\boldsymbol{w}+\boldsymbol{f}-(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}+\operatorname{div}[(1+|\mathcal{D}\boldsymbol{u}|)^{p-2}\mathcal{D}\boldsymbol{u}]$ in the distribution sense. Observing that the right-hand side of the previous identity belongs to $L^2(\Omega)$, we obtain that $\nabla\eta\in L^2(\Omega)$.

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