



# Solvability of nondensely defined partial functional integrodifferential equations using the integrated resolvent operators

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Received 27 March 2019, appeared 9 December 2019

Communicated by Tibor Krisztin

**Abstract.** In this work, we study the existence and regularity of solutions for a class of nondensely defined partial functional integrodifferential equations. We suppose that the undelayed part admits an integrated resolvent operator in the sense given by Oka [*J. Integral Equations Appl.* 7(1995), 193–232.]. We give some sufficient conditions ensuring the existence, uniqueness and regularity of solutions. The continuous dependence on the initial data of solutions is also proved. Some examples are provided to illustrate our abstract theory.

**Keywords:** partial functional integrodifferential equations, integrated semigroup, integrated resolvent operator, integral solution, strict solution.

**2010 Mathematics Subject Classification:** 34K30, 45K05, 47D62.

## 1 Introduction

The aim of this work is to study the existence and regularity of solutions for the following partial functional integrodifferential equation:

$$\begin{cases} u'(t) = Au(t) + \int_0^t B(t-s)u(s)ds + F(t, u_t) & \text{for } t \geq 0 \\ u_0 = \varphi \in C = C([-r, 0]; X). \end{cases} \quad (1.1)$$

where  $F : \mathbb{R}^+ \times C \rightarrow X$  is a continuous function,  $A$  is not necessarily densely defined linear operator, satisfies the Hille–Yosida condition on a Banach space  $X$ , and  $(B(t))_{t \geq 0}$  is a family linear operators in  $X$  with  $D(A) \subset D(B(t))$  for  $t \geq 0$  and of bounded linear operators from  $D(A)$  into  $X$ .  $C([-r, 0]; X)$  is the space of all continuous functions on  $[-r, 0]$  with values in  $X$ , provided with the uniform norm topology. For  $u \in C([-r, +\infty); X)$  and for every  $t \geq 0$ , the history function  $u_t \in C$  is defined by

$$u_t(\theta) = u(t + \theta) \quad \text{for } \theta \in [-r, 0].$$

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As a model for this class one may take the hyperbolic one-dimensional integrodifferential equation with delay

$$\begin{cases} \frac{\partial}{\partial t} w(t, x) = -a(x) \frac{\partial}{\partial x} w(t, x) + \int_0^t p(t-s, x) \frac{\partial}{\partial x} w(s, x) ds + f(t, w(t-\tau, x)) \\ w(t, 0) = w(t, 1) \text{ for } t \geq 0 \\ w(\theta, x) = w_0(\theta, x) \text{ for } \theta \in [-r, 0] \text{ and } x \in [0, 1], \end{cases} \quad \text{for } t \geq 0 \text{ and } x \in [0, 1]$$

where  $w_0$  is a given initial function,  $a, f$  are continuous functions,  $\tau$  is a positive real number, and  $p \in BV_{\text{loc}}(\mathbb{R}^+, C([0, 1]; \mathbb{R}))$ . Here  $BV_{\text{loc}}(\mathbb{R}^+, C([0, 1]; \mathbb{R}))$  denotes the set of all functions of locally bounded variation from  $\mathbb{R}^+$  to  $C([0, 1]; \mathbb{R})$ .

The theory of partial functional integrodifferential equations has been emerging as an important area of investigation in recent years. In particular, when operator  $A$  generates a strongly continuous semigroup, or equivalently, when a closed linear operator  $A$  satisfies

- (i)  $\overline{D(A)} = X$ ,
- (ii) the Hille–Yosida condition; that is, there exist  $M \geq 0$  and  $w \in \mathbb{R}$  such that  $(w, +\infty) \subset \rho(A)$  and  $|(\lambda I - A)^{-n}| \leq M/(\lambda - w)^n$  for  $\lambda > w$  and  $n \in \mathbb{N}^*$ ,

where  $\rho(A)$  is the resolvent set of  $A$  and  $I$  is the identity operator.

Then Eq. (1.1) has been studied extensively. In this case, the classical resolvent operator theory developed by Grimmer in [15] ensures the well-posedness of Eq. (1.1), namely existence, uniqueness and regularity among other things, are derived; we refer to [9–11, 13, 14] and references therein. We refer also to [19, 23, 24] for the study of Eq. (1.1) when  $B = 0$  using the semigroups theory.

However, in application, there are many examples in concrete situations where operators are not densely defined. Only hypothesis (ii) holds. For example, in the work of Da Prato and Sinestrari [7], the authors studied one-dimensional heat equation with Dirichlet conditions on  $[0, 1]$  and consider  $A = \frac{\partial^2}{\partial x^2}$  in  $C([0, 1], \mathbb{R})$  in order to measure the solutions in the sup-norm, the domain

$$D(A) = \{\phi \in C^2([0, 1], \mathbb{R}) : \phi(0) = \phi(1) = 0\}$$

is not dense in  $C([0, 1], \mathbb{R})$  with the sup-norm. Further examples involving hyperbolic equations can also be found in [17]. One can refer to [4, 5, 16, 18, 21, 22] for more examples and remarks concerning nondensely defined operators.

Recall that, in the case where the operator  $B$  is equal to zero, Eq. (1.1) can be handled by using the classical integrated semigroups (see Ezzinbi et al. in [1–3, 8, 12] for further details). But, if  $B \neq 0$ , the integrated semigroups theory may not be useful to study Eq. (1.1). In [20], Oka considered the following integrodifferential equation

$$\begin{cases} x'(t) = Ax(t) + \int_0^t B(t-s)x(s)ds + q(t) \text{ for } t \geq 0 \\ x(0) = x_0 \in X, \end{cases} \quad (1.2)$$

where  $q : [0, +\infty) \rightarrow X$  is a continuous function. The main hypothesis in [20] was that the operator  $A$  satisfies the Hille–Yosida condition (ii). The author introduce the concept of integrated resolvent operators theory. The principal goals in this theory interact in the

following way. To every integrodifferential equation (1.2) (when  $q = 0$  and some conditions on  $B$ ) he associates a unique integrated resolvent operator. Moreover, a variation of constants formula is obtained to prove some results concerning existence and regularity of solutions to Eq. (1.2).

Motivated by the work of Oka [20], we extend the problem (1.2) to functional type equation. We use the integrated resolvent operators theory to prove existence, uniqueness, regularity and continuous dependence on the initial data of solutions of Eq. (1.1) when the operator  $A$  is nondensely defined. The obtained results generalize the well-known results developed in many papers.

The paper is organized as follows. In Section 2, we recall some basic results concerning integrated semigroups, resolvent operators and integrated resolvent operators theory. In Section 3, we study the existence of integral and strict solutions of Eq. (1.1) by using the integrated resolvent operators. The continuity dependence on the initial data is also established. Finally, in Section 4, we propose applications to illustrate the main results of this work.

## 2 Preliminaries and basic results

In this section, we summarize basic results on integrated semigroups, resolvent operators and integrated resolvent operators. Throughout this work, we denote by  $X$  the Banach space with norm  $|\cdot|$ ,  $A : D(A) \subseteq X \rightarrow X$  a closed linear operator and  $Y$  the Banach space  $D(A)$  equipped with the graph norm  $|y|_Y := |Ay| + |y|$  for  $y \in Y$ .  $\mathcal{B}(Y, X)$  the Banach space of bounded linear operators from  $Y$  into  $X$  endowed with the operator norm and we abbreviate to  $\mathcal{B}(X)$  when  $Y = X$ .

### 2.1 Integrated semigroups and differential equations with nondense domain

In this section, we introduce some definitions and preliminary facts on integrated semigroups theory which are used in the study of differential equations with nondense domain. For further information and results, we refer the reader to [1, 2, 5, 18] and references therein.

**Definition 2.1** ([5]). Let  $X$  be a Banach space. A family  $(S(t))_{t \geq 0} \subset \mathcal{B}(X)$  is called an integrated semigroup if the following conditions are satisfied:

- (i)  $S(0) = 0$ .
- (ii) For any  $x \in X$ ,  $S(t)x$  is a continuous function of  $t \geq 0$  with values in  $X$ .
- (iii) For any  $t, s \geq 0$  and  $x \in X$ ,  $S(s)S(t)x = \int_0^s (S(t+r) - S(r))x dr$ .

**Definition 2.2** ([5]). An integrated semigroup  $(S(t))_{t \geq 0}$  is called exponentially bounded, if there exist constants  $\beta \geq 1$  and  $w \in \mathbb{R}$  such that

$$|S(t)| \leq \beta e^{wt} \quad \text{for } t \geq 0.$$

Moreover,  $(S(t))_{t \geq 0}$  is called nondegenerate if,  $S(t)x = 0$  for all  $t \geq 0$ , implies  $x = 0$ .

**Definition 2.3** ([5]). An operator  $A$  is called a generator of an integrated semigroup, if there exists  $\omega \in \mathbb{R}$  such that  $(\omega, +\infty) \subset \rho(A)$ , and there exists a strongly continuous exponentially bounded family  $(S(t))_{t \geq 0}$  of linear bounded operators such that  $S(0) = 0$  and  $(\lambda I - A)^{-1} = \lambda \int_0^{+\infty} e^{-\lambda s} S(s) ds$  for all  $\lambda > \omega$ .

**Proposition 2.4** ([5]). Let  $A$  be the generator of an integrated semigroup  $(S(t))_{t \geq 0}$ . Then, for all  $x \in X$  and  $t \geq 0$ , we have

$$\int_0^t S(s)x ds \in D(A) \quad \text{and} \quad S(t)x = A\left(\int_0^t S(s)x ds\right) + tx.$$

Moreover, for all  $x \in D(A)$ ,  $t \geq 0$

$$S(t)x \in D(A), \quad AS(t)x = S(t)Ax \quad \text{and} \quad S(t)x = \int_0^t S(s)Ax ds + tx.$$

**Theorem 2.5** ([5]). Let  $A$  be the generator of an integrated semigroup  $(S(t))_{t \geq 0}$ . Then, for all  $x \in X$  and  $t \geq 0$  one has  $S(t)x \in \overline{D(A)}$ . Moreover, for  $x \in X$ ,  $S(\cdot)x$  is differentiable in  $t \geq 0$  if and only if  $S(t)x \in D(A)$ . In this case

$$S'(t)x = AS(t)x + x.$$

An important special case is when the integrated semigroup is locally Lipschitz continuous with respect to time.

**Definition 2.6** ([18]). An integrated semigroup  $(S(t))_{t \geq 0}$  is called locally Lipschitz continuous, if for all  $\tau > 0$ , there exists a constant  $C(\tau) > 0$  such that

$$|S(t) - S(s)| \leq C(\tau)|t - s| \quad \text{for } t, s \in [0, \tau].$$

The following Theorem shows that the Hille–Yosida operators characterize the generators of locally Lipschitz continuous integrated semigroups.

**Theorem 2.7** ([18]). The following are equivalent:

- (i)  $A$  is the generator of a locally Lipschitz continuous integrated semigroup.
- (ii)  $A$  satisfies the Hille–Yosida condition.

In the sequel, we recall some results obtained in [1] for the existence of solutions of the following differential equation:

$$\begin{cases} \frac{du}{dt} = Au(t) + F(t, u_t) \text{ for } t \geq 0 \\ u_0 = \varphi \in C, \end{cases} \quad (2.1)$$

where  $F$  is a continuous function and  $A$  satisfies the Hille–Yosida condition.

**Definition 2.8** ([1]). A function  $u : [-r, +\infty) \rightarrow X$  is said to be an integral solution of Eq. (2.1) if the following hold:

- (i)  $u$  is continuous on  $[0, +\infty)$ .
- (ii)  $\int_0^t u(s) ds \in D(A)$  for  $t \geq 0$ .
- (iii)  $u(t) = \begin{cases} \varphi(0) + A \int_0^t u(s) ds + \int_0^t F(s, u_s) ds & \text{for } t \geq 0 \\ \varphi(t) & \text{for } -r \leq t \leq 0. \end{cases}$

According to [1], if the integral solution of Eq. (2.1) exists, then it is given by the following variation of constants formula:

$$u(t) = S'(t)\varphi(0) + \frac{d}{dt} \int_0^t S(t-s)F(s, u_s)ds \quad \text{for } t \geq 0.$$

**Definition 2.9** ([1]). A function  $u : [-r, +\infty) \rightarrow X$  is said to be a strict solution of Eq. (2.1) if the following hold:

- (i)  $u \in C^1([0, +\infty); X) \cap C([0, +\infty); D(A))$ .
- (ii)  $u$  satisfies Eq. (2.1) on  $[-r, +\infty)$ .

**Proposition 2.10** ([1]). *If the integral solution  $u$  of Eq. (2.1) is continuously differentiable on  $[0, +\infty)$  or belongs to  $C([0, +\infty); D(A))$ , then  $u$  is strict solution for Eq. (2.1) on  $[-r, +\infty)$ .*

Applying the above argument, the authors show in [1] the following results concerning the existence and regularity of solutions of Eq. (2.1).

**Theorem 2.11** ([1]). *Assume that  $A$  is a Hille–Yosida operator,  $F$  is continuous and Lipschitzian with respect the second argument. Let  $\varphi \in C$  be such that  $\varphi(0) \in \overline{D(A)}$ . Then, Eq. (2.1) has a unique integral solution defined on  $[-r, +\infty)$ .*

**Theorem 2.12** ([1]). *Assume that  $A$  is a Hille–Yosida operator,  $F$  is continuously differentiable and the partial derivatives are locally Lipschitz with respect the second argument. Let  $\varphi \in C^1([-r, 0], X)$  be such that*

$$\varphi(0) \in D(A), \quad \varphi'(0) \in \overline{D(A)} \quad \text{and} \quad \varphi'(0) = A\varphi(0) + F(0, \varphi).$$

*Then, the integral solution of Eq. (2.1) becomes a strict solution.*

## 2.2 Resolvent operators

The resolvent operators play an important role in the study of the well-posedness of Eq. (1.1) in the weak and strict sense, it generalize the notion of strongly continuous semigroup; see [6, 13–15] for more details. Consider the following integrodifferential equation:

$$\begin{cases} x'(t) = Ax(t) + \int_0^t B(t-s)x(s)ds & \text{for } t \geq 0 \\ x(0) = x_0 \in X, \end{cases} \quad (2.2)$$

where  $A$  is densely defined, closed linear operator on  $X$  and  $B(t), t \geq 0$ , is closed linear operator on  $X$ .

**Definition 2.13** ([15]). A resolvent operator for Eq. (2.2) is a bounded linear operator valued function  $\mathcal{R}(t) \in \mathcal{B}(X)$  for  $t \geq 0$ , having the following properties:

- (a)  $\mathcal{R}(0) = I$  and  $|\mathcal{R}(t)| \leq Me^{\tilde{\beta}t}$  for some constants  $M$  and  $\tilde{\beta}$ .
- (b) For each  $x \in X$ ,  $\mathcal{R}(t)x$  is strongly continuous for  $t \geq 0$ .
- (c)  $\mathcal{R}(t) \in \mathcal{B}(Y)$  for  $t \geq 0$ . For  $x \in Y$ ,  $\mathcal{R}(\cdot)x \in C^1([0, +\infty); X) \cap C([0, +\infty); Y)$  and

$$\begin{aligned} \mathcal{R}'(t)x &= A\mathcal{R}(t)x + \int_0^t B(t-s)\mathcal{R}(s)xds \\ &= \mathcal{R}(t)Ax + \int_0^t \mathcal{R}(t-s)B(s)xds \quad \text{for } t \geq 0. \end{aligned}$$

Throughout this section, we assume that:

- (I)  $A$  is the infinitesimal generator of a strongly continuous semigroup on  $X$ .
- (II) For all  $t \geq 0$ ,  $B(t)$  is closed linear operator from  $D(A)$  to  $X$  and  $B(t) \in \mathcal{B}(Y, X)$ . For any  $y \in Y$ , the map  $t \mapsto B(t)y$  is bounded, differentiable and the derivative  $t \mapsto B'(t)y$  is bounded uniformly continuous on  $\mathbb{R}^+$ .

The following theorem provides sufficient conditions for the existence of the resolvent operator for Eq. (2.2).

**Theorem 2.14** ([15]). *Assume that (I) and (II) hold. Then, there exists a unique resolvent operator of Eq. (2.2).*

Now, we shall introduce the notions of mild and strict solutions to Eq. (1.1) and we give some existence results using the resolvent operator theory.

**Definition 2.15** ([14]). A continuous function  $u : [-r, +\infty) \rightarrow X$  is called a strict solution of Eq. (1.1) if:

- (i)  $u \in C^1([0, +\infty); X) \cap C([0, +\infty); Y)$ .
- (ii)  $u$  satisfies Eq. (1.1) on  $[-r, +\infty)$ .

**Theorem 2.16** ([14]). *Assume that (I) and (II) hold. If  $u$  is a strict solution of Eq. (1.1), then*

$$u(t) = \mathcal{R}(t)\varphi(0) + \int_0^t \mathcal{R}(t-s)F(s, u_s)ds \quad \text{for } t \geq 0. \quad (2.3)$$

**Remark 2.17.** The converse is not true. In fact if  $u$  satisfies Eq. (2.3),  $u$  may be not differentiable, that is why we distinguish between mild and strict solutions.

**Definition 2.18** ([14]). A continuous function  $u : [-r, +\infty) \rightarrow X$  is called a mild solution of Eq. (1.1) if  $u$  satisfies:

$$u(t) = \begin{cases} \mathcal{R}(t)\varphi(0) + \int_0^t \mathcal{R}(t-s)F(s, u_s)ds & \text{for } t \geq 0 \\ \varphi(t) & \text{for } -r \leq t \leq 0. \end{cases}$$

**Theorem 2.19** ([14, Theorem 3.4]). *Assume that (I)–(II) hold, and  $F$  is continuous and Lipschitzian with respect to the second argument. Let  $\varphi \in C$ . Then Eq. (1.1) has a unique mild solution defined on  $[-r, +\infty)$ .*

The next Theorem provides sufficient conditions for the regularity of solutions of Eq. (1.1).

**Theorem 2.20** ([14, Theorem 4.1]). *Assume that (I)–(II) hold,  $F$  is continuously differentiable and the partial derivatives are locally Lipschitz with respect to the second argument. Let  $\varphi \in C^1([-r, 0], X)$  be such that*

$$\varphi(0) \in D(A) \quad \text{and} \quad \varphi'(0) = A\varphi(0) + F(0, \varphi).$$

*Then, the corresponding mild solution becomes a strict solution of Eq. (1.1).*

### 2.3 Integrated resolvent operators

Throughout the remainder of this work, we shall assume that  $D(A)$  is not necessarily densely defined. In the sequel, we collect some basic results developed in [20] on integrated resolvent operators theory and integrodifferential equations with nondense domain. Here, we consider the integrodifferential equation (2.2) with  $A : D(A) \subseteq X \rightarrow X$  a closed linear operator whose domain is not necessarily densely defined and  $B$  is a closed linear operator on  $X$ .

**Definition 2.21** ([20]). An integrated resolvent operator for Eq. (2.2) is a bounded operator valued function  $R(t) \in \mathcal{B}(X)$  for  $t \geq 0$ , having the following properties:

(r<sub>1</sub>) For all  $x \in X$ ,  $R(\cdot)x \in C([0, +\infty); X)$ .

(r<sub>2</sub>) For all  $x \in X$ ,  $\int_0^\cdot R(s)x ds \in C([0, +\infty); Y)$ .

(r<sub>3</sub>)  $R(t)x - tx = A \int_0^t R(s)x ds + \int_0^t B(t-s) \int_0^s R(r)x dr ds$  for all  $x \in X$  and  $t \geq 0$ .

(r<sub>4</sub>)  $R(t)x - tx = \int_0^t R(s)Ax ds + \int_0^t \int_0^s R(s-r)B(r)x dr ds$  for all  $x \in D(A)$  and  $t \geq 0$ .

**Definition 2.22** ([20]). An integrated resolvent operator  $(R(t))_{t \geq 0}$  in  $\mathcal{B}(X)$  is called locally Lipschitz continuous, if for all  $a > 0$ , there exists a constant  $C_a = C(a) > 0$  such that

$$|R(t) - R(s)| \leq C_a |t - s| \quad \text{for } t, s \in [0, a].$$

**Theorem 2.23** ([20]). Suppose that  $(R(t))_{t \geq 0}$  is a locally Lipschitz continuous integrated resolvent operator. Then for all  $x \in \overline{D(A)}$ ,  $t \mapsto R(t)x$  is  $C^1$ -function on  $[0, +\infty)$ .

**Remark 2.24.**

- (a) Notice that from the Definition 2.21, we know that the integrated resolvent operator of Eq. (2.2) is the integrated semigroup of  $A$  when  $B = 0$ .
- (b) The notion of integrated resolvent operator of Eq. (2.2) coincides with that of resolvent operator introduced by Grimmer [15] in the case where  $\overline{D(A)} = X$  and  $\rho(A) \neq \emptyset$  (see [20, Theorem 2.9]).

To show the existence of locally Lipschitz continuous integrated resolvent operator for Eq. (2.2), let us recall the following assumptions:

**(H0)**  $A$  satisfies the Hille–Yosida condition.

**(H1)**  $(B(t))_{t \geq 0}$  is a family linear operators in  $X$  with  $D(A) \subset D(B(t))$  for all  $t \geq 0$  and, of bounded linear operators from  $Y$  to  $X$  such that the functions  $B(\cdot)x$  are of strong bounded variation on each finite interval  $[0, a]$ ,  $a > 0$ , for  $x \in D(A)$ .

**Theorem 2.25** ([20, Theorem 3.2]). Assume that **(H0)** and **(H1)** hold. Then there exists a unique locally Lipschitz continuous integrated resolvent operator of the problem (2.2).

**Remark 2.26.** If  $B = 0$ , Theorem 2.25 shows that the Hille–Yosida condition characterizes a locally Lipschitz continuous integrated semigroup (see [18]).

**Theorem 2.27** ([20, Theorem 2.9]). Assume that  $\overline{D(A)} = X$  and  $\rho(A) \neq \emptyset$ . Let  $q \in L^1_{\text{loc}}(0, +\infty; X)$ . The following statements are equivalent:

- (i) Eq. (2.2) admits a locally Lipschitz continuous integrated resolvent operator  $(R(t))_{t \geq 0}$ .
- (ii) Eq. (2.2) admits a resolvent operator  $(T(t))_{t \geq 0}$ .
- (iii) For all  $x_0 \in X$ , there exists a unique integral solution  $x$  to Eq. (1.2).
- (iv) For all  $x_0 \in X$ , there exists a unique mild solution  $x$  to Eq. (1.2). In this case

$$\begin{aligned} R(t)x_0 &= \int_0^t \mathcal{R}(s)x_0 ds \quad \text{for } t \geq 0 \text{ and } x_0 \in X; \\ x(t) &= \frac{d}{dt} \left( R(t)x_0 + \int_0^t R(t-s)q(s)ds \right) \quad \text{for } t \geq 0, \\ &= \mathcal{R}(t)x_0 + \int_0^t \mathcal{R}(t-s)q(s)ds \quad \text{for } t \geq 0 \text{ and } x_0 \in X. \end{aligned}$$

For later use, let us recall the fundamental results obtained by Oka [20] on the initial value problem (1.2).

**Definition 2.28** ([20]). Let  $q \in L^1_{\text{loc}}(0, +\infty; X)$  and  $x_0 \in X$ . A function  $x : [0, +\infty) \rightarrow X$  is called an integral solution of Eq. (1.2) if the following conditions hold:

- (i)  $x \in C([0, +\infty); X)$ .
- (ii)  $\int_0^\cdot x(s)ds \in C([0, +\infty); Y)$ .
- (iii)  $x(t) = x_0 + A \int_0^t x(s)ds + \int_0^t B(t-s) \int_0^s x(r)dr ds + \int_0^t q(s)ds$  for  $t \geq 0$ .

**Definition 2.29** ([20]). A function  $x : [0, +\infty) \rightarrow X$  is called a strict solution of Eq. (1.2) if the following conditions hold:

- (i)  $x \in C^1([0, +\infty); X) \cap C([0, +\infty); Y)$ ,
- (ii)  $x$  satisfies Eq. (1.2) on  $[0, +\infty)$ .

The next theorem plays a key role in this work.

**Theorem 2.30.** [20, Lemma 2.6]. Let a family  $(U(t))_{t \geq 0}$  in  $\mathcal{B}(X)$  be locally Lipschitz continuous with  $U(0) = 0$ . Then, the following hold.

- (i) If  $q \in L^1(0, a; X)$ , then  $\int_0^\cdot U(\cdot - s)q(s)ds \in C^1([0, a]; X)$ .

Putting  $Q(t) := \frac{d}{dt} \int_0^t U(t-s)q(s)ds$  for  $t \in [0, a]$ , we have

$$|Q(t)| \leq C_a \int_0^t |q(s)|ds \quad (2.4)$$

where  $C_a$  is the Lipschitz constant of  $U(t)$  on  $[0, a]$ . Moreover, if  $|q(t)| \leq K$  for  $t \in [0, a]$ , then

$$|Q(t+s) - Q(t)| \leq KC_a s + C_a \int_0^t |q(s+r) - q(r)|dr \quad \text{for } s, t, t+s \in [0, a]. \quad (2.5)$$

- (ii) If a function  $q : [0, a] \rightarrow X$  is of strong bounded variation, the function  $Q(\cdot)$  defined in (i) is Lipschitz continuous on  $[0, a]$ .



**Remark 2.31.** The results reported in Theorem 2.30 hold for any locally Lipschitz continuous family of bounded linear operators. In particular, these results are true for the integrated resolvent operators.

**Remark 2.32.** If  $x$  is an integral solution of Eq. (1.2) then, according to Definition 2.28,  $x(t) \in \overline{D(A)}$  for all  $t \geq 0$ . In fact,  $x(t) = \lim_{h \rightarrow 0} \frac{1}{h} \int_t^{t+h} x(s) ds$  and  $\int_t^{t+h} x(s) ds \in D(A)$ . In particular  $x(0) \in \overline{D(A)}$  is a necessary condition for existence of an integral solution of Eq. (1.2).

The following theorem gives sufficient conditions for the existence and regularity of solutions of Eq. (1.2).

**Theorem 2.33** ([20, Theorem 2.7]). *Assume that Eq. (2.2) has an integrated resolvent operator  $(R(t))_{t \geq 0}$  that is locally Lipschitz continuous and  $\rho(A) \neq \emptyset$ . Then, the following hold.*

(i) *If  $x_0 \in \overline{D(A)}$  and  $q \in L^1(0, a; X)$ , then there exists a unique integral solution  $x(\cdot)$  of Eq. (1.2) which is given by the variation of constants formula*

$$x(t) = R'(t)x_0 + \frac{d}{dt} \int_0^t R(t-s)q(s)ds \quad \text{for } t \in [0, a].$$

Moreover, we have

$$|x(t)| \leq C_a \left( |x_0| + \int_0^t |q(s)| ds \right) \quad \text{for } t \in [0, a]. \quad (2.6)$$

(ii) *If  $x_0 \in D(A)$ ,  $q \in W^{1,1}(0, a; X)$  and  $Ax_0 + q(0) \in \overline{D(A)}$ , then there exists a unique strict solution  $x(\cdot)$  of Eq. (1.2). Moreover, we have*

$$|x'(t)| \leq C_a \left( |Ax_0 + q(0)| + \int_0^t |B(s)x_0 + q'(s)| ds \right) \quad \text{for } t \in [0, a]. \quad (2.7)$$

### 3 Existence and regularity of solutions for Eq. (1.1)

In this section, we prove the existence, continuous dependence on the initial data and regularity of solutions of problem (1.1) using the integrated resolvent operators theory. We give the definitions of the so-called integral and strict solutions of Eq. (1.1).

**Definition 3.1.** Let  $\varphi \in C$ . A function  $u : [-r, +\infty) \rightarrow X$  is called an integral solution of Eq. (1.1) if:

- (i)  $u \in C([0, +\infty); X)$ .
- (ii)  $\int_0^\cdot u(s) ds \in C([0, +\infty); Y)$ .
- (iii)  $u(t) = \begin{cases} \varphi(0) + A \int_0^t u(s) ds + \int_0^t B(t-s) \int_0^s u(r) dr ds + \int_0^t F(s, u_s) ds & \text{for } t \geq 0 \\ \varphi(t) & \text{for } -r \leq t \leq 0. \end{cases}$

**Definition 3.2.** Let  $\varphi \in C$ . A function  $u : [-r, +\infty) \rightarrow X$  is called a strict solution of Eq. (1.1) if:

- (i)  $u \in C^1([0, +\infty); X) \cap C([0, +\infty); Y)$ .
- (ii)  $u$  satisfies Eq. (1.1) on  $[-r, +\infty)$ .

**Remark 3.3.**

- (a) From Definition 3.1, if  $u$  is an integral solution of Eq. (1.1) in  $[-r, +\infty)$ , then  $u(t) \in \overline{D(A)}$  for all  $t \geq 0$ . In particular  $\varphi(0) \in \overline{D(A)}$ .
- (b) If  $u$  is an integral solution of Eq. (1.1) in  $[-r, +\infty)$  such that  $u$  belongs to  $C^1([0, +\infty); X)$  or  $C([0, +\infty); Y)$ , then  $u$  becomes a strict solution of Eq. (1.1) in  $[-r, +\infty)$ .

**3.1 Existence and uniqueness of the integral solution**

To establish the existence and uniqueness of the integral solution, we assume the following condition:

- (H2)  $F : \mathbb{R}^+ \times C \rightarrow X$  is continuous and Lipschitzian with respect to the second argument. Let  $L_F > 0$  be such that

$$|F(t, \varphi) - F(t, \psi)| \leq L_F |\varphi - \psi| \quad \text{for } t \geq 0 \text{ and } \varphi, \psi \in C.$$

**Theorem 3.4.** Assume that Eq. (2.2) has an integrated resolvent operator  $(R(t))_{t \geq 0}$  that is locally Lipschitz continuous and  $\rho(A) \neq \emptyset$ . Let  $F$  satisfy (H2) and  $\varphi \in C$  be such that  $\varphi(0) \in \overline{D(A)}$ . Then, Eq. (1.1) has a unique integral solution  $u = u(\cdot, \varphi)$  defined on  $[-r, +\infty)$ . Moreover,

$$u(t) = \begin{cases} R'(t)\varphi(0) + \frac{d}{dt} \int_0^t R(t-s)F(s, u_s)ds & \text{for } t \geq 0 \\ \varphi(t) & \text{for } -r \leq t \leq 0. \end{cases}$$

*Proof.* Let  $a > 0$  be fixed and  $C([0, a]; X)$  be the space of continuous functions from  $[0, a]$  into  $X$  endowed with the uniform norm topology. Let  $\varphi \in C$  such that  $\varphi(0) \in \overline{D(A)}$ . Consider the nonempty closed subset of  $C([0, a]; X)$

$$\mathbb{D}_a(\varphi) := \{u \in C([0, a]; X) : u(0) = \varphi(0)\}.$$

and the mapping  $\Psi$  defined on  $\mathbb{D}_a(\varphi)$  by

$$(\Psi u)(t) = R'(t)\varphi(0) + \frac{d}{dt} \int_0^t R(t-s)F(s, \tilde{u}_s)ds \quad \text{for } t \in [0, a],$$

where the extension  $\tilde{u} : [-r, a] \rightarrow X$  is such that

$$\tilde{u}(t) = \begin{cases} u(t) & \text{for } 0 \leq t \leq a, \\ \varphi(t) & \text{for } -r \leq t \leq 0. \end{cases}$$

First, we shall prove  $\Psi$  maps  $\mathbb{D}_a(\varphi)$  into  $\mathbb{D}_a(\varphi)$ . Since  $R(\cdot)y \in C^1([0, a]; X)$  for  $y \in D(A)$  by (r4), the local Lipschitz continuity of  $R(\cdot)$  implies that the function  $t \mapsto R(t)\varphi(0)$  is  $C^1([0, a]; X)$ . Moreover, by virtue of condition (H2), the function  $s \mapsto F(s, \tilde{u}_s)$  is continuous on  $[0, a]$ . Then, Theorem 2.30 implies  $t \mapsto \int_0^t R(t-s)F(s, \tilde{u}_s)ds$  is  $C^1([0, a]; X)$ . Thus,  $\Psi(\mathbb{D}_a(\varphi)) \subset \mathbb{D}_a(\varphi)$ .

Next, we shall show that  $\Psi$  is a strict contraction from  $\mathbb{D}_a(\varphi)$  into  $\mathbb{D}_a(\varphi)$ . Let  $u, v \in \mathbb{D}_a(\varphi)$  and  $t \in [0, a]$ . Then by Theorem 2.30 and (H2), we have

$$\begin{aligned}
|(\Psi u)(t) - (\Psi v)(t)| &= \left| \frac{d}{dt} \int_0^t R(t-s) [F(s, \tilde{u}_s) - F(s, \tilde{v}_s)] ds \right| \\
&\leq C_a \int_0^t |F(s, \tilde{u}_s) - F(s, \tilde{v}_s)| ds \\
&\leq C_a L_F \int_0^t |\tilde{u}_s - \tilde{v}_s| ds \\
&\leq C_a L_F t |u - v|.
\end{aligned}$$

A similar reasoning, we obtain that

$$\begin{aligned}
|(\Psi^2 u)(t) - (\Psi^2 v)(t)| &= |\Psi(\Psi u)(t) - \Psi(\Psi v)(t)| \\
&\leq C_a L_F \int_0^t |\Psi \tilde{u}_s - \Psi \tilde{v}_s| ds \\
&\leq \frac{(C_a L_F t)^2}{2!} |u - v|.
\end{aligned}$$

Consequently, we have for all  $n \geq 1$

$$|(\Psi^n u)(t) - (\Psi^n v)(t)| \leq \frac{(C_a L_F t)^n}{n!} |u - v|.$$

Thus it follows that for all  $n \geq 1$

$$|\Psi^n u - \Psi^n v| \leq \frac{(C_a L_F a)^n}{n!} |u - v|.$$

Let  $n_0$  be such that  $\frac{(C_a L_F a)^{n_0}}{n_0!} < 1$ . Then  $\Psi^{n_0}$  is a strict contraction in  $\mathbb{D}_a(\varphi)$ . Consequently,  $\Psi$  is a strict contraction in  $\mathbb{D}_a(\varphi)$ , and the fixed point of  $\Psi$  gives a unique integral solution  $u = u(\cdot, \varphi)$  on  $[0, a]$ . This is true for any  $a > 0$ , which means that we have a global existence of the integral solution on  $\mathbb{R}^+$ .  $\square$

Now, we give another existence result under more restrictive assumptions on the operators  $A$  and  $B(\cdot)$ .

**Corollary 3.5.** *Assume that (H0)–(H2) hold. Let  $\varphi \in C$  be such that  $\varphi(0) \in \overline{D(A)}$ . Then, Eq. (1.1) has a unique integral solution on  $[-r, +\infty)$ .*

The proof follows from Theorem 2.25 and Theorem 3.4.

### 3.2 Continuous dependence with the initial data

Next, we show that the integral solution of Eq. (1.1) depends continuously on the initial data. Let  $\varphi \in C$  be such that  $\varphi(0) \in \overline{D(A)}$  and  $u(\cdot, \varphi)$  the integral solution of (1.1) starting from  $\varphi$ . We have the following result.

**Theorem 3.6.** *The integral solution of (1.1) depends continuously on the initial data in the following sense: for all  $a > 0$ , there exists a constant  $\sigma(a) > 0$  such that for any  $\varphi, \psi \in C$  satisfying  $\varphi(0), \psi(0) \in \overline{D(A)}$  we have*

$$|u_t(\cdot, \varphi) - u_t(\cdot, \psi)| \leq \sigma(a) |\varphi - \psi| \quad \text{for } t \in [0, a].$$

*Proof.* Let  $u = u(\cdot, \varphi)$  and  $v = u(\cdot, \psi)$ . Then, for  $t \in [0, a]$ , we have

$$u(t) - v(t) = R'(t)[\varphi(0) - \psi(0)] + \frac{d}{dt} \int_0^t R(t-s)[F(s, u_s) - F(s, v_s)]ds.$$

Using the estimations of Theorem 2.33, we get that

$$\begin{aligned} |u(t) - v(t)| &\leq C_a \left( |\varphi(0) - \psi(0)| + \int_0^t |F(s, u_s) - F(s, v_s)|ds \right) \\ &\leq C_a \left( |\varphi - \psi| + L_f \int_0^t |u_s - v_s|ds \right). \end{aligned}$$

Without loss of generality, we assume that  $C_a \geq 1$ . This implies that

$$|u_t - v_t| \leq C_a |\varphi - \psi| + C_a L_f \int_0^t |u_s - v_s|ds \quad \text{for } t \in [0, a].$$

By Gronwall's Lemma, we deduce that

$$|u_t - v_t| \leq C_a e^{C_a L_f a} |\varphi - \psi|.$$

This completes the proof.  $\square$

### 3.3 Regularity of the integral solution

In this section, we prove the existence of strict solution of Eq. (1.1). Assume the following condition:

**(H3)**  $F \in C^1(\mathbb{R}^+ \times C; X)$  and the partial derivatives  $D_t F(\cdot, \cdot)$  and  $D_\varphi F(\cdot, \cdot)$  are locally Lipschitzian with respect to the second argument.

**Theorem 3.7.** Assume that Eq. (2.2) has an integrated resolvent operator  $(R(t))_{t \geq 0}$  that is locally Lipschitz continuous and  $\rho(A) \neq \emptyset$ . Let  $F$  satisfy **(H2)**, **(H3)** and  $\varphi \in C^1([-r, 0], X)$  be such that

$$\varphi(0) \in D(A), \quad \varphi'(0) \in \overline{D(A)} \quad \text{and} \quad \varphi'(0) = A\varphi(0) + F(0, \varphi). \quad (3.1)$$

Then, the integral solution of Eq. (1.1) given by Theorem 3.4 is a strict solution on  $[-r, +\infty)$ .

*Proof.* Let  $u$  be the unique integral solution of Eq. (1.1) and  $a > 0$ . Then

$$u(t) = \begin{cases} R'(t)\varphi(0) + \frac{d}{dt} \int_0^t R(t-s)F(s, u_s)ds & \text{for } 0 \leq t \leq a, \\ \varphi(t) & \text{for } -r \leq t \leq 0. \end{cases}$$

Consider the following problem

$$\begin{cases} v'(t) = Av(t) + \int_0^t B(t-s)v(s)ds + D_t F(t, u_t) + D_\varphi F(t, u_t)v_t + B(t)\varphi(0) & \text{for } t \in [0, a] \\ v_0 = \varphi'. \end{cases} \quad (3.2)$$

By the same argument used in the proof of Theorem 3.4, we can prove that Eq. (3.2) has a unique integral solution  $v$  and that

$$v(t) = \begin{cases} R'(t)\varphi'(0) + \frac{d}{dt} \int_0^t R(t-s)[D_t F(s, u_s) + D_\varphi F(s, u_s)v_s + B(s)\varphi(0)]ds & \text{for } 0 \leq t \leq a, \\ \varphi'(t) & \text{for } -r \leq t \leq 0. \end{cases}$$

Let  $w : [-r, a] \rightarrow X$  be the function defined by

$$w(t) = \begin{cases} \varphi(0) + \int_0^t v(s)ds & \text{for } t \in [0, a] \\ \varphi(t) & \text{for } t \in [-r, 0]. \end{cases}$$

Then,

$$w_t = \varphi + \int_0^t v_s ds \quad \text{for } t \in [0, a].$$

Now, we will prove that the function  $w = u$ . By the expression of  $v(t)$  for  $t \in [0, a]$ , we mean that

$$w(t) = \varphi(0) + R(t)\varphi'(0) + \int_0^t R(t-s)[D_t F(s, u_s) + D_\varphi F(s, u_s)v_s + B(s)\varphi(0)]ds. \quad (3.3)$$

Since  $\varphi(0) \in D(A)$ ,  $\varphi'(0) \in \overline{D(A)}$  and  $\varphi'(0) = A\varphi(0) + F(0, \varphi)$ , then

$$R(t)\varphi'(0) = R(t)A\varphi(0) + R(t)F(0, \varphi).$$

Using the derivative of  $(r_4)$  in Definition 2.21, we obtain

$$R(t)\varphi'(0) = R'(t)\varphi(0) - \varphi(0) - \int_0^t R(t-s)B(s)\varphi(0)ds + R(t)F(0, \varphi). \quad (3.4)$$

On the other hand, since the map  $t \mapsto w_t$  is continuously differentiable, then the map

$$t \mapsto \int_0^t R(t-s)F(s, w_s)ds$$

is also continuously differentiable and we have

$$\begin{aligned} \frac{d}{dt} \int_0^t R(t-s)F(s, w_s)ds &= \frac{d}{dt} \int_0^t R(s)F(t-s, w_{t-s})ds \\ &= R(t)F(0, \varphi) + \int_0^t R(t-s)[D_t F(s, w_s) + D_\varphi F(s, w_s)v_s]ds. \end{aligned}$$

This implies that

$$R(t)F(0, \varphi) = \frac{d}{dt} \int_0^t R(t-s)F(s, w_s)ds - \int_0^t R(t-s)[D_t F(s, w_s) + D_\varphi F(s, w_s)v_s]ds. \quad (3.5)$$

By combining (3.3), (3.4) and (3.5), we obtain that

$$\begin{aligned} w(t) &= R'(t)\varphi(0) + \frac{d}{dt} \int_0^t R(t-s)F(s, w_s)ds \\ &\quad - \int_0^t R(t-s)[D_t F(s, w_s) - D_t F(s, u_s)]ds \\ &\quad - \int_0^t R(t-s)[D_\varphi F(s, w_s) - D_\varphi F(s, u_s)]v_s ds. \end{aligned}$$

Hence,

$$\begin{aligned} u(t) - w(t) &= \frac{d}{dt} \int_0^t R(t-s)[F(s, u_s) - F(s, w_s)]ds \\ &\quad - \int_0^t R(t-s)[D_t F(s, u_s) - D_t F(s, w_s)]ds \\ &\quad - \int_0^t R(t-s)[D_\varphi F(s, u_s) - D_\varphi F(s, w_s)]v_s ds. \end{aligned}$$



In order to rewrite (4.1) as the abstract Eq. (1.1), we introduce the following spaces, let  $X = C([0, 1]; \mathbb{R})$  be the space of continuous functions from  $[0, 1]$  to  $\mathbb{R}$  provided with the supremum norm. We define the operator  $A$  and  $B(t)$  by

$$\begin{cases} D(A) = \{z \in C^1([0, 1]; \mathbb{R}) : z(0) = z(1)\} \\ Az = -a(x)z' \text{ for } x \in [0, 1], \end{cases}$$

and

$$\begin{cases} D(B(t)) = C^1([0, 1]; \mathbb{R}) \\ (B(t)z)(x) = p(t, x)z'(x) \text{ for } t \geq 0 \text{ and } x \in [0, 1]. \end{cases}$$

**Lemma 4.1** ([20]). *A is a Hille–Yosida operator on  $X$  with  $\overline{D(A)} = \{z \in X : z(0) = z(1)\}$ . Moreover,  $(B(t))_{t \geq 0}$  is a family of bounded linear operators from  $D(A)$  to  $X$ , where  $D(A)$  is equipped with graph norm and  $B(\cdot)z \in BV_{\text{loc}}(\mathbb{R}^+; X)$  for any  $z \in D(A)$ .*

Consider the linear equation

$$\begin{cases} v'(t) = Av(t) + \int_0^t B(t-s)v(s)ds \text{ for } t \geq 0 \\ v(0) = v_0 \in X \end{cases} \quad (4.2)$$

**Lemma 4.2** ([20]). *Equation (4.2) has a unique integrated resolvent operator that is locally Lipschitz continuous.*

Let

$$\begin{aligned} v(t)(x) &= w(t, x) && \text{for } t \geq 0 \text{ and } x \in [0, 1], \\ \varphi(\theta)(x) &= w_0(\theta, x) && \text{for } \theta \in [-r, 0] \text{ and } x \in [0, 1], \\ F(t, \varphi)(x) &= f(t, \varphi(-\tau)(x)) && \text{for } x \in [0, 1], t \geq 0 \text{ and } \varphi \in C([-r, 0]; X). \end{aligned}$$

Then Eq. (4.1) can be transformed as the following abstract form

$$\begin{cases} v'(t) = Av(t) + \int_0^t B(t-s)v(s)ds + F(t, v_t) \text{ for } t \geq 0 \\ v_0 = \varphi. \end{cases} \quad (4.3)$$

One can see that  $F : \mathbb{R}^+ \times C([-r, 0]; X) \rightarrow X$  is continuous and

$$|F(t, \varphi) - F(t, \psi)| \leq k|\varphi - \psi| \quad \text{for } t \geq 0 \text{ and } \varphi, \psi \in C([-r, 0]; X).$$

In fact, for all  $t \geq 0$  and  $\varphi_1, \varphi_2 \in C([-r, 0]; X)$ , we have

$$\begin{aligned} |F(t, \varphi_1) - F(t, \varphi_2)| &= \sup_{0 \leq x \leq 1} |f(t, \varphi_1(-\tau)(x)) - f(t, \varphi_2(-\tau)(x))| \\ &\leq k \sup_{0 \leq x \leq 1} |\varphi_1(-\tau)(x) - \varphi_2(-\tau)(x)| \\ &\leq k|\varphi_1 - \varphi_2|. \end{aligned}$$

Consequently, by Theorem 3.4 we obtain the following existence result.

**Proposition 4.3.** *Assume that  $\varphi(0) \in \overline{D(A)}$  which is equivalent to  $w_0(0, 0) = w_0(0, 1)$ . Then Eq. (4.1) has a unique integral solution defined on  $\mathbb{R}^+$ .*

For the regularity of the integral solution, we make the following assumption.

- (i)  $f : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$  is  $C^1$ -function and  $\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}$  are locally Lipschitzian with respect to the second argument. Let  $\rho > 0$  and  $k_1, k_2$  two positives constants be such that

$$\left| \frac{\partial f}{\partial t}(t, x_1) - \frac{\partial f}{\partial t}(t, x_2) \right| \leq k_1 |x_1 - x_2| \quad \text{for } |x_1|, |x_2| \leq \rho$$

$$\left| \frac{\partial f}{\partial x}(t, x_1) - \frac{\partial f}{\partial x}(t, x_2) \right| \leq k_2 |x_1 - x_2| \quad \text{for } |x_1|, |x_2| \leq \rho.$$

**Lemma 4.4.** *The function  $F : \mathbb{R}^+ \times C([-r, 0]; X) \rightarrow X$  is also  $C^1$ -function. Moreover, the partial derivatives  $D_t F$  and  $D_\varphi F$  are locally Lipschitzian with respect to the second argument.*

*Proof.* The assumption (i) implies that  $F$  is continuously differentiable. Moreover, for  $\phi, \psi \in C([-r, 0]; X)$  and  $x \in [0, 1]$ , the following formula hold

$$D_t F(t, \varphi)(x) = \frac{\partial}{\partial t} f(t, \varphi(-\tau)(x))$$

$$D_\varphi F(t, \varphi)(\psi)(x) = \frac{\partial}{\partial x} f(t, \varphi(-\tau)(x))(\psi)(-\tau)(x).$$

$D_t F$  and  $D_\varphi F$  are locally Lipschitzian with respect to the second argument. In fact, for all  $\rho > 0$  and  $|\varphi_1| \leq \rho, |\varphi_2| \leq \rho$ , we have

$$\left| D_t F(t, \varphi_1)(x) - D_t F(t, \varphi_2)(x) \right| \leq \left| \frac{\partial}{\partial t} f(t, \varphi_1(-\tau)(x)) - \frac{\partial}{\partial t} f(t, \varphi_2(-\tau)(x)) \right|$$

$$\leq k_1 |\varphi_1(-\tau)(x) - \varphi_2(-\tau)(x)| \leq k_1 |\varphi_1 - \varphi_2|.$$

Moreover,

$$\left| D_\varphi F(t, \varphi_1)(\psi)(x) - D_\varphi F(t, \varphi_2)(\psi)(x) \right|$$

$$\leq \left| \frac{\partial}{\partial x} f(t, \varphi_1(-\tau)(x))(\psi)(-\tau)(x) - \frac{\partial}{\partial x} f(t, \varphi_2(-\tau)(x))(\psi)(-\tau)(x) \right|$$

$$\leq k_2 |\varphi_1(-\tau)(x) - \varphi_2(-\tau)(x)| |(\psi)(-\tau)(x)| \leq k_2 |\varphi_1 - \varphi_2| |\psi|. \quad \square$$

Now, to prove that the integral solution is a strict one, we recall that  $\varphi : [-r, 0] \rightarrow X$  is defined as above by  $\varphi(\theta) = w_0(\theta, \cdot)$ . We assume the following assumptions:

- (ii)  $\frac{\partial w_0}{\partial \theta} : [-r, 0] \times [0, 1] \rightarrow \mathbb{R}$  is continuous.

- (iii)  $\varphi'(0) \in \overline{D(A)}$  which means that

$$\frac{\partial}{\partial \theta} w_0(0, 0) = \frac{\partial}{\partial \theta} w_0(0, 1).$$

- (iv)  $\varphi'(0) = A\varphi(0) + F(0, \varphi)$  which means that

$$\frac{\partial}{\partial \theta} w_0(0, x) = -a(x) \frac{\partial}{\partial x} w_0(0, x) + F(0, w_0(-\tau, x)).$$

**Lemma 4.5.** *The function  $\varphi$  is  $C^1([-r, 0]; X)$ .*





where  $w_0 : [-r, 0] \times [0, 1] \rightarrow \mathbb{R}$  is a given continuous function,  $p \in BV_{loc}(\mathbb{R}^+, C([0, 1]; \mathbb{R}))$  and  $g : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous. We take the space  $X = C([0, 1]; \mathbb{R})$  the space of continuous functions from  $[0, 1]$  to  $\mathbb{R}$  provided with the supremum norm. Consider the linear operator  $A : D(A) \subset X \rightarrow X$  defined by

$$\begin{cases} D(A) = \{z \in C^2([0, 1]; \mathbb{R}) : z(0) = z(1) = 0\} \\ Az = z''. \end{cases}$$

It is well known from [2] that  $A$  satisfies the Hille–Yosida condition. Moreover, we can see that

$$\overline{D(A)} = \{z \in X : z(0) = z(1) = 0\} \neq X.$$

Thus (H0) is true. Let  $B : D(A) \subset X \rightarrow X$  be the operator defined by

$$\begin{cases} D(B(t)) = D(A) \\ (B(t)z)(x) = p(t, x)z''(x) \quad \text{for } t \geq 0 \text{ and } x \in [0, 1]. \end{cases}$$

From [20],  $(B(t))_{t \geq 0}$  is family of bounded linear operators from  $D(A)$  to  $X$  and  $B(\cdot)z \in BV_{loc}(\mathbb{R}^+; X)$  for any  $z \in D(A)$ . Then (H1) is satisfied and hence, by Theorem 2.25, Eq. (4.2) has a unique locally Lipschitz continuous integrated resolvent operator  $(R(t))_{t \geq 0}$  on  $X$ . Moreover, we assume that

(i)  $w_0(0, 0) = w_0(0, 1) = 0$ .

(ii) There exists a constant  $k > 0$ , such that

$$|g(t, x) - g(t, y)| \leq k|x - y| \quad \text{for } t \geq 0 \text{ and } x, y \in \mathbb{R}.$$

Let  $C = C([-r, 0]; X)$  and define the operator  $G : \mathbb{R}^+ \times C \rightarrow X$  by

$$G(t, \phi)(x) = \int_{-r}^0 g(t, \phi(\theta)(x)) d\theta \quad \text{for } x \in [0, 1] \text{ and } \phi \in C.$$

If we put

$$\begin{aligned} u(t) &= w(t, x) \quad \text{for } t \geq 0 \text{ and } x \in [0, 1], \\ \varphi(\theta)(x) &= w_0(\theta, x) \quad \text{for } \theta \in [-r, 0] \text{ and } x \in [0, 1]. \end{aligned}$$

Then Eq. (4.1) takes the following abstract form

$$\begin{cases} u'(t) = Au(t) + \int_0^t B(t-s)u(s)ds + G(t, u_t) \text{ for } t \geq 0 \\ u_0 = \varphi. \end{cases} \quad (4.5)$$

As a consequence of (i), we have  $\varphi(0) \in \overline{D(A)}$ . Moreover, the continuity of  $g$  means that  $G$  is continuous on  $\mathbb{R}^+ \times C$  with values in  $X$ , and (ii) imply (H2). Then, Corollary 3.5 ensures the existence and uniqueness of the integral solution of Eq. (4.4).

To establish the existence of strict solutions of Eq. (4.4), we assume the following:

(iii)  $g : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$  is  $C^1$ -function and  $\frac{\partial g}{\partial t}, \frac{\partial g}{\partial x}$  are locally Lipschitzian with respect to the second argument.

By (iii), the function  $G : \mathbb{R}^+ \times C \rightarrow X$  is  $C^1$ -function. Moreover, we have the following:

$$D_t G(t, \varphi)(x) = \int_{-r}^0 \frac{\partial}{\partial t} g(t, \varphi(\theta)(x)) d\theta$$

$$D_\varphi G(t, \varphi)(\psi)(x) = \int_{-r}^0 \frac{\partial}{\partial x} g(t, \varphi(\theta)(x))(\psi)(\theta)(x) d\theta.$$

It follows that the partial derivatives  $D_t G$  and  $D_\varphi G$  are locally Lipschitzian with respect to the second argument. Then (H3) holds.

- (iv)  $\frac{\partial w_0}{\partial \theta} : [-r, 0] \times [0, 1] \rightarrow \mathbb{R}$  is continuous.
- (v)  $\frac{\partial}{\partial \theta} w_0(0, 0) = \frac{\partial}{\partial \theta} w_0(0, 1) = 0$ .
- (vi)  $\frac{\partial}{\partial \theta} w_0(0, x) = \frac{\partial^2}{\partial x^2} w_0(0, x) + \int_{-r}^0 g(0, w_0(\theta, x)) d\theta$ .

From (v) and (vi) one has  $\varphi'(0) \in \overline{D(A)}$  and  $\varphi'(0) = A\varphi(0) + G(0, \varphi)$  respectively. Moreover, (iv) implies that the function  $\varphi$  is  $C^1([-r, 0]; X)$ . Consequently, all the conditions in Corollary 3.8 are satisfied. Hence  $u$  is a strict solution of Eq. (4.4) and the function  $w$  defined by  $w(t, x) = u(t)(x)$  for  $t \geq 0$  and  $x \in [0, 1]$  is a solution of Eq. (4.4).

**Remark 4.7.** Note that if  $A$  is densely defined and satisfies the Hille–Yosida conditions, then  $A$  generates a strongly continuous semigroup. So, our main results are also true in the resolvent operators context obtained in [14].

**Remark 4.8.** The results obtained here extend [14] to the case of nondensely defined operator  $A$ .

**Remark 4.9.** Our results extend the results proved in [1] for Eq. (1.1) in the case where  $B = 0$  and  $A$  satisfies the Hille–Yosida condition (i.e.  $A$  generates an integrated semigroup).

## Acknowledgements

The authors would like to thank the referees for their careful reading of the manuscript. Their valuable suggestions made numerous improvements throughout the paper.

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