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Differentiation of Solutions of Nonlocal Boundary Value Problems with Respect to Boundary Data

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Abstract

In this paper, we investigate boundary data smoothness for solutions of the nonlocal boundary value problem, $y^{(n)} = f(x, y, y', \dots, y^{(n-1)}), y^{(i)}(x_j) = y_{ij}$ and $y^{(i)}(x_k) - \sum_{p=1}^{m} r_{ip} y(\eta_{ip}) = y_{ik}$. Essentially, we show under certain conditions that partial derivatives of the solution to the problem above exist with respect to boundary conditions and solve the associated variational equation. Lastly, we provide a corollary and nontrivial example.

Keywords: Nonlinear boundary value problem, Variational equation, Ordinary differential equation, Nonlocal boundary condition, Uniqueness, Existence

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1 Introduction

Interest in nonlocal or multipoint boundary values problems for ordinary differential equations has been on the rise in recent years as can be seen in [1], [8], [19], [20], [26], and [27]. For dynamic equations on time scales, we refer the reader to [2]-[6], [9], [11], [13]-[14], [16], [18], [21]-[25]. The result of this paper is an extension and perhaps culmination of publications [7], [10], [12], and [15]. The astute reader may wish to investigate further the recent publication [17] which presents a similar result to the theorems presented here for difference equations.

2 Preliminaries

Our concern is characterizing partial derivatives with respect to the boundary data of solutions to the nth order nonlocal boundary value problem

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}),\ a < x < b, \tag{1}$$

satisfying

$$\begin{aligned} &y^{(i)}(x_j) = y_{ij},\ 0 \le i \le m_j - 1,\ 1 \le j \le k-1, \\ &y^{(i)}(x_k) - \sum_{p=1}^{m} r_{ip} y(\eta_{ip}) = y_{ik},\ 0 \le i \le m_k - 1, \end{aligned} \tag{2}$$

where $2 \le k \le n,\ m \in \mathbb{N},\ m_1, \dots, m_k$ are positive integers such that $\sum_{i=1}^{k} m_i = n,\ a < x_1 < x_2 < \cdots < x_{k-1} < \eta_{01} < \cdots < \eta_{m_k-1,m} < x_k < b$, and $y_{01}, \dots, y_{m_k-1,k}, r_{01}, \dots, r_{m_k-1,m} \in \mathbb{R}$.

We establish a few conditions that are imposed upon (1):

(i) $f(x, y_1, \dots, y_n) : (a, b) \times \mathbb{R}^n \to \mathbb{R}$ is continuous,

(ii) $\dfrac{\partial f}{\partial y_i}(x, y_1, \dots, y_n) : (a, b) \times \mathbb{R}^n \to \mathbb{R}$ are continuous, $i = 1, 2, \dots, n$, and

(iii) solutions of initial value problems for (1) extend to (a, b).

Remark 2.1 *Note that* (iii) *is not a necessary condition but lets us avoid continually making statements about maximal intervals of existence inside* (a, b).

The theorem presented in this work relies heavily upon the definition for the variational equation which we now give.

Definition 2.1 *Given a solution* $y(x)$ *of* (1)*, we define the variational equation along* $y(x)$ *by*

$$z^{(n)} = \sum_{i=1}^{n} \frac{\partial f}{\partial y_i}(x, y(x), y'(x), \dots, y^{(n-1)}(x)) z^{(i-1)}. \tag{3}$$

We seek an analogue of the following theorem that Hartmann, [9], attributes to Peano for (1), (2).

Theorem 2.1 *[A Peano Theorem] Assume that, with respect to* (1)*, conditions* (i)-(iii) *are satisfied. Let* $x_0 \in (a, b)$ *and* $y(x) := y(x, x_0, c_1, c_2, \dots, c_n)$ *denote the solution of* (1) *satisfying the initial conditions* $y^{(i-1)}(x_0) = c_i,\ 1 \le i \le n$*. Then,*

(a) *for each* $1 \le j \le n$, $\dfrac{\partial y}{\partial c_j}(x)$ *exists on* (a, b)*, and* $\alpha_j(x) := \dfrac{\partial y}{\partial c_j}(x)$ *is the solution of the variational equation* (3) *along* $y(x)$ *satisfying the initial conditions*

$$\alpha_j^{(i-1)}(x_0) = \delta_{ij},\ 1 \le i \le n.$$

(b) $\dfrac{\partial y}{\partial x_0}(x)$ *exists on* (a, b)*, and* $\beta(x) := \dfrac{\partial y}{\partial x_0}(x)$ *is the solution of the variational equation* (3) *along* $y(x)$ *satisfying the initial conditions*

$$\beta^{(i-1)}(x_0) = -y^{(i)}(x_0),\ 1 \le i \le n.$$

(c) $\dfrac{\partial y}{\partial x_0}(x) = -\displaystyle\sum_{i=1}^{n} y^{(i)}(x_0) \frac{\partial y}{\partial c_i}(x).$

The next condition guarantees uniqueness of solutions of (1), (2) and is a nonlocal analogue of $(m_1, \dots, m_k)$-disconjugacy:

(iv) Let $2 \le k \le n$, $m \in \mathbb{N}$, and $m_1, \dots, m_k$ be positive integers such that $\sum_{i=1}^{k} m_i = n$. Given $a < x_1 < x_2 < \cdots < x_{k-1} < \eta_{01} < \cdots < \eta_{m_k-1,m} < x_k < b$ and $r_{01}, \dots, r_{m_k-1,m} \in \mathbb{R}$, if, for $0 \le i \le m_j - 1,\ 1 \le j \le k - 1$,

$$y^{(i)}(x_j) = z^{(i)}(x_j),$$

and, for $0 \le i \le m_k - 1$,

$$y^{(i)}(x_k) - \sum_{p=1}^{m} r_{ip} y(\eta_{ip}) = z^{(i)}(x_k) - \sum_{p=1}^{m} r_{ip} z(\eta_{ip}),$$

where $y(x)$ and $z(x)$ are solutions of (1), then, on (a, b),

$$y(x) \equiv z(x).$$

The last condition provides uniqueness of solutions of (3) along all solutions of (1) and again is a nonlocal analogue of $(m_1, \dots, m_k)$-disconjugacy:

(v) Let $2 \le k \le n$, $m \in \mathbb{N}$,, and $m_1, \ldots, m_k$ be positive integers such that $\sum_{i=1}^{k} m_i = n$. Given $a < x_1 < x_2 < \cdots < x_{k-1} < \eta_{01} < \cdots < \eta_{m_k-1,m} < x_k < b$ and $r_{01}, \ldots, r_{m_k-1,m} \in \mathbb{R}$, and a solution $y(x)$ of (1), if, for $0 \le i \le m_j - 1$, $1 \le j \le k-1$,

$$u^{(i)}(x_j) = 0,$$

and, for $0 \le i \le m_k - 1$,

$$u^{(i)}(x_k) - \sum_{p=1}^{m} r_{ip} u(\eta_{ip}) = 0,$$

where $u(x)$ is a solution of (3) along $y(x)$, then, on (a, b),

$$u(x) \equiv 0.$$

We also make much use of a well known continuous dependence result which is an application of the Brouwer Invariance of Domain Theorem.

Theorem 2.2 *Assume* (i)-(iv) *are satisfied with respect to* (1). *Let* $2 \le k \le n$, $m \in \mathbb{N}$, *and* $m_1, \ldots, m_k$ *be positive integers such that* $\sum_{i=1}^{k} m_i = n$. *Let* $u(x)$ *be a solution of* (1) *on* (a, b), *and let* $a < c < x_1 < x_2 < \cdots < x_{k-1} < \eta_{01} < \cdots < \eta_{m_k-1,m} < x_k < d < b$ *and* $r_{01}, \ldots, r_{m_k-1,m} \in \mathbb{R}$ *be given. Then, there exists a* $\delta > 0$ *such that, for*

$$|x_j - t_j| < \delta,\ 1 \le j \le k,$$

$$|\eta_{ip} - \tau_{ip}| < \delta \text{ and } |r_{ip} - \rho_{ip}| < \delta,\ 0 \le i \le m_k - 1,\ 1 \le p \le m,$$

$$|u^{(i)}(x_j) - y_{ij}| < \delta,\ 0 \le i \le m_j - 1,\ 1 \le j \le k-1,$$

and

$$|u^{(i)}(x_k) - \sum_{p=1}^{m} r_{ip} u(\eta_{ip}) - y_{ik}| < \delta,\ 0 \le i \le m_k - 1,$$

there exists a unique solution $u_\delta(x)$ *of* (1) *such that*

$$u_\delta^{(i)}(t_j) = y_{ij},\ 0 \le i \le m_j - 1,\ 1 \le j \le k-1,$$

$$u_\delta^{(i)}(t_k) - \sum_{p=1}^{m} \rho_{ip} u_\delta(\tau_{ip}) = y_{ik},\ 0 \le i \le m_k - 1,$$

and, for $0 \le i \le n-1$, $\{u_\delta^{(i)}(x)\}$ *converges uniformly to* $u^{(i)}(x)$ *as* $\delta \to 0$ *on* $[c, d]$.

3 Analogue of Peano's Theorem

In this section, we present our analogue to Theorem 2.1. The result is stated in four parts, but each proof is essentially the same. Thus, in the interest of time and space, we only prove part (b).

Theorem 3.1 *Assume conditions* (i)-(v) *are satisfied. Let* $n \ge 2$, $m \in \mathbb{N}$, *and* $2 \le k \le n$ *be given and* $m_1, \ldots, m_k$ *be positive integers such that* $\sum_{i=1}^{k} m_i = n$. *Let* $u(x)$ *be a solution of* (1) *on* (a, b). *Let* $a < x_1 < \cdots < x_{k-1} < \eta_{01} < \cdots < \eta_{m_k-1,m} < x_k < b$ *and* $u_{01}, \ldots, u_{m_k-1,k}, r_{01}, \ldots, r_{m_k-1,m} \in \mathbb{R}$ *be given so that*

$$u(x) = u(x, x_1, \ldots, x_k, u_{01}, \ldots, u_{m_k-1,k}, \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m}),$$

where

$$u^{(i)}(x_j) = u_{ij},\ 0 \le i \le m_j - 1,\ 1 \le j \le k-1,$$

and

$$u^{(i)}(x_k) - \sum_{p=1}^{m} r_{ip} u(\eta_{ip}) = u_{ik},\ 0 \le i \le m_k - 1.$$

Then,

(a) *for each* $1 \le l \le k-1$ *and* $0 \le r \le m_l - 1$, $\dfrac{\partial u}{\partial u_{rl}}(x)$ *exists on* (a,b), *and* $y_{rl}(x) := \dfrac{\partial u}{\partial u_{rl}}(x)$ *is the solution of the variational equation* (3) *along* $u(x)$ *satisfying the boundary conditions*

$$\begin{aligned} y_{rl}^{(i)}(x_j) &= 0, \qquad 0 \le i \le m_j - 1,\ 1 \le j \le k-1,\ j \ne l, \\ y_{rl}^{(i)}(x_l) &= 0, \qquad 0 \le i \le m_j - 1,\ i \ne r, \\ y_{rl}^{(r)}(x_l) &= 1, \\ y_{rl}^{(i)}(x_k) &- \sum_{p=1}^{m} r_{ip} y_{rl}(\eta_{ip}) = 0, \qquad 0 \le i \le m_k - 1, \end{aligned}$$

and for $0 \le r \le m_k - 1$, $y_{rk}(x) := \dfrac{\partial u}{\partial u_{rk}}(x)$ *exists on* (a,b) *and solves* (3) *along* $u(x)$ *satisfying the boundary conditions*

$$\begin{aligned} y_{rk}^{(i)}(x_j) &= 0, \qquad 0 \le i \le m_j - 1,\ 1 \le j \le k-1, \\ y_{rk}^{(i)}(x_k) &- \sum_{p=1}^{m} r_{ip} y_{rk}(\eta_{ip}) = 0, \qquad 0 \le i \le m_k - 1,\ i \ne r, \\ y_{rk}^{(r)}(x_k) &- \sum_{p=1}^{m} r_{rp} y_{rk}(\eta_{rp}) = 1. \end{aligned}$$

(b) *for each* $1 \le l \le k-1$, $\dfrac{\partial u}{\partial x_l}(x)$ *exists on* (a,b), *and* $z_l(x) := \dfrac{\partial u}{\partial x_l}(x)$ *is the solution of the variational equation* (3) *along* $u(x)$ *satisfying the boundary conditions*

$$\begin{aligned} z_l^{(i)}(x_j) &= 0, \qquad 0 \le i \le m_j - 1,\ 1 \le j \le k-1,\ j \ne l, \\ z_l^{(i)}(x_l) &= -u^{(i+1)}(x_l), \qquad 0 \le i \le m_l - 1, \\ z_l^{(i)}(x_k) &- \sum_{p=1}^{m} r_{ip} z_l(\eta_{ip}) = 0, \qquad 0 \le i \le m_k - 1, \end{aligned}$$

and $z_k(x) := \dfrac{\partial u}{\partial x_k}(x)$ *exists on* (a,b) *and solves* (3) *along u(x) satisfying the boundary conditions*

$$\begin{aligned} z_k^{(i)}(x_j) &= 0, \qquad 0 \le i \le m_j - 1,\ 1 \le j \le k-1, \\ z_k^{(i)}(x_k) &- \sum_{p=1}^{m} r_{ip} z_k(\eta_{ip}) = -u^{(i+1)}(x_k), \qquad 0 \le i \le m_k - 1. \end{aligned}$$

(c) *for* $0 \le r \le m_k - 1$ *and* $1 \le s \le m$, $\dfrac{\partial u}{\partial \eta_{rs}}(x)$ *exists on* (a,b), *and* $w_{rs}(x) := \dfrac{\partial u}{\partial \eta_{rs}}(x)$ *is the solution of* (3) *along* $u(x)$ *satisfying the boundary conditions*

$$\begin{aligned} w_{rs}^{(i)}(x_j) &= 0, \qquad 0 \le i \le m_j - 1,\ 1 \le j \le k-1, \\ w_{rs}^{(i)}(x_k) &- \sum_{p=1}^{m} r_{ip} w_{rs}(\eta_{ip}) = 0, \qquad 0 \le i \le m_k - 1,\ i \ne r, \\ w_{rs}^{(r)}(x_k) &- \sum_{p=1}^{m} r_{rp} w_{rs}(\eta_{rs}) = r_{rs} u'(\eta_{rs}). \end{aligned}$$

(d) *for* $0 \le r \le m_k - 1$ *and* $1 \le s \le m$, $\dfrac{\partial u}{\partial r_{rs}}(x)$ *exists on* (a, b), *and* $v_{rs}(x) := \dfrac{\partial u}{\partial r_{rs}}(x)$ *is the solution of* (3) *along* $u(x)$ *satisfying the boundary conditions*

$$\begin{aligned} v_{rs}^{(i)}(x_j) &= 0, \qquad 0 \le i \le m_j - 1,\ 1 \le j \le k-1, \\ v_{rs}^{(i)}(x_k) &- \sum_{p=1}^{m} r_{ip} v_{rs}(\eta_{ip}) = 0, \qquad 0 \le i \le m_k - 1,\ i \ne r, \\ v_{rs}^{(r)}(x_k) &- \sum_{p=1}^{m} r_{rp} v_{rs}(\eta_{rp}) = u(\eta_{rs}). \end{aligned}$$

Proof: We will only prove part (b) as the proofs associated with (a), (c), and (d) follow similarly. Let $1 \le l \le k-1$, and consider $\frac{\partial u}{\partial x_l}$. Since the argument for $\frac{\partial u}{\partial x_k}$ is essentially the same, we omit its proof.

In the interests of conserving space and lessening the tedious notation, we will denote $u(x, x_1, \ldots, x_l, \ldots, x_k, u_{01}, \ldots, u_{m_k-1,k}, \eta_{01}, \ldots, \eta_{m_k-1,m}, r_{01}, \ldots, r_{m_k-1,m})$ by $u(x, x_l)$ as x_l is the parameter of interest. Let $\delta > 0$ be as in Theorem 2.2, $0 < |h| < \delta$ be given, and define

$$z_{lh}(x) = \frac{1}{h}[u(x, x_l + h) - u(x, x_l)].$$

Note that for every $h \ne 0$ and $1 \le i \le m_l - 1$,

$$\begin{aligned} z_{lh}^{(i)}(x_l) &= \frac{1}{h}[u^{(i)}(x_l, x_l + h) - u^{(i)}(x_l, x_l] \\ &= \frac{1}{h}[u^{(i)}(x_l, x_l + h) - u^{(i)}(x_l + h, x_l + h) + u^{(i)}(x_l + h, x_l + h) - u^{(i)}(x_l, x_l,)] \\ &= -\frac{1}{h}[u^{(i+1)}(c_{x_l,h}, x_l + h) \cdot h + u_{il} - u_{il}] \\ &= -u^{(i+1)}(c_{x_l,h}, x_l + h), \end{aligned}$$

where $c_{x_l,h}$ lies between x_l and $x_l + h$.

Also, for every $h \ne 0$, $0 \le i \le m_j - 1$, $1 \le j \le k-1$, and $j \ne l$,

$$\begin{aligned} z_{lh}^{(i)}(x_j) &= \frac{1}{h}[u^{(i)}(x_j, x_l + h) - u^{(i)}(x_j, x_l)] \\ &= \frac{1}{h}[u_{ij} - u_{ij}] \\ &= 0, \end{aligned}$$

and for every $h \ne 0$ and $0 \le i \le m_k - 1$,

$$\begin{aligned} z_{lh}^{(i)}(x_k) &- \sum_{p=1}^{m} r_{ip} z_{lh}(\eta_{ip}) \\ &= \frac{1}{h}[u^{(i)}(x_k, x_l + h) - u^{(i)}(x_k, x_l,)] - \sum_{p=1}^{m} \frac{r_{ip}}{h}[u(\eta_{ip}, x_l + h) - u(\eta_{ip}, x_l)] \\ &= \frac{1}{h}[u_{ik} - u_{ik}] \\ &= 0. \end{aligned}$$

Now that we have established the boundary conditions, for $m_l \le i \le n-1$, let

$$\beta_i = u^{(i)}(x_l, x_l),$$

and

$$\epsilon_i = \epsilon_i(h) = u^{(i)}(x_l, x_l + h) - \beta_i.$$

By Theorem 2.2, for $m_l \le i \le n-1,\ \epsilon_i = \epsilon_i(h) \to 0$ as $h \to 0$. Using the notation of Theorem 2.1 for solutions of initial value problems for (1), viewing $u(x)$ as the solution of an initial value problem, and denoting a solution $u(x) = y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \ldots, \beta_{n-1})$, we have

$$\begin{aligned} z_{lh}(x) \quad &= \quad \frac{1}{h}[y(x, x_l + h, u_{0l}, \ldots, \ldots, u_{m_l-1,l}, \beta_{m_l} + \epsilon_{m_l}, \\ &\qquad \beta_{m_l+1} + \epsilon_{m_l+}, \ldots, \beta_{n-1} + \epsilon_{n-1}) \\ &\qquad -y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1} \ldots, \beta_{n-1})]. \end{aligned}$$

Then, by utilizing a telescoping sum, we have

$$\begin{aligned} z_{lh}(x) \quad = \quad & \frac{1}{h}\big\{[y(x, x_l + h, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \epsilon_{m_l}, \\ & \qquad \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1}) \\ & -y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \epsilon_{m_l}, \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})] \\ & +[y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \epsilon_{m_l}, \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1}) \\ & -y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})] \\ & +[y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1}) \\ & -y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})] \\ & + - \cdots \\ & +[y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1}) \\ & -y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1}, \ldots, \beta_{n-1})]\big\}. \end{aligned}$$

By Theorem 2.1 and the Mean Value Theorem, we obtain

$$\begin{aligned} z_{lh}(x) \quad = \quad & \beta(x, y(x, x_l + \bar{h}, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \epsilon_{m_l}, \\ & \qquad \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})) \\ & + \frac{\epsilon_{m_l}}{h}\alpha_{m_l}(x, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \bar{\epsilon}_{m_l}, \\ & \qquad \qquad \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})) \\ & + \frac{\epsilon_{m_l+1}}{h}\alpha_{m_l+1}(x, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \\ & \qquad \qquad \beta_{m_l+1} + \bar{\epsilon}_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})) \\ & + \cdots \\ & + \frac{\epsilon_{n-1}}{h}\alpha_{n-1}(x, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \\ & \qquad \qquad \beta_{m_l+1}, \ldots, \beta_{n-1} + \bar{\epsilon}_{n-1}), \end{aligned}$$

where $\beta(x, y(\cdot))$ is the solution of the variational equation (1) along $y(\cdot)$ satisfying

$$\beta^{(i)}(x_l, y(\cdot)) = -y^{(i+1)}(x_l),\ 0 \le i \le n-1,$$

and where, for $0 \le j \le n-1,\ \alpha_j(x, y(\cdot))$ is the solution of the variational equation (1) along $y(\cdot)$ satisfying

$$\alpha_j^{(i)}(x_l) = \delta_{ij}, \quad 0 \le i \le n-1.$$

Furthermore, $x_l + \bar{h}$ is between x_l and $x_l + h$, and for $m_l \le i \le n-1,\ \beta_i + \bar{\epsilon}_i$ is between β_i and $\beta_i + \epsilon_i$.

Thus, to show $\lim\limits_{h\to 0} z_{lh}(x)$ exists, it suffices to show, for $m_l \le i \le n-1,\ \lim\limits_{h \to 0} \dfrac{\epsilon_i}{h}$ exists.

Now, from the construction of $z_{lh}(x)$, we have

$$z_{lh}^{(i)}(x_j) = 0, \qquad 0 \le i \le m_j - 1,\ 1 \le j \le k-1,\ j \ne l,$$

and

$$z_{lh}^{(i)}(x_k) - \sum_{p=1}^{m} r_{ip} z_{lh}(\eta_{ip}) = 0, \qquad 0 \le i \le m_k - 1.$$

Hence, we have a system of $n - m_l$ linear equations with $n - m_l$ unknowns:

$$\begin{aligned}
- &\quad \beta^{(i)}(x_j, y(x, x_l + \bar{h}, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \epsilon_{m_l}, \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})) \\
= &\quad \frac{\epsilon_{m_l}}{h} \alpha_{m_l}^{(i)}(x_j, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \bar{\epsilon}_{m_l}, \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})) \\
&\quad + \cdots \\
&\quad + \frac{\epsilon_{n-1}}{h} \alpha_{n-1}^{(i)}(x_j, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1}, \ldots, \beta_{n-1} + \bar{\epsilon}_{n-1})), \\
&\quad 0 \le i \le m_j - 1,\ 1 \le j \le k-1,\ j \ne l,
\end{aligned}$$

and

$$\begin{aligned}
- &\quad \beta^{(i)}(x_k, y(x, x_l + \bar{h}, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \epsilon_{m_l}, \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})) \\
&\quad + \sum_{p=1}^{m} r_{ip} \beta(\eta_{ip}, y(x, x_l + \bar{h}, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \epsilon_{m_l}, \\
&\qquad\qquad \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})) \\
= &\quad \frac{\epsilon_{m_l}}{h} \Big[\alpha_{m_l}^{(i)}(x_k, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \bar{\epsilon}_{m_l}, \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})) \\
&\quad - \sum_{p=1}^{m} r_{ip} \alpha_{m_l}(\eta_{ip}, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l} + \bar{\epsilon}_{m_l}, \\
&\qquad\qquad \beta_{m_l+1} + \epsilon_{m_l+1}, \ldots, \beta_{n-1} + \epsilon_{n-1})) \Big] \\
&\quad + \cdots \\
&\quad + \frac{\epsilon_{n-1}}{h} \Big[\alpha_{n-1}^{(i)}(x_k, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1}, \ldots, \beta_{n-1} + \bar{\epsilon}_{n-1})) \\
&\quad - \sum_{p=1}^{m} r_{ip} \alpha_{n-1}(\eta_{ip}, y(x, x_l, u_{0l}, \ldots, u_{m_l-1,l}, \beta_{m_l}, \beta_{m_l+1}, \ldots, \beta_{n-1} + \bar{\epsilon}_{n-1})) \Big], \\
&\quad 0 \le i \le m_k - 1.
\end{aligned}$$

At this point in the proof, we will occasionally suppress the arguments of α and β as well as the subscripts of r and η, and limits of the summation. In the system of equations above, we notice that $y(\cdot)$ is not always the same. Therefore, we must consider the matrix

$$M := \begin{pmatrix} \alpha_{m_l}(x_1,u(x)) & \alpha_{m_l+1}(x_1,u(x)) & \cdots & \alpha_{n-1}(x_1,u(x)) \\ \alpha'_{m_l}(x_1,u(x)) & \alpha'_{m_l+1}(x_1,u(x)) & \cdots & \alpha'_{n-1}(x_1,u(x)) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m_l}^{(m_1-1)}(x_1,u(x)) & \alpha_{m_l+1}^{(m_1-1)}(x_1,u(x)) & \cdots & \alpha_{n-1}^{(m_1-1)}(x_1,u(x)) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m_l}^{(m_{l-1}-1)}(x_{l-1},u(x)) & \alpha_{m_l+1}^{(m_{l-1}-1)}(x_{l-1},u(x)) & \cdots & \alpha_{n-1}^{(m_{l-1}-1)}(x_{l-1},u(x)) \\ \alpha_{m_l}(x_{l+1},u(x)) & \alpha_{m_l+1}(x_{l+1},u(x)) & \cdots & \alpha_{n-1}(x_{l+1},u(x)) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m_l}(x_k,u(x))- & \alpha_{m_l+1}(x_k,u(x))- & & \alpha_{n-1}(x_k,u(x))- \\ \sum r\alpha_{m_l}(\eta,u(x)) & \sum r\alpha_{m_l+1}(\eta,u(x)) & \cdots & \sum r\alpha_{n-1}(\eta,u(x)) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m_l}^{(m_k-1)}(x_k,u(x))- & \alpha_{m_l+1}^{(m_k-1)}(x_k,u(x))- & & \alpha_{n-1}^{(m_k-1)}(x_k,u(x))- \\ \sum r\alpha_{m_l}(\eta,u(x)) & \sum r\alpha_{m_l+1}(\eta,u(x)) & \cdots & \sum r\alpha_{n-1}(\eta,u(x)) \end{pmatrix}.$$

We claim $\det(M) \neq 0$. Suppose to the contrary that $\det(M) = 0$. Then there exist $p_i \in \mathbb{R}$, $m_l \leq i \leq n-1$, not all zero such that

$$p_{m_l} \begin{pmatrix} \alpha_{m_l}(x_1,u(x)) \\ \alpha'_{m_l}(x_1,u(x)) \\ \vdots \\ \alpha_{m_l}^{(m_{l-1}-1)}(x_{l-1},u(x)) \\ \alpha_{m_l}(x_{l+1},u(x)) \\ \vdots \\ \alpha_{m_l}^{(m_k-1)}(x_k,u(x))- \\ \sum r\alpha_{m_l}(\eta,u(x)) \end{pmatrix} + \cdots + p_{n-1} \begin{pmatrix} \alpha_{n-1}(x_1,u(x)) \\ \alpha'_{n-1}(x_1,u(x)) \\ \vdots \\ \alpha_{n-1}^{(m_{l-1}-1)}(x_{l-1},u(x)) \\ \alpha_{n-1}(x_{l+1},u(x)) \\ \vdots \\ \alpha_{n-1}^{(m_k-1)}(x_k,u(x))- \\ \sum r\alpha_{n-1}(\eta,u(x)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ \\ 0 \end{pmatrix}.$$

Set

$$w(x,u(x)) := p_{m_l}\alpha_{m_l}(x,u(x)) + \cdots + p_{n-1}\alpha_{n-1}(x,u(x)).$$

Then, $w(x,u(x))$ is a nontrivial solution of (3), but

$$w^{(i)}(x_j,u(x)) = 0,\ 0 \leq i \leq m_j - 1,\ 1 \leq j \leq k-1,$$

and

$$w^{(i)}(x_k,u(x)) - \sum_{p=1}^{m} r_{ip} w(\eta_{ip},u(x)) = 0,\ 0 \leq i \leq m_k - 1,$$

which when coupled with hypothesis (v) implies $w(x,u(x)) = 0$. Thus, $p_{m_l} = p_{m_{l+1}} = \cdots = p_{n-1} = 0$ which is a contradiction to the choice of the $p_i's$. Hence $\det(M) \neq 0$. Thus, as a result of continuous dependence, for $h \neq 0$ and sufficiently small, $\det(M(h)) \neq 0$ implying $M(h)$ has an inverse where $M(h)$ is the appropriately defined matrix from the system of equations. Therefore, for each $m_l \leq i \leq n-1$, we can solve $\epsilon_i(h)/h$ by using Cramer's rule:

$$\begin{aligned} \frac{\epsilon_i(h)}{h} &= \frac{1}{|M(h)|} \times \\ &\begin{vmatrix} \alpha_{m_l} & \cdots & \alpha_{i-2} & -\beta & \alpha_i & \cdots & \alpha_{n-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{m_l}^{(m_k-1)}- & & \alpha_{i-2}^{(m_k-1)} & -\beta^{(m_k-1)}+ & \alpha_i^{(m_k-1)}- & & \alpha_{n-1}^{(m_k-1)}- \\ \sum r\alpha_{m_l} & \cdots & \sum r\alpha_{i-2} & \sum r\beta & \sum r\alpha_i & \cdots & \sum r\alpha_{n-l} \end{vmatrix} \end{aligned}$$

Note as $h \to 0$, $\det(M(h)) \to \det(M)$, and so for $m_l \le i \le n-1$, $\epsilon_i(h)/h \to \det(M_i)/\det M := B_i$ as $h \to 0$, where M_i is the $n - m_l \times n - m_l$ matrix found by replacing the appropriate column of the matrix defining M by

$$\begin{aligned}\text{col}\Big[&- \beta(x_1, u(x)), \ldots, -\beta^{(m_1-1)}(x_1, u(x)), \ldots, \\ &-\beta(x_{l-1}, u(x)), \ldots, -\beta^{(m_{l-1}-1)}(x_{l-1}, u(x)), \\ &-\beta(x_{l+1}, u(x)), \ldots, -\beta^{(m_{l+1}-1)}(x_{l+1}, u(x)), \ldots, \\ &-\beta(x_k, u(x)) - \sum_{p=1}^{m} r_{0p}\beta(\eta_{0p}, u(x)), \ldots, \\ &-\beta^{(m_k-1)}(x_k, u(x)) - \sum_{p=1}^{m} r_{m_k-1,p}\beta(\eta_{m_k-1,p}, u(x))\Big].\end{aligned}$$

Now let $z_l(x) = \lim_{h\to 0} z_{lh}(x)$, and note by construction of $z_{lh}(x)$,

$$z_l(x) = \frac{\partial u}{\partial x_l}(x).$$

Furthermore,

$$z_l(x) = \lim_{h\to 0} z_{lh}(x) = \beta(x, u(x)) + \sum_{i=m_l}^{n-1} B_i\alpha_i(x, u(x))$$

which is a solution of the variational equation (3) along $u(x)$. In addition,

$$z_l^{(i)}(x_j) = \lim_{h\to} z_{lh}^{(i)}(x_j) = -u^{(i+1)}(x_j)\delta_{jl},\ 0 \le i \le m_j - 1,\ 1 \le j \le k-1,$$

and

$$z_l^{(i)}(x_k) - \sum_{p=1}^{m} r_{ip}z_l(\eta_{ip}) = \lim_{h\to 0}\left[z_{lh}^{(i)}(x_k) - \sum_{p=1}^{m} r_{ip}z_{lh}(\eta_{ip})\right] = 0,\ 0 \le i \le m_k - 1.$$

This completes the argument for $\frac{\partial u}{\partial x_l}$. □

4 Corollary and Nontrivial Example

We now present a corollary that follows from Theorem 3.1. The proof is immediate from the n-dimensionality of the solution space for the variational equation (3) along solutions of (1), and also creates a nice analogue to part (c) of Theorem 2.1 of Peano.

Corollary 4.1 *Assume the conditions of Theorem 3.1. Then,*

(*a*) *for each* $1 \le l \le k$,

$$\frac{\partial u}{\partial x_l}(x) = -\sum_{r=0}^{m_l-1} u^{(r+1)}(x_l)\frac{\partial u}{\partial u_{rl}}(x).$$

(*b*) *for* $0 \le r \le m_k - 1$ *and* $1 \le s \le m$,

$$\frac{\partial u}{\partial \eta_{rs}}(x) = r_{rs}\frac{u'(\eta_{rs})}{u(\eta_{rs})}\frac{\partial u}{\partial r_{rs}}(x).$$

Finally, we give a nontrivial example.

Example 4.1 *Consider the BVP*

$$y'' - y = 0, \tag{4}$$

$$y(x_1) = y_1, \qquad y(x_2) - ry(\eta) = y_2, \tag{5}$$

where $x_1, x_2, \eta, y_1, y_2, r \in \mathbb{R}$ *with* $x_1 < \eta < x_2$.

If we impose the condition $r \neq \dfrac{\sinh(x_2 - x_1)}{\sinh(\eta - x_1)}$, then (4), (5) satisfy condtions (i)-(v), and the results stated in 3.1 hold. Verification is left to the reader.

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