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On the non-autonomous Hopf bifurcation problem: systems with rapidly varying coefficients

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Abstract. We consider a 2-dimensional ordinary differential equation (ODE) depending on a parameter ϵ. If the ODE is autonomous the supercritical Andronov–Hopf bifurcation theory gives sufficient conditions for the genesis of a repeller–attractor pair, made up by a critical point and a stable limit cycle respectively. We give assumptions that enable us to reproduce the analogous phenomenon in a non-autonomous context, assuming that the coefficients of the system are subject to fast oscillations, and have very weak recurrence properties, e.g. they are almost periodic (in fact we just need that the associated base flow is uniquely ergodic). In this context the critical point is replaced by a trajectory which is a copy of the base and the limit cycle by an integral manifold. The dynamics inside the attractor becomes much richer and, if one asks for stronger recurrence assumptions, e.g. the coefficients are quasi periodic, it can be (partially) analyzed by the methods of [M. Franca, R. Johnson, V. Muñoz-Villarragut, *Discrete Contin. Dyn. Syst. Ser. S* **9**(2016), No. 4, 1119–1148].

The problem is in fact studied as a two parameters problem: we use ϵ to describe the size of the perturbation and $1/\mu$ to describe the speed of oscillations, but the results allows to set $\epsilon = \mu$.

Keywords: Andronov–Hopf bifurcation, fast varying coefficients, integral manifolds, non-autonomous bifurcation.

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Dedication The ideas for this article and the proofs of the results came out from the common work of the two authors. Unfortunately professor Johnson suddenly passed away on the 22^{nd} of July 2017, before the manuscript was finished. Therefore (unluckily for the readers) M. Franca has written most of it. M. Franca wishes to express all his gratitude to R. Johnson, who was the one who introduced him to research activities. Working with him has always been a pleasure, a stimulating experience, and an opportunity to learn and to improve as a mathematician.

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1 Introduction

In this paper, along with [12,13], we aim to give some insight to the rising field of bifurcation theory for non-autonomous systems, which nowadays is a well investigated topic, see e.g. [10, 11, 19, 21–23, 27, 28, 31, 32]. In particular in [21, 23, 27, 30] the generalization to a non-autonomous context of transcritical, saddle-node and pitchfork bifurcation was considered. In fact in [30, Section 7.3] Rasmussen considered also the case of the Andronov–Hopf bifurcation (AH bifurcation for short) assuming that the system is asymptotically autonomous. In [12] we have already analyzed the effect of a small non-autonomous perturbation on an autonomous system exhibiting an AH bifurcation: we mainly used the methods of [32], and we showed the existence of an exponentially stable integral manifold, which plays the role of the stable limit cycle of the autonomous case. Then we analyzed the dynamics on the stable limit cycle, motivated by [6, 7, 19]. In [13] we considered the effect of a fast varying perturbation on the typical one-dimensional bifurcation patterns: transcritical, saddle-node, pitchfork. In that case the analysis was built up on the change of variables constructed in [1,2]. In this article we focus on non autonomous systems subject to fast oscillations, assuming that the average undergoes to AH bifurcation, and again we mainly rely on [12,32] for the construction of the asymptotically stable integral manifold, and on [12] for the analysis of the dynamics in it.

Let us briefly recall what an AH bifurcation is, see [12,25] for details. In the supercritical case one looks for conditions sufficient to guarantee that system

$$\dot{x} := \frac{dx}{dt} = f(x,\epsilon)\,, \qquad t \in \mathbb{R}\,, \quad x \in \mathbb{R}^2 \tag{1.1}$$

where $f(0,\epsilon) \equiv 0$, f sufficiently smooth, has the following features

a) the origin $x = 0$ is an exponentially asymptotically stable critical point of (1.1) for $\epsilon < 0$;

b) there is an exponentially asymptotically orbitally stable periodic solution of (1.1) for $\epsilon > 0$, whose graph is denoted by Γ,

see [12] for more details.

Our purpose is to reproduce a pattern with characteristics analogous to **a)**, **b)** for an equation of the form:

$$\dot{x} = f(x,\epsilon) + \epsilon g\left(\frac{t}{\epsilon}, x, \epsilon\right) \qquad t \in \mathbb{R}\,,\ x \in \mathbb{R}^2, \tag{1.2}$$

where g is bounded, smooth and it has very weak recurrence properties.

Problems with rapid oscillations, besides their intrinsic mathematical interest, have a great relevance in applications. In fact it is well known that when the coefficients of a mechanical system are subject to a 0 average oscillation, this may alter in a substantial way the stability of the system. Some examples in this directions are given by, but non limited to, the phenomenon of stabilization of a planar pendulum via vertical oscillations [3], the elimination of a Van der Pol oscillation, and the large-scale alteration of a stability diagram in a catalytic reactor [1,2].

As already pointed out in [12] the very concept of AH bifurcation itself is not immediately clear in a non-autonomous context: this in fact happens for all the types of bifurcations. Roughly speaking critical points are likely to be replaced by trajectories which are copies of the base, i.e. they have the same recurrence properties as the function g (e.g. periodic of period ϵT if g is periodic of period T), while the graph of a periodic trajectory should be replaced by a t-dependent attracting integral manifold $W_\epsilon(t)$. Further, the analysis of the dynamics on the integral manifold $W_\epsilon(t)$ will become rather problematic, rarely resulting in the simple "addition of a frequency" to the frequency moduli of g.

Our scheme is to go from mild requirements which allow to reprove the existence of an attractor–repeller pair, to more stringent ones, which enable us to give some enlight on the dynamics of the "attractor" i.e. the set that plays the role of the orbitally stable limit cycle. We recall that the dynamics of the "attractor" then organizes the long time behavior of all the trajectories in a neighborhood of the origin (apart from the one in the repeller).

We assume that the autonomous system (1.1) exhibits a AH bifurcation and that g has 0 time-average, see Section 2 for details. The starting point of our analysis is the classical Bebutov hull construction which allows to apply the well developed machinery of skew-product semiflows, see e.g. [10, 19, 22, 24], see also [12, 13] and Section 2 of this article. We start by assuming that the dynamics in the base is uniquely ergodic (see Definition 2.1) and that $g(t, 0, \epsilon) = O(\sqrt{\epsilon})$: these conditions permit us to prove the existence of a repeller which is a copy of the base (Proposition 3.8), and of a positively invariant annulus $\mathfrak{A}$. The annulus $\mathfrak{A}$ contains a time dependent set $W_\epsilon(t)$ which is a set topologically equivalent to S^1 (See Definition 3.1), which, roughly speaking, "plays the role of the attractor". This is the content of Proposition 3.9, which is based on Wazewski's principle and a topological lemma borrowed from [29].

Afterwards we ask for stronger conditions to gain a better insight on the dynamics: we require that $g(t, 0, \epsilon, \mu) \equiv 0$ so that the repeller is the origin and that g has stronger recurrence properties, i.e. the base flow has non-positive Lyapunov exponents, e.g. g is almost periodic, cf Remark 3.13. Hence in Theorem 3.12, using the methods of [32], we show that $W_\epsilon(t)$ is indeed an exponentially stable integral manifold, lying $o(\sqrt{\epsilon})$ close to Γ. Namely for any fixed $t \in \mathbb{R}$, $W_\epsilon(t)$ is the graph of a continuous function $v_t(\theta, \epsilon) : S^1 \times [0, \epsilon_0] \to \mathbb{R}^2$ where $S^1 = \mathbb{R}/[0, 2\pi]$. In fact v_t is of class C^r if $f \in C^r$ and $g \in C^{r+1}$ when $(\theta, \epsilon) \in S^1 \times (0, \epsilon_0]$ but we just have continuity when $\epsilon = 0$. Obviously, if $g \equiv 0$, then $W_\epsilon(t) \equiv \Gamma$ for any $t \in \mathbb{R}$. However, in order to have some hints on the dynamics taking place in $W_\epsilon(t)$, we need the integral manifold $W_\epsilon(t)$ to depend smoothly on t as well. This is obtained assuming, e.g., that the function g is quasi periodic, and it is proved in Theorem 3.14, which is an adaption of [12, Theorem 3.9], which is in fact based on [20, 26]. Then, following [12, §4], it will be possible to have some clue on the dynamics in the integral manifold $W_\epsilon(t)$ using the concept of suspension flow and circle extension of the base flow.

The whole argument is carried on embedding (1.2) in the following two parameters problem

$$\frac{dx}{dt} = f(x, \epsilon) + \epsilon^s g\left(\frac{t}{\mu}, x, \epsilon, \mu\right) \qquad t \in \mathbb{R}, \; x \in \mathbb{R}^2. \tag{1.3}$$

Obviously (1.3) reduces to (1.2) by setting $\epsilon = \mu$ and $s = 1$. In (1.3) we distinguish between the parameter ϵ determining the "size" of the perturbation, and a parameter μ which controls "the speed of oscillation" of the non-autonomous perturbation. This framework, besides providing slightly more general results, is essential in simplifying the argument of our proof, helping us to introduce exponential dichotomy tools and to gain enough hyperbolicity to apply the methods of [26, 32].

Setting $\mu = 1$ we recover the regular perturbation problem discussed in [12] (which however is not contained in the present article since we always need $\mu \ll 1$). In fact, in [12] the existence of the AH pattern, and in particular the existence and the smoothness of the exponentially stable integral manifold $W_\epsilon(t)$, has been obtained just in the case $s > 1$ (and $\mu = 1$). Here, in this apparently more difficult setting, quite surprisingly, we may relax the assumption and allow $s = 1$ in (1.3) and study

$$\frac{dx}{dt} = f(x, \epsilon) + \epsilon g\left(\frac{t}{\mu}, x, \epsilon, \mu\right) \qquad t \in \mathbb{R}\,,\ x \in \mathbb{R}^2\,. \tag{1.4}$$

Such a gain is obtained by applying the averaging method developed by Fink in [10], and adapted in [22]. As we will see in section 2, the method consists in the construction of a change of variables which allows to recast the problem in such a way that the term $\epsilon^s g$ in (1.3) is replaced by a term of the form $\epsilon^s \zeta(\mu)\tilde{g}$, where $\tilde{g}$ is bounded and $\zeta(\mu)$ is a continuous monotone increasing function such that $\zeta(0) = 0$. We stress that we do not have any control on how small $\zeta(\mu)$ in fact is: it might go to 0 even as a logarithm, see [12, Appendix B]. However, even setting $s = 1$ and $\epsilon = \mu$ as in (1.2), we can regard the term $\epsilon^s\zeta(\mu)g$ as a $o(\epsilon)$ perturbation of f: this is enough to apply our argument and to built up the AH pattern.

On the other hand this approach has an important drawback. We are able to produce the AH pattern for (1.3) when $0 < \epsilon \le \epsilon_0$ and $0 < \mu \le \mu_0$, where $\epsilon_0 = \epsilon_0(s) > 0$. **The integral manifold $W_\mu(t)$ is of size $O(\sqrt{\epsilon})$**, so the whole phenomenon is hard to be detected if $\epsilon_0 > 0$ is too small. If we set $s = 1$ and $\epsilon = \mu$ as in (1.2), the whole argument works as long as the terms of order $O(\zeta(\epsilon))$ are negligible with respect to $\sqrt{\zeta(\epsilon)}$: this may result in asking for very small (unprecisely small) values of ϵ_0, and correspondingly for very small integral manifold $W_\mu(t)$ (of size $O(\sqrt{\epsilon})$) which might become invisible in real applications (indistinguishable from the repeller, which we usually assume to be the origin).

To make our argument more robust one could choose $s > 1$, e.g. $s = \frac{3}{2}$: this will result in neglecting terms of order $O(\epsilon^{s-1}\zeta(\mu))$ when compared to $\sqrt{\zeta(\mu)}$, and at the end should allow for "visible ϵ_0" even when setting $\mu = \epsilon$, see Section 3.3 and in particular Remark 3.18 and Corollary 3.17.

The introduction of a second parameter μ describing the speed of variation of the perturbation terms is of help also in the investigation of the dynamical properties of the stable integral manifold $W_\mu(t)$, using the methods of [12, §4]. One might expect that if g is almost periodic the dynamics on $M^*_\mu = \cup_{t\in\mathbb{R}}(W_\mu(t) \times \{t\})$ will be obtained simply by adding a frequency to the frequency modulus of g. This is not always the case due to possible resonances which are difficult to be avoided.

As pointed out in [12, §4], which was motivated by [19], in this context a key role is played by the bounded mean motion property, see Definition 5.2: if the base flow is quasi-periodic (i.e. if g is quasi periodic in t), and the integral manifold actually has the bounded mean motion property (e.g. if g is periodic in t), the resulting flow will be either a quasi-periodic flow obtained by adding a frequency to the frequency modulus of g, or it will be a Cantorus and laminates in almost periodic minimal flows. However in general, due to resonances, the resulting flow may be even weakly mixing [8] or mixing [9], and intermittency phenomena have to be expected. In fact for $\epsilon > 0$ and $\mu = 0$ the periodic trajectory of the autonomous system will have period, say, $T(\epsilon)$, while for $\mu > 0$, even in the simplest case, i.e. if g is 2π-periodic, the forcing term will be of period $2\pi\mu$. Hence we cannot hope for the flow on M_μ to be simply quasi-periodic, for the whole range $0 < \mu \le \mu_0$, but just for specific sequences of values, far enough from resonances.

The structure of the article is as follows. In Section 2 we introduce the language of skew-product semi-flow, and Fink averaging in infinite intervals. In Section 3 we introduce some definitions and we state the results: in Section 3.1 the ones with weaker assumptions which regards the existence of the repeller and of the positively invariant annulus, in Section 3.2 the existence and the smoothness of the asymptotically stable integral manifold. Then in Section

3.3 we give some Remarks on the problem of the robustness of the pattern and smallness of the parameters. In Section 4 we carry on the proofs of the main results. In Section 5 we recall some of the results described in [12] concerning the dynamics on $W_\mu(t)$, for convenience of the reader: they have been stated for the regular perturbation case and are trivially adapted to this context. Finally in the Appendix we briefly discuss the change of variables developed by Bellman et al. in [1,2], which allows to show a shift of the bifurcation value for the averaged system. In fact this analysis was started in [13] with the example of a Van der Pol oscillator, here it is extended to the general case of AH bifurcation.

2 Preliminaries

In this section we briefly explain how we can translate the non-autonomous equation (1.2) in the language of skew product flows. Then we apply the infinite interval averaging technique developed by Fink and Hale [10,17] (see also [22] and Remark 2.2 below) to see how a system with fast varying coefficients can be seen as a perturbation of its average. In the whole section we will be rather sketchy, remanding the interested reader to [13, §2] where a more detailed explanation can be found.

Let P be a topological space. A *flow* on P is a family $\{\phi_t \mid t \in \mathbb{R}\}$ of homeomorphisms of P with the following properties:

- $\phi_0(p) = p$ for all $p \in P$;
- $\phi_t \circ \phi_s = \phi_{t+s}$ for all $t, s \in \mathbb{R}$;
- $\phi : \mathbb{R} \times P \to P$: $(t, p) \to \phi_t(p)$ is continuous.

Suppose that P is a compact metric space, and let $\{\phi_t\}$ be a flow on P.

Let us consider a differential system with fast varying dependence i.e.

$$\frac{dx}{dt} := \dot{x} = \mathfrak{f}\left(\frac{t}{\mu}, x, \mu\right) \qquad t \in \mathbb{R}\,,\ x \in \mathbb{R}^2, \tag{2.1}$$

where $\mathfrak{f}$ is as smooth as needed and $\mu > 0$ is small, or equivalently

$$\frac{dx}{d\tau} := x' = \mu\mathfrak{f}(\tau, x, \mu) \qquad \tau \in \mathbb{R}\,,\ x \in \mathbb{R}^2\,. \tag{2.2}$$

Following [13, §2] let $l = (l_1, l_2) \in \mathbb{N}^2$ be such that $0 \le l_1, l_2 \le l_1 + l_2 = |l| \le r$. One requires that $\mathfrak{f}$, together with all its partial derivatives $D^l_x\mathfrak{f} = D^{l_1}_{x_1}, D^{l_2}_{x_2}\mathfrak{f}$ of order $|l| \le r$, are uniformly continuous on sets of the form $\mathbb{R} \times K$ where $K \subset \mathbb{R}^2$ is compact. Then there exist:

(i) a compact metric space P with a flow $\{\phi_t\}$;

(ii) a continuous function $\mathfrak{f}_* : P \times \mathbb{R}^2 \to \mathbb{R}^2$ such that $D^l_x\mathfrak{f}_* : P \times \mathbb{R}^2 \to \mathbb{R}^2$ exists and is continuous for each $l = (l_1, l_2) \in \mathbb{N}^2$ with $|l| \le r$;

(iii) a point $p_* \in P$ such that $\mathfrak{f}(t, x, \mu) = \mathfrak{f}_*(\phi_t(p_*), x, \mu)$ for all $t \in \mathbb{R}$, $x \in \mathbb{R}^2$.

The flow $\{\phi_t\}$ is induced by the translation in t, and the points of P are actually functions $p(t, x) = \lim_{n\to\infty} \mathfrak{f}(t + t_n, x, \mu)$ for appropriate sequences $\{t_n\} \subset \mathbb{R}$. Here the limit is taken in the compact-open topology on $\mathbb{R} \times \mathbb{R}^2$. One usually abuses notation at this point and writes

$\mathfrak{f}$ instead of $\mathfrak{f}_*$ (but not p for p_*). This way equation (2.2) (and consequently (2.1)) has been embedded into the family of differential equations

$$x' = \mu\mathfrak{f}(\phi_t(p), x, \mu) \qquad p \in P,\ x \in \mathbb{R}^2,\ 0 < \mu \le \mu_0 \tag{2.2p}$$

where (2.2) coincides with (2.2p_*).

Suppose that each equation (2.2p) admits a unique global solution $x(t; x_0, p)$ for each initial value $x_0 \in \mathbb{R}^n$. Then the family of homeomorphisms

$$\psi_t : P \times \mathbb{R}^2 \to P \times \mathbb{R}^2 : (p, x_0) \to (\phi_t(p), x(t; x_0, p))$$

defines a flow on $P \times \mathbb{R}^2$. One speaks of a skew-product flow because the first factor does not depend on x_0, and the flow $(P, \{\phi_t\})$ is usually referred to as the base.

Let $\triangle$ denote the the usual symmetric difference of sets, i.e. $A\triangle B = (A\backslash B) \cup (B\backslash A)$. A regular Borel probability measure ξ on P is said to be ϕ_t-invariant if $\xi(\phi_t(B)) = \xi(B)$ for each Borel set $B \subset P$ and for each $t \in \mathbb{R}$. An invariant measure is said to be $\{\phi_t\}$-*ergodic* if, in addition, the following indecomposibility condition holds: if $B \subset P$ is a Borel set, and if $\xi(B\triangle\phi_t(B)) = 0$ for each $t \in \mathbb{R}$, then either $\xi(B) = 0$ or $\xi(B) = 1$.

We recall that any flow on P admits an ergodic invariant measure, but such a measure is not always unique.

Definition 2.1. We say that $(P, \{\phi_t\})$ is **uniquely ergodic** if it admits a unique ergodic invariant measure.

Following [13] page 221–222 (which is in fact based on [22], [10, Lemma 14.1]), if $(P, \{\phi_t\})$ is uniquely ergodic, and ξ is its unique invariant measure, we can fix $p \in P$ and set $\overline{\mathfrak{f}_\mu}(x) = \int_P \mathfrak{f}(p, x) d\xi(p)$, to define

$$\begin{aligned} \mathfrak{F}_\mu(0, x) :=& \mathfrak{F}(p, x, \mu) = \int_{-\infty}^{0} \mathrm{e}^{\mu s} \left\{\mathfrak{f}(\phi_s(p), x, \mu) - \overline{\mathfrak{f}_\mu}(x)\right\} ds, \\ \mathfrak{F}_\mu(t, x) :=& \mathfrak{F}(\phi_t(p), x, \mu) = \mathrm{e}^{-\mu t} \int_{-\infty}^{t} \mathrm{e}^{\mu s} \left\{\mathfrak{f}(\phi_s(p), x, \mu) - \overline{\mathfrak{f}_\mu}(x)\right\} ds. \end{aligned} \tag{2.3}$$

From [13, Proposition 2.2] for all $p \in P$, $x \in \mathbb{R}^2$, and for each $l \in \mathbb{N}^2$ with $|l| \le r$ we find

$$|\mu D_x^l \mathfrak{F}_\mu(t, x)| \le \zeta(\mu), \tag{2.4}$$

where $\zeta(\mu)$ is a continuous increasing function such that $\zeta(0) = 0$. Of course $\mathfrak{F}_\mu$ depends on p as well but the estimate (2.4) is uniform in P (since it is compact). We emphasize that $\zeta(\mu)$ need not tend to zero at a prescribed rate, e.g. we cannot say $\zeta(\mu) = c\mu^s$ for some $s > 0$, see [13, Appendix B].

For each fixed $p \in P$, and $|x| \le 1$, $0 < \mu \le \mu_0$, we apply the following C^∞ (invertible) change of variables, cf. [13, Proposition 2.3],

$$x = y + \mu\mathfrak{F}_\mu(t, y) \tag{2.5}$$

and we pass from (2.2p) to the following:

$$\left(I + \mu\frac{\partial\mathfrak{F}_\mu}{\partial y}\right) y' = \mu \left\{\overline{\mathfrak{f}_\mu}(y) + \mu\mathfrak{F}_\mu(t, y) + \mathfrak{f}(\phi_t(p), x, \mu) - \mathfrak{f}(\phi_t(p), y, \mu)\right\}. \tag{2.6}$$

Then, using (2.4) and the expansion

$$\mathfrak{f}(p,x,\mu) - \mathfrak{f}(p,y,\mu) = \frac{\partial \mathfrak{f}_\mu}{\partial y}(p,y)\mu\mathfrak{F}_\mu + O(|\mu\mathfrak{F}_\mu|^2) \tag{2.7}$$

we recast (2.6), hence (2.2p), as

$$y' = \mu \left\{\overline{\mathfrak{f}_\mu}(y) + \zeta(\mu)A_1(\phi_t(p),y) + R(\phi_t(p),y)\right\} \tag{2.8}$$

where $R(\phi_t(p),y) = o(\zeta(\mu))$ uniformly in all the variables and A_1 is bounded and explicitly known, i.e.

$$\zeta(\mu)A_1(\phi_t(p),y) := -\mu\frac{\partial\mathfrak{F}_\mu(t,y)}{\partial y}\overline{\mathfrak{f}_\mu}(y) + \left(\mathbb{I} + \frac{\partial \mathfrak{f}_\mu}{\partial y}(\phi_t(p),y)\right)\mu\mathfrak{F}_\mu(t,y). \tag{2.9}$$

This way we have rewritten (2.1) as a $\zeta(\mu)$ small non-autonomous perturbation of its averaged autonomous equation $\dot{x} = \overline{\mathfrak{f}_\mu}(x)$, i.e. as a system of the form

$$\dot{y} = \overline{\mathfrak{f}_\mu}(y) + \zeta(\mu)\tilde{g}(\phi_{\frac{t}{\mu}}(p),y,\mu) \tag{2.10}$$

where the function $\tilde{g}$ is bounded in all its components. Hence we can apply the techniques developed in [12] for small non-autonomous perturbations adapting them to a context of rapidly varying coefficients.

Remark 2.2. Notice that, passing from (2.1) to (2.8), we have lost one order of smoothness due to (2.7), i.e. if $\mathfrak{f} \in C^{r+1}$ then $\overline{\mathfrak{f}_\mu} \in C^{r+1}$ but $\tilde{g} \in C^r$. Further we need at least $\mathfrak{f} \in C^{r+1}$, $r \geq 0$, to infer $R(\phi_t(p),y) = o(\zeta(\mu))$; moreover if $r \geq 1$ we get $R(\phi_t(p),y) = O(\zeta(\mu))^2$. In fact if $\mathfrak{f}$ is smooth enough we can proceed to expand the term R further to any order, getting explicitly known terms.

We emphasize that, from [11, Propositions 2.5 and 2.6], if x^* is such that the solution $x(t;x^*,p,\mu)$ of (2.1) and the solution $x_0(t;x^*,p)$ of its autonomous averaged equation both exist for any $t \in \mathbb{R}$ and stay in the ball of radius 1, then

$$\lim_{\mu\to 0^+} x(t;x^*,p,\mu) = x_0(t;x^*,p)$$

along with all its derivative with respect to x, see [13, Proposition 2.4].

We introduce the following notation which is in force in the whole paper: *we denote by $x(t,T;Q)$ the trajectory of (2.1) which is in Q at $t = T$, and by $x(t,p;Q)$ the trajectory of (2.2p) which is in Q at $t = 0$.* We use analogous notation for all the equations to be introduced in the article. Notice that by construction $x(t,T;Q) = x(t-T,\phi_T(p)\,;Q)$, for any $t,T \in \mathbb{R}$.

3 Statement of the main results

In Sections 3.1, 3.2 we state the main results of the article, and we consider (1.3); so we make use of two small nonnegative parameters: ϵ which measures the size of the perturbations, and μ which measures the rapidity of the variation of the coefficients. Further we always assume $s \geq 1$ without further mentioning, i.e. the non-autonomous perturbation is of the same size or smaller than the autonomous perturbation giving rise to AH bifurcation for the averaged system. Then in Section 3.3 we see that we can set $s = 1$ and $\epsilon = \mu$, i.e. we consider (1.2), so that we can regard the whole problem as a 1-parameter system, and restate all the results in this setting, see Corollary 3.17. At this stage we prefer to keep them distinguished: this will help us in the proofs and allows for slightly more general statements; further it also to underlines their different "physical" meaning. In Section 3.3 we also discuss the dependence of the robustness of the results on the choice of $s \geq 1$. All the proofs are postponed to Section 4.

We begin by recalling some standard facts concerning the AH bifurcation pattern. Let us consider the equation

$$\dot{x} = f(x,\epsilon) = L(\epsilon)x + N(x,\epsilon)\,, \qquad N(x,\epsilon) = O(x^2) \tag{3.1}$$

and assume $f(0,\epsilon) = 0$ (i.e in (1.3) we set $g \equiv 0$). Following [12] which is based on [18, 25], we can recast (3.1) in its normal form, i.e.

$$f(x,\epsilon) = \begin{pmatrix} \epsilon & -1 \\ 1 & \epsilon \end{pmatrix} x - \begin{pmatrix} 1 & \varpi(\epsilon) \\ -\varpi(\epsilon) & 1 \end{pmatrix} |x|^2 x + W(x,\epsilon) \tag{3.2}$$

where $W(x,\epsilon) = O(|x|^5)$. Notice that passing from (3.1) to (3.2) we have made a polynomial change of variables (see [12] or [18] for more details), but, abusing the notation, the unknown is still denoted by x.

Passing to polar coordinates (r,θ) it is easy to see that, for $\epsilon > 0$, (3.2) admits a stable limit cycle Γ which can be written in the form

$$\Gamma = \{\Gamma(\theta) = (r_\Gamma(\theta),\theta) \mid 0 \leq \theta \leq 2\pi\} \qquad r_\Gamma(\theta) = \sqrt{\epsilon} + O(\epsilon)$$

where $r(\cdot)$ is a smooth 2π-periodic function. In fact, for $\epsilon > 0$, the dynamics in a small neighborhood of the origin is ruled by the presence of the attractor–repeller pair made up by Γ and the origin, which are respectively an asymptotically stable limit cycle and an unstable focus.

Now we switch on the non-autonomous perturbation, and we turn to consider system (1.3). We go back to the notation of Section 2 so, using Fink's averaging techniques we introduce a base flow $(P, \{\phi_t\})$ describing the dynamics of the coefficients, and we embed (1.3) in a family of type (2.2p) as follows

$$\dot{x} = f(x,\epsilon) + \epsilon^s g\left(\phi_{\frac{t}{\mu}}(p), x, \epsilon, \mu\right) \qquad t \in \mathbb{R}\,,\ x \in \mathbb{R}^2, \tag{3.3}$$

where $f \in C^r$ and $g \in C^{r+1}$ in x, ϵ, μ and $r \geq 4$, uniformly with respect to $p \in P$ (smoothness requirement might be relaxed but they are not the main issue in this paper). Notice that here and afterwards, abusing the notation, g stands for g_*.

In the whole article we assume that P is a compact metric space and the base flow $(P, \{\phi_t\})$ is uniquely ergodic, see Definitions 2.1: this is a very weak recurrence property on the t dependence of the original function $g(t,x,\epsilon,\mu)$, and it is satisfied, e.g., if $g(t,x,\epsilon,\mu)$ is almost periodic, see [12] or [10] for more details. As we said in the Introduction, more stringent recurrence properties will be required in some results: this way we will get some insight on the topological and dynamical characteristics of the set which replaces the stable limit cycle of the classical autonomous AH bifurcation. Further **we will always assume that all the trajectories are continuable for any $t \in \mathbb{R}$, since we are just interested in local dynamics**.

Then we are in the position to apply the Fink's averaging procedure developed in section 2 and we pass to the new variable y, see (2.5); hence we find a monotone increasing continuous function $\zeta(\mu)$ such that $\zeta(0) = 0$, so that (3.3) can be recast as an equation of type (2.8). We assume w.l.o.g. that the time-average of g and its derivative are 0. This amounts to ask that they are already contained in the autonomous term f, which we assume to be in its normal form (3.2). So we rewrite (3.3) as follows

$$\dot{y} = f(y, \epsilon) + \epsilon^s \zeta(\mu) \tilde{g}\left(\phi_{\frac{t}{\mu}}(p), y, \epsilon, \mu\right). \tag{3.4}$$

and $\tilde{g}$ is just C^r in (y, ϵ, μ) uniformly for $p \in P$, see Remark 2.2. This way we have gained something in the smallness of the non-autonomous perturbation. The drawback is that we have lost one order of smoothness and that in fact we cannot precisely say how small $\zeta(\mu)$ is, again see [13, Appendix B].

We collect here some dynamical definitions.

Definition 3.1. In the whole paper we say that $W \subset (\mathbb{R}^2 \setminus \{(0,0)\})$ is **topologically equivalent to** S^1 if there is an homotopy between W and $\mathcal{S}^1$ in $\mathbb{R}^2 \setminus \{(0,0)\}$. More precisely if there is a continuous function $H : (t, x) \in [0,1] \times (\mathbb{R}^2 \setminus \{(0,0)\}) \to (\mathbb{R}^2 \setminus \{(0,0)\})$ such that $H(0, x) = x$ for any $x \in W$ and $H(1, y) = y$ for any $y \in S^1$.

Roughly speaking W is topologically equivalent to S^1 if it can be continuously deformed into $\mathcal{S}^1$, so it may be thick in some parts or everywhere.

Definition 3.2. A function f is almost periodic if every sequence $\{f(T_n + t)\}$ of translations of f has a subsequence that converges uniformly for $t \in \mathbb{R}$.

Definition 3.3. Let P be a compact metric space and $(P, \{\phi_t\})$ be a flow. We say that $(P, \{\phi_t\})$ is (Bohr) almost periodic if there is a metric d on P, which is compatible with the topology on P, such that

$$d(\phi_t(p_1), \phi_t(p_2)) = d(p_1, p_2).$$

Definition 3.4. We say that a surface M in $\mathbb{R} \times \mathbb{R}^2$ is an **integral manifold** for

$$\dot{x} = \mathfrak{f}(t, x) \tag{3.5}$$

if and only if, for any $(T, x_0) \in M$, the solution $x(t, T; x_0)$ which is in x_0 at $t = T$ is such that $(t, x(t, T; x_0)) \in M$ for all $t \in \mathbb{R}$.

The existence of an integral manifold $M \subset (\mathbb{R} \times \mathbb{R}^2)$ in the (t, x) variables is in fact obtained constructing first an integral manifold $M_* \subset (P \times \mathbb{R}^2)$ in the (p, x) variables.

Definition 3.5. We say that M_* in $P \times \mathbb{R}^2$ is an **integral manifold** for

$$\dot{x} = \mathfrak{f}_*(\phi_t(p), x) \tag{3.6}$$

if and only if, for any $(\phi_t(p), x_0) \in M_*$, the solution $x(t; x_0, p)$ is such that $(\phi_t(p), x(t; x_0, p)) \in M_*$ for all $t \in \mathbb{R}$.

Definition 3.6. We say that $E \subset \mathbb{R}^2$ is **positively invariant** (respectively **negatively invariant)** either for Eq. (3.5) or for Eq. (3.6) if for any $T \in \mathbb{R}$ and any $x_0 \in E$ we have $x(t, T; x_0) \in E$ for any $t \geq T$ (respectively for any $t \leq T$), or equivalently if $x(t; x_0, \phi_T(p)) \in E$ for any $t \geq 0$ (respectively for any $t \leq 0$).

We recall that if we rephrase the time dependence of $\mathfrak{f}$ in the language of skew-product flows as in Section 2 we have $x(t, T; x_0) = x(t - T, \phi_T(p); x_0)$.

In Sections 3.1, 3.2 we enumerate some results addressed to reconstruct the AH bifurcation pattern in our context. Their proofs are postponed to Section 4.

3.1 Existence of positively invariant annulus and negatively invariant disc

Let us consider (3.4) and introduce polar coordinates as follows

$$r = \sqrt{y_1^2 + y_2^2}, \qquad \theta = \arctan\left(\frac{y_2}{y_1}\right). \tag{3.7}$$

Following [12, §2], the first step in order to construct the attractor–repeller pair is the following Lemma, compare with [12, Lemma 2.2].

Lemma 3.7. *Assume that $g(t,0,\epsilon,\mu) = O(\sqrt{\epsilon})$, so that $\tilde{g}(p,0,\epsilon,\mu) = O(\sqrt{\epsilon})$, uniformly in all the variables, and set $\bar{\delta} = \sqrt{\zeta(\mu)+\epsilon}$. Then there are μ_0 and ϵ_0 such that the annulus $\mathfrak{A}$ and the disc $\mathfrak{D}$*

$$\mathfrak{A} := \{(r,\theta) \mid \sqrt{\epsilon}(1-\bar{\delta}) \le r \le \sqrt{\epsilon}(1+\bar{\delta}),\ 0 \le \theta \le 2\pi\},$$
$$\mathfrak{D} := \{(r,\theta) \mid r \le \bar{\delta}\sqrt{\epsilon},\ 0 \le \theta \le 2\pi\}$$

are respectively positively and negatively invariant for (3.4), for any $0 < \mu \le \mu_0$, $0 < \epsilon \le \epsilon_0$.

The full fledged proof of this result as well as the ones of the others of this subsection are postponed to §4.1. Using a result by Shen and Yi borrowed from [31] it is easy to prove that $\mathfrak{D}$ contains a repeller (a set attractive under time reversal).

Proposition 3.8. *Assume that $g(t,0,\epsilon,\mu) = O(\sqrt{\epsilon})$ uniformly in all the variables, and that the flow $(P,\{\phi_t\})$ is* almost periodic*; then there are $\epsilon_0 > 0$, $\mu_0 > 0$ such that $\mathfrak{D}$ contains a unique trajectory $y_o(t) := y_o(t;p)$ of (3.3) such that $y_o(t) \in \mathfrak{D}$ for any $t \in \mathbb{R}$, and any $0 < \epsilon \le \epsilon_0$, $0 < \mu \le \mu_0$. Further $y_o(t)$ is* almost periodic *and, if $Q \in \mathfrak{D}$ then $\|y_o(t) - y(t,p;Q)\| \to 0$ as $t \to -\infty$.*

In fact we can relax the assumptions on the base flow $(P,\{\phi_t\})$, see Remark 4.2 below.

Using Wazewski's principle it is easy to prove the following.

Proposition 3.9. *Assume that $g(t,0,\epsilon,\mu) = O(\sqrt{\epsilon})$ uniformly in all the variables. Then then there are $\epsilon_0 > 0$, $\mu_0 >$ such that the set $\mathfrak{A}$ contains an invariant set $W_\mu(0)$, i.e if $Q \in W_\mu(0)$ then $y(t,p;Q) \in \mathfrak{A}$ for any $t \in \mathbb{R}$, and any $0 < \epsilon \le \epsilon_0$, $0 < \mu \le \mu_0$. The set $W_\mu(0)$ is a compact connected set and its image through the flow, i.e. $W_\mu(T) = \{x(T,p;Q) \mid Q \in W_\mu(0)\} \subset \mathfrak{A}$. Further, for any $T \in \mathbb{R}$, $W_\mu(T)$ is topologically equivalent to S^1, see Definition 3.1.*

The set $W_\mu(T)$, $T \in \mathbb{R}$, plays the role of the attracting limit cycle of the classical AH autonomous perturbation. In fact in the next subsection, requiring stronger assumptions, we show that $W_\mu(T)$ is an exponentially stable integral manifold.

Remark 3.10. To prove Propositions 3.8 and 3.9 we just need f and g in (1.3) to be respectively C^1 and C^2 so that $\tilde{g}$ in (3.4) is C^1. However, to put in normal form a generic system of the form (1.4), so that f is as in (3.2), we need $f \in C^r$ and $g \in C^{r+1}$, $r \ge 4$, so that $f \in C^r$ and $\tilde{g} \in C^r$.

3.2 Stronger assumptions: existence of an asymptotically stable integral manifold

In this subsection we always assume, for simplicity, that $\tilde{g}(t,0,\epsilon,\mu) \equiv 0$, so that the repeller of Proposition 3.8 is in fact the origin. Here we aim to prove that $M_\mu^* = \cup_{T\in\mathbb{R}}\big(\{T\} \times W_\mu(T)\big)$ is indeed an asymptotically stable integral manifold: for this purpose we adapt the argument developed in [12], so that we can apply the results of [32] and [26].

Obviously, for $\mu = 0$ and $0 < \epsilon < \epsilon_0$, (3.4) admits an attracting integral manifold $M_0 = \Gamma \times \mathbb{R}$ which is independent of T but depends on ϵ, i.e. we have $W_0(T) \equiv \Gamma$ for any $T \in \mathbb{R}$.

Since we need a better look at a neighborhood of Γ, following [12, §2], we introduce polar coordinates which evaluate the displacement from Γ, i.e.

$$\rho = \frac{r(\theta) - r_\Gamma(\theta)}{\sqrt{\epsilon}}, \qquad \theta = \arctan\left(\frac{y_2}{y_1}\right), \qquad -\frac{1}{2} \le \rho \le 2. \tag{3.8}$$

We emphasize that ρ is not a radius but it measures the radial displacement with respect to Γ, therefore it might be negative. In fact $\rho = 0$ corresponds to Γ in the original (y_1, y_2) variables, and ρ is negative when (y_1, y_2) lies in the bounded set enclosed by Γ. Further (3.8) is not well defined when $\rho < -1/2$, i.e. when (y_1, y_2) is too close to the origin and in particular when $(y_1, y_2) \in \mathfrak{D}$.

This way (3.4) is turned into the following:

$$\begin{aligned} \dot\rho &= -2\epsilon\rho - 3\epsilon\rho^2 + \epsilon^{3/2}\rho A(\rho,\theta) + \epsilon^s \zeta(\mu) B(\phi_{t/\mu}(p), \rho, \theta), \\ \dot\theta &= 1 + \epsilon\varpi(\epsilon)(1+\rho)^2 + \epsilon^{3/2} C(\rho,\theta) + \epsilon^s \zeta(\mu) D(\phi_{t/\mu}(p), \rho, \theta). \end{aligned} \tag{3.9}$$

Once again (3.9) is defined just on the stripe $-1/2 \le \rho \le 2$ which contains the image of $\mathfrak{A}$ through (3.8).

Remark 3.11. Since Γ is invariant for (3.4) for $\mu = 0$ we see that $\rho = 0$ is invariant for (3.9), so A and C are bounded; further B and D are bounded because $\tilde g(t, 0, \epsilon, \mu) \equiv 0$. We give more details of the derivation of (3.9) in Section 4 (however compare with (2.9), (2.10) in [12]).

Reasoning as in the proof of Lemma 3.7 we can set $\tilde\delta := c\sqrt{\zeta(\mu)}$ where $c > 0$ is fixed (independently of ϵ and μ), so that the stripe $|\rho| \le \tilde\delta$ is positively invariant for (3.9). Further the set $|\rho| \le \tilde\delta$ in the new variables introduced for (3.9), corresponds to a set $\tilde{\mathfrak{A}}$ in the old (y_1, y_2) variables of (3.4) such that $\Gamma \subset \tilde{\mathfrak{A}} \subset \mathfrak{A}$ (if $c > 0$ is small enough).

Now we put ourselves in the setting of [32], so we assume that P is endowed with a distance d, and we state the result analogous to [12, Proposition 3.2], which is in fact based on [32, Theorem 6.1], keeping the same notation to help a comparison.

Theorem 3.12. *Assume that f and $\tilde g$ in (3.4) are Lipschitz continuous, so that the functions B and D of (3.9) are Lipschitz continuous in p, uniformly for all relevant values of ρ and θ. Suppose that the metric d satisfies the following condition:*

$$\sup_{p_1 \ne p_2 \in P} \left\{ \limsup_{t\to+\infty} \frac{1}{t} \ln\left(\frac{d(\phi_t(p_1), \phi_t(p_2))}{d(p_1,p_2)} \right) \right\} = 0. \tag{3.10}$$

Then there are numbers $\epsilon_0 > 0$, $\mu_0 > 0$ and a continuous function $v : P \times S^1 \times [0, \epsilon_0] \times (0, \mu_0) \to [0, +\infty)$ such that

$$M_\mu = \{(p, [\Gamma(\theta) + \sqrt{\epsilon} v(p,\theta,\epsilon,\mu)] e^{i\theta}) \mid p \in P,\ 0 \le \theta \le 2\pi\} \subset P \times \mathbb{R}^2 \tag{3.11}$$

is invariant with respect to the flow of (3.4), i.e. if $(\bar p, \bar y) \in M_\mu$ then $\psi_t(\bar p, \bar y) := (\phi_t(p), y(t, p; \bar y)) \in M_\mu$ for any $t \in \mathbb{R}$, $0 \le \epsilon \le \epsilon_0$, $0 < \mu \le \mu_0$. Moreover $v(p,\theta,\epsilon,\mu)$ is Lipschitz in (θ,ϵ) uniformly for any $p \in P$, $0 \le \theta \le 2\pi$, $0 \le \epsilon \le \epsilon_0$, $0 < \mu \le \mu_0$.

Further assume that f and $\tilde g$ in (3.4) are C^r (i.e. $f \in C^r$, and $g \in C^{r+1}$ in (1.2)) so that B and D in (3.9) are C^r, $r \ge 1$, and let $C = C(\rho,\theta)$ be the coefficient in (3.9). If

$$\sqrt{\epsilon_0} \left| \frac{\partial C}{\partial\theta}(0,\theta) \right| + \zeta(\mu_0)\epsilon^{s-1} \left| \frac{\partial D}{\partial\theta}(p,0,\theta) \right| \le \frac{1}{r+1}, \tag{3.12}$$

for any $0 \le \theta \le 2\pi$ and any $p \in P$, then $v(p,\theta,\epsilon,\mu)$ is of class C^r in (θ,ϵ).

Here and afterwards $e^{i\theta}$ stands for $(\cos(\theta), \sin(\theta))$, i.e we identify $\mathbb{R}^2$ and $\mathbb{C}$, for notational purposes. The proof of Theorem 3.12 is postponed to Section 4.2.

Remark 3.13. Condition (3.10) amounts to ask for the Lyapunov exponents of the slow flow $(P, \{\phi_t\})$ (which have the same sign of the fast flow $(P, \{\phi_{\tau=t/\mu}\})$) to be non-positive: it is satisfied by almost periodic and quasi periodic flows, see [12, §2] for details and definitions.

From Theorem 3.12 we can write M_μ as follows

$$M_\mu = \bigcup_{p \in P} \left(\{p\} \times \bar{W}_\mu(p)\right) \tag{3.13}$$

where $\bar{W}_\mu(p)$ is a compact 1 dimensional C^r manifold homoeomorphic to S^1.

Let $p_* \in P$ be the function such that $g_*(\phi_t(p_*), x, \epsilon, \mu) \equiv g(t, x, \epsilon, \mu)$, see Section 2, and set $W_\mu(T) := \bar{W}_\mu(\phi_T(p_*))$. We emphasize that $W_\mu(t)$ is a C^r manifold homeomorphic to S^1 and coincides with the invariant set $W_\mu(t)$ constructed in Proposition 3.9. Further,

$$M^*_\mu = \bigcup_{t \in \mathbb{R}} \left(\{\phi_t(p)\} \times W_\mu(t)\right) \subset M_\mu \tag{3.14}$$

and M^*_μ is dense in M_μ. An unsatisfactory aspect of Theorem 3.12 is that it does not provide any regularity of M_μ with respect to the p (or equivalently t) variable, i.e. we do not have much information about the regularity of $\bar{W}_\mu(p)$ and $W_\mu(T)$ with respect to p and T. In fact in this setting we could expect for at most Lipschitz regularity. To overcome this problem we need to ask for a little bit more concerning the properties of the base flow $(P, \{\phi_t\})$; then we can adapt the argument of [12, Theorem 3.9], which is in fact based on [26], to get the needed regularity. So let $\phi_\tau(p)$ be the solution on the smooth compact manifold P of the equation $\dot{p} = h(p)$ where $h \in C^r$, and consider the following extended equation:

$$\begin{aligned}
\dot{\rho} &= -2\epsilon\rho - 3\epsilon\rho^2 + \epsilon^{3/2}\rho A(\rho, \theta, \epsilon, \mu) + \epsilon^s \zeta(\mu) B(p, \rho, \theta, \epsilon, \mu), \\
\dot{\theta} &= 1 + \epsilon\varpi(\epsilon)(1+\rho)^2 + \epsilon^{3/2} C(\rho, \theta, \epsilon, \mu) + \epsilon^s \zeta(\mu) D(p, \rho, \theta, \epsilon, \mu), \\
\mu\dot{p} &= h(p).
\end{aligned} \tag{3.15}$$

We consider the following assumption borrowed from [12]:

H Let $\phi_\tau(p)$ be a solution of $\dot{p} = h(p)$, and let $\mathcal{A}(t)$ be the solution of the variational equation $\dot{\chi} = \frac{\partial h}{\partial p}(\phi_\tau(p))\chi$, such that $\mathcal{A}(0) = \mathbb{I}$. Then all the eigenvalues of $\mathcal{A}(\tau)$ have modulus 1.

Assumption **H** is satisfied e.g. if P is a d-Torus and $(P, \{\phi_t\})$ is the Kronecker flow with rationally independent frequencies, i.e. the original g is quasi periodic in the t variable.

Theorem 3.14. *Assume that hypothesis **H** holds, and that $f \in C^r$ and $g \in C^{r+1}$ in all their variables. Then there are $\epsilon_0 > 0$, $\mu_0 > 0$ and a continuous function $v : P \times S^1 \times [0, \epsilon_0] \times (0, \mu_0] \to [0, +\infty)$, such that the manifold M_μ defined in (3.11) is invariant for the flow of (3.4). Further v is C^r in (p, θ) for any $p \in P$, $0 \le \theta \le 2\pi$, $0 \le \epsilon \le \epsilon_0$, $0 < \mu \le \mu_0$.*

M_μ is asymptotically stable, i.e. if $y_0 \in \mathfrak{A}$ then there is $(\bar{p}, \bar{y}) \in M_\mu$ such that $\|y(t, p; y_0) - y(t, \bar{p}; \bar{y})\| \to 0$ as $t \to +\infty$ exponentially.

Moreover there is $c > 0$ (independent of ϵ and μ) such that $|v(p, \theta, \epsilon, \mu)| \le c\sqrt{\zeta(\mu)}$ and $\|\frac{\partial v}{\partial p, \partial \theta}(p, \theta, \epsilon, \mu)\|_\infty \le \sqrt{\zeta(\mu)}$.

The proof of Theorem 3.14 is postponed to Section 4.2.

Corollary 3.15. *Assume that the hypotheses of Theorem 3.14 hold. Then the continuous function* $v^* : \mathbb{R} \times S^1 \times [0, \epsilon_0] \times (0, \mu_0] \to [0, +\infty)$ *defined by*

$$v^*(t, \theta, \epsilon, \mu) := v(\phi_t(p^*), \theta, \epsilon, \mu)$$

is C^r *in* (t, θ, ϵ) *for any* $t \in \mathbb{R}$, $0 \leq \theta \leq 2\pi$, $0 \leq \epsilon \leq \epsilon_0$, *and* $0 < \mu \leq \mu_0$. *Further the set* M^*_μ *defined by* (3.14) *is invariant for the flow of* (1.3) *and the* C^r *manifold* $W_\mu(t)$ *depend in a* C^r *way from* (t, ϵ) *for any* $(t, \epsilon) \in \mathbb{R} \times [0, \epsilon_0]$ *but it is just continuous in* μ.

Remark 3.16. The main contributions of Theorem 3.14 with respect to Theorem 3.12 is that, using assumption **H**, it is able to guarantee that the function v is C^r in the p variable. Hence, by Corollary 3.15, $W_\mu(t)$ is C^r in t and M^*_μ is a C^r manifold: this fact will allow to apply directly some classical tools such as Diliberto's maps, suspension flows, and KAM theory, to get information on the dynamics inside M_μ, see [12, §4].

A priori $W_\mu(t)$ is not defined when $\mu = 0$ but we can set $W_0(t) \equiv \Gamma$. Since $v \to 0$ as $\mu \to 0$ uniformly in all the variables and its size is controlled by $c\sqrt{\zeta(\mu)}$ we see that $W_\mu(t)$ is continuous in μ for $0 \leq \mu \leq \mu_0$.

3.3 Some comments on the size of the parameters

We emphasize once again that all the results of the previous section can be reformulated also in the case $s = 1$ and $\epsilon = \mu$, i.e. for equation (1.2).

Corollary 3.17. *Let us consider* (1.2). *Then there is* $\epsilon_0 > 0$ *such that the results in Lemma 3.7, Propositions 3.8 and 3.9, Theorems 3.12, 3.14 and Corollary 3.15 hold for any* $0 < \epsilon < \epsilon_0$ *without any further change.*

From Corollary 3.17 we see that we can handle (3.3), hence (1.4), as a 1-parameter bifurcation problem (even if for the proofs a 2 parameters argument is in fact needed to recover the presence of uniform hyperbolicity, i.e exponential dichotomy). In fact also for (1.2) we can construct a negatively invariant disc $\mathfrak{D}$ containing a repeller, and a positively invariant annulus $\mathfrak{A}$ containing an invariant set $W_\epsilon(T)$ which is topologically equivalent to S^1, just assuming $|g(t, 0, \epsilon, \epsilon)| \leq c\sqrt{\epsilon}$ for a certain $c > 0$ (hence $\tilde{g}(\phi_{t/\epsilon}(p), 0, \epsilon, \epsilon) = o(\sqrt{\epsilon})$ for any $t \in \mathbb{R}$), cf. Propositions 3.8, 3.9. Further, assuming condition (3.10) on the base flow, we find that $W_\epsilon(T)$ is in fact a compact C^r integral manifold diffeomorphic to a circle, for any $T \in \mathbb{R}$, cf. Theorem 3.12. Finally, if assumption **H** holds, from Theorem 3.14 we see that $W_\epsilon(T)$ changes smoothly with $T \in \mathbb{R}$ i.e. M_ϵ defined in (3.11) is smooth in p as well.

However in this case the results are not very robust, and it might be very difficult to detect the bifurcation pattern in applications.

Remark 3.18. Let us consider (1.2). We emphasize that the value of ϵ_0 is possibly very small. In fact in Lemma 3.7 such a value depends on (4.2), (4.3) in which it is required that $2\epsilon\sqrt{\epsilon\zeta(\epsilon)} + O(\epsilon\zeta(\epsilon)\sqrt{\epsilon}) > \epsilon\sqrt{\epsilon\zeta(\epsilon)}$. Estimates of the same order are required several times through the proofs of the results of the whole sections 3.1, 3.2. Since we do not have a real control on the smallness of $\zeta(\epsilon)$, which might go to 0 even as $|\ln(\epsilon)|^{-1}$ or slowlier, the value ϵ for which (4.2) holds might be very small. We recall that the annulus $\mathfrak{A}$, in which the manifold $W_\epsilon(T)$ is contained, is centered on the circle of radius $r = \sqrt{\epsilon}$. Hence if ϵ_0 is very small also $W_\epsilon(T)$ might be so small that it might be difficult to distinguish it from the origin, thus making the bifurcation pattern hard to be detected.

To overcome the lack of robustness explained in Remark 3.18 we could act in two possible ways. Firstly we could go back to (1.4) and separate the parameters ϵ and μ; then we should ask for $\mu \ll \epsilon$, e.g. $\mu = \epsilon^2$. However, once again, we cannot really say how small $\zeta(\mu)$ is, and it might happen that $|\ln(\epsilon)|^{-1} \ll \zeta(\mu)$, see [13, Appendix B], so this approach should be discarded.

A more appropriate approach could be to ask for more stringent smallness conditions on the function g, i.e. to start from (1.3) with $s > 1$.

In this setting we could also avoid to apply Fink's averaging and the change of variables (2.5). The drawback is that we lose something in the smallness of the perturbation but we recover one order of regularity, and the function g we are dealing with is directly known.

In this case the smallness condition on ϵ_0 and μ_0 is needed to guarantee that terms of order ϵ^{s-1} (or of order $\zeta(\mu)\epsilon^{s-1}$ if we apply (2.5)) may be neglected with respect to constant terms. So, setting e.g. $s = 3/2$, we can expect for a visible attracting manifold $W_\mu(T)$.

4 Proofs

4.1 Proof of Propositions 3.8 and 3.9.

We begin this section by sketching the proof of Lemma 3.7, which is in fact a translation of the proof of [12, Lemma 2.2]. Consider (3.4) and pass to polar coordinates as in (3.7), then

$$\begin{aligned} \dot{r} &= \epsilon r - r^3 + \gamma_1(r,\theta,\epsilon) + \epsilon^s \zeta(\mu) \bar{\gamma}_1(\phi_{t/\mu}(p), r, \theta, \epsilon, \mu), \\ \dot{\theta} &= 1 + \varpi(\epsilon) r^2 + \gamma_2(r,\theta,\epsilon) + \epsilon^s \zeta(\mu) \bar{\gamma}_2(\phi_{t/\mu}(p), r, \theta, \epsilon, \mu) \end{aligned} \tag{4.1}$$

where $\bar{\gamma}_1(r,\theta,\epsilon) = O(r^5)$, $\gamma_2(r,\theta,\epsilon) = O(r^4)$ and $\bar{\gamma}_i(\phi_{t/\mu}(p), r, \theta, \epsilon, \mu)$ are bounded for $i = 1, 2$.

Proof of Lemma 3.7. In the assumption of Lemma 3.7 we see that $\bar{\gamma}_1(\phi_{t/\mu}(p), 0, \theta, \epsilon, \mu) = O(\sqrt{\epsilon})$. Hence if we set $r_0 = \bar{\delta}\sqrt{\epsilon}$ (where $\bar{\delta} := \sqrt{\zeta(\mu) + \epsilon}$) from (4.1) we easily see that

$$\dot{r}(r_0, \theta) = \epsilon r_0 \left[1 + O(\bar{\delta}^2 + \epsilon^{s-1}\sqrt{\zeta(\mu)}\right] > \epsilon r_0/2 > 0 \tag{4.2}$$

for any θ, if ϵ and μ are small enough. Hence it follows that $\mathfrak{D}$ is negatively invariant. Analogously let us set $r_1 = \sqrt{\epsilon}(1 - \bar{\delta})$ and $r_2 = \sqrt{\epsilon}(1 + \bar{\delta})$: from a straightforward computation we get

$$\dot{r}(r_1, \theta) > \epsilon\sqrt{\epsilon}[(1 - \bar{\delta}) - (1 - \bar{\delta})^3 + O(\epsilon + \epsilon^{s-1}\zeta(\mu))] > \epsilon^{3/2}\bar{\delta} > 0 \tag{4.3}$$

and similarly $\dot{r}(r_2, \theta) < -\epsilon^{3/2}\bar{\delta} < 0$, for any θ, if ϵ and μ are small enough. Hence we see that $\mathfrak{A}$ is positively invariant. □

In order to prove Proposition 3.8 we need to show that all the trajectories of (3.4) approach each other, and then to apply the argument of [31, Part II, §2.4].

Proof of Proposition 3.8. Fix $p \in P$, $Q^1, Q^2 \in \mathfrak{D}$ and set $d(t) = y(t, p; Q^2) - y(t, p; Q^1)$. Notice that $y(t, p; Q^i) \in \mathfrak{D}$ for any $t \leq 0$ by Lemma 3.7, for $i = 1, 2$; hence $R(t) := |d(t)| \leq 2\bar{\delta}\sqrt{\epsilon}$ for any $t \leq 0$, where $\bar{\delta} = \sqrt{\zeta(\mu) + \epsilon}$.

Observe now that $N_x(\bar{y}, \epsilon) = O(\bar{\delta}^2\epsilon)$ and $\tilde{g}_x(t, \bar{y}, \epsilon, \mu)$ is bounded uniformly for any $t \leq 0$, $\bar{y} \in \mathfrak{D}$, $0 < \epsilon < \epsilon_0$, $0 < \mu < \mu_0$. Hence, linearizing (3.4) in $y(t, p; Q^1)$ we see that, for any $t \leq 0$, $d(t)$ solves an equation of the form

$$\dot{d} = [L + O(\epsilon^s \zeta(\mu))]d$$

Hence for any $t \le 0$ we find

$$\dot{R} = \epsilon(1 + O(\zeta(\mu))R > \frac{\epsilon}{2}R > 0. \tag{4.4}$$

It follows that the distance in $\mathbb{R}^2$ itself is a Lyapunov function in backward time and the flow of (3.4) is contractive, again in backward time, see [31, Part II, §2.4] at page 30. So applying [31, Part II, Theorem 2.9], we get the existence of an almost periodic trajectory $y_o(t)$ which is a repeller, and we conclude the proof of Proposition 3.8. □

Proof of Proposition 3.9. Let $W_\mu(\tau)$ be the set of all the points Q such that $y(t,\tau;Q) \in \mathfrak{A}$ for any $t \in \mathbb{R}$: we need to show first that $W_\mu(\tau)$ is non-empty, then that it is a compact connected set topologically equivalent to S^1. Let $Q \in \mathfrak{A} \setminus W_\mu(\tau)$ and set $T(Q) := \inf\{s \mid y(t;\tau,Q) \in \mathfrak{A}, \text{ for any } t > s\}$. Let $\partial\mathfrak{A}^i$ and $\partial\mathfrak{A}^o$ be the inner and outer circle defining the border of $\mathfrak{A}$ and set

$$E^i := \{Q \in \mathfrak{A} \setminus W_\mu(\tau) \mid y(T(Q);\tau,Q) \in \partial\mathfrak{A}^i\},$$
$$E^o := \{Q \in \mathfrak{A} \setminus W_\mu(\tau) \mid y(T(Q);\tau,Q) \in \partial\mathfrak{A}^o\}$$

so that we have partitioned $\mathfrak{A}$ as $\mathfrak{A} = E^i \cup E^o \cup W_\mu(\tau)$.

Since the flow on $\partial\mathfrak{A}$ aims transversally inside $\mathfrak{A}$ for any $t \in \mathbb{R}$, we easily get that $T(Q)$ is continuous and that E^i, E^0 are relatively open. Now we show that E^i and E^o are nonempty so $W_\mu(\tau)$ is non empty and compact as well.

Let $\Upsilon(s) : [0,1] \to \mathfrak{A}$ be a smooth path such that $\Upsilon(0) \in \partial\mathfrak{A}^i$ and $\Upsilon(1) \in \partial\mathfrak{A}^o$; then by construction $\Upsilon(0) \in E^i$ while $\Upsilon(1) \in E^o$. It follows that E^i, E^o are non-empty; hence $W_\mu(\tau)$ is non-empty and compact. Further we have shown that each path connecting $\partial\mathfrak{A}^i$ and $\partial\mathfrak{A}^o$ intersects $W_\mu(\tau)$; then Proposition 3.9 follows by a straightforward application of [29, Lemma 4]. □

Remark 4.1. In fact, applying [29, Lemma 4] we find that $W_\mu(\tau)$ is a continuum i.e. there is a 1 dimensional manifold $\tilde{\Gamma}$ homeomorphic to a circle such that for any $\delta > 0$ and any $Q_1 \in \tilde{\Gamma}$ there is $Q_2 \in W_\mu(\tau)$ such that $|Q_2 - Q_1| < \delta$. Further $W_\mu(\tau)$ is topologically equivalent to S^1, i.e. $W_\mu(\tau)$ might be thick in some or in all the parts so it might not be a manifold, in this setting, cf. Definition 3.1.

Remark 4.2. We emphasize that Proposition 3.8 can be trivially generalized to the case where the base flow $(P, \{\phi_t\})$ is distal, i.e. satisfying $\inf\{|\phi_t(p_2) - \phi_t(p_1)| \mid t \in \mathbb{R}\} > 0$ for any $p_1 \neq p_2$ in P. In this more general case we can still apply [31, Part II, Theorem 2.9] and we find that $y_o(t)$ is a copy of the base (i.e. its dynamics is isomorphic to $\{\phi_t\}$), see [31, Part II,§2] for more details.

Proposition 3.9 needs even weaker recurrence properties; in fact it is enough that $(P, \{\phi_t\})$ is uniquely ergodic so that we can apply the machinery described in section 2 and write equation (3.4).

4.2 Proof of Theorems 3.12 and 3.14

The proofs of Theorems 3.12 and 3.14 are obtained with some simple modifications of the arguments of [12, Proposition 3.2] and [12, Theorem 3.9], therefore we will be rather sketchy, remanding the reader to [12] for more details. The key idea is to consider (3.4), to fix $\epsilon > 0$ and to regard $\zeta(\mu)$ as a bifurcation parameter so that we can use exponential dichotomy and uniform hyperbolicity, and put ourselves in the framework of [12, §3].

Proof of Theorem 3.12. First of all, following [12, pages 1127-1128], we see that setting $\rho = [r(\theta) - \Gamma(\theta)]/\sqrt{\epsilon}$ we pass from (4.1) to (3.9). In fact $\Gamma(\theta) = \sqrt{\epsilon}[1 + \sqrt{\epsilon}K(\theta, \epsilon)]$ where $K(\theta, \epsilon)$ is a function bounded with its derivatives. Hence $r = \sqrt{\epsilon}\,[\rho + 1 + \sqrt{\epsilon}K(\theta, \epsilon)]$ and, substituting in (4.1), we get

$$\dot{\rho} + \sqrt{\epsilon}\frac{\partial K}{\partial \theta}(\theta, \epsilon)\dot{\theta} = \epsilon[\rho + 1 + \sqrt{\epsilon}K(\theta, \epsilon)] - \epsilon[\rho + 1 + \sqrt{\epsilon}K(\theta, \epsilon)]^3 + \epsilon^2 H_1(\rho, \theta, \epsilon) \\ + \epsilon^s \zeta(\mu)\bar{H}_1(\phi_{t/\mu}(p), \rho, \theta, \epsilon, \mu)$$

where H_1 and $\bar{H}_1$ are bounded, and $\dot{\theta}$ is given by (3.9). Since $\rho = 0$ is invariant for $\mu = 0$, we see that we have no 0-order term in ρ for $\epsilon = 0$, hence

$$\dot{\rho} = -2\epsilon\rho - 3\epsilon\rho^2 + \epsilon^{3/2}\rho A(\rho, \theta, \epsilon) + \epsilon^s \zeta(\mu) B(\phi_{t/\mu}(p), \rho, \theta, \epsilon, \mu) \tag{4.5}$$

where the remainder term $A(\rho, \theta, \epsilon)$ is bounded and its leading term comes from $\frac{\partial K}{\partial \theta}(\theta, \epsilon)\dot{\theta}$. So we passed from (4.1) to (3.9).

Now is easy to check that the set E defined below

$$E = \{(\rho, \theta, p) \mid |\rho| \le \delta\,,\ (\theta, p) \in S^1 \times P\}\,, \qquad \delta = \sqrt{\zeta(\mu)} > 0 \tag{4.6}$$

is positively invariant for the flow of (3.9) (in fact it is a slight deformation of $\mathfrak{A} \times P$ and their sizes are of the same order if $\epsilon = O(\zeta(\mu))$).

Following [12, Proposition 3.2] we consider (4.5) as a perturbation of its linear part. Then, for any $0 < \epsilon < \epsilon_0$, $0 < \mu < \mu_0$, for each orbit $\phi_\tau(p)$ we see that the linearized ρ equation admits exponential dichotomy with dichotomy constants (K, α) where $\alpha > \epsilon$, see Proposition 1 of [5, §4]. Next we can directly apply the argument of [32, Theorem 6.1] and if

$$\sup_{0 \le \theta \le 2\pi} \epsilon^{3/2}\left|\frac{\partial C}{\partial \theta}(0, \theta)\right| + \zeta(\mu)\epsilon^s \left|\frac{\partial D}{\partial \theta}(p, 0, \theta)\right| \le \frac{\epsilon}{(r+1)}\,, \tag{4.7}$$

we get the existence of the manifold M_μ with the desired properties. So the Theorem is proved simply noticing that (4.7) follows from (3.12). □

For the proof of Theorem 3.14 we borrow some notation from [12], since it is obtained by repeating the argument of the proof of [12, Theorem 3.9].

We recall that the set E defined in (4.6) is positively invariant for the flow of (3.9). We denote by $\Pi, \Pi_\theta, \Pi_p, \Pi_\rho$ the projection $\Pi(\rho, \theta, p) = (\theta, p)$, $\Pi_\rho(\rho, \theta, p) = \rho$, $\Pi_\theta(\rho, \theta, p) = \theta$, $\Pi_p(\rho, \theta, p) = p$. The first step is to discretize time, so let us fix $T > 0$; we invite the reader to think of T as a time close to the first return time for $\mu = 0$, i.e. $T = 2\pi$, even if it is not needed. Let $F_{\mu,t}(\rho_0, \theta_0, p_0)$ be the solution of (3.15) such that $F_{\mu,0}(\rho_0, \theta_0, p_0) = (\rho_0, \theta_0, p_0)$ and consider $F_{\mu,T} : E \to E$. Observe that $V_{0,T} = \{0\} \times S^1 \times P$ is an invariant centre manifold for $F_{0,T}$, and it corresponds to the invariant manifold $M_0 = \Gamma \times P$ in the original coordinates.

Let $\sigma : (S^1 \times P) \to E$ be a C^r function (a section), i.e. $\sigma(\theta, p) = (\theta, p, s(\theta, p))$ and $s : (S^1 \times P) \to [-\delta, \delta]$ is C^r. We define the *slope* of a section σ as

$$\|\sigma\|_{sl} := \sup\left\{\left|\frac{\partial s(\theta, p)}{\partial \theta, \partial p}\right| \,\middle|\, (\theta, p) \in S^1 \times P\right\}$$

and we consider the set $\Sigma := \{\sigma : (S^1 \times P) \to E \mid \|\sigma\|_{sl} \le \delta\}$. We endow Σ with the C^0 norm, which makes it complete, see [26] for details.

Our aim is to apply the results of [26] to obtain the following result on the discrete map $F_{\mu,T}$, rephrased from [26], analogous to [12, Proposition 3.3]. Once again we just sketch the proof remanding to [12, Proposition 3.3] for details.

Lemma 4.3. *Assume that f and $\tilde{g}$ are C^r, $r \geq 1$, in all their variables; assume further* **H** *and that $\tilde{g}(t, 0, \epsilon, \mu) \equiv 0$. Then there is a C^r function $v_{\mu,T} : S^1 \times P \to \mathbb{R}$, such that the manifold*

$$V_{\mu,T} = \left\{ (v_{\mu,T}(\theta, p), \theta, p) \mid \theta \in S^1, \; p \in P \right\} \tag{4.8}$$

is invariant for forward and backward iterates of $F_{\mu,T}$. Moreover $\|v_{\mu,T}\| = O(\delta)$ and $\left\| \frac{\partial v_{\mu,T}}{\partial\theta, \partial p} \right\|_\infty \leq \delta$.

All the quantities in this proof depend on ϵ and μ which are small positive constants which will be fixed at the end of the proof: we omit to write explicitly these dependencies. Further, we will write F for $F_{\mu,T}$ to avoid cumbersome notation (T is fixed in the whole proof).

Let $\mu > 0$; for any $\sigma \in \Sigma$ we define the map $H_\sigma : (S^1 \times P) \to (S^1 \times P)$ by $H_\sigma = \Pi \circ F \circ \sigma$. Such a map is surjective since F is a diffeomorphism. Roughly speaking a section σ maps each point $(\bar{\theta}, \bar{p}) \in (S^1 \times P)$ in a point $\bar{e} = (\bar{\rho}, \bar{\theta}, \bar{p}) \in E$. The section H_σ maps $(\bar{\theta}, \bar{p})$ in a point $\tilde{e} = (\tilde{\rho}, \tilde{\theta}, \tilde{p}) \in E$, where $\tilde{e}$ is the evolution of $\bar{e}$ through system (3.15), after time T.
We emphasize that when $\mu = 0$ the map F is not properly defined in its P component, so we arbitrarily decide that $F_{0,T}$ acts as the identity in P, i.e. $\Pi_p F_{0,T}(\rho, \theta, p) \equiv p$ for any $(\rho, \theta, p) \in E$. However this means that the map $F_{\mu,T}$ is not continuous in μ when $\mu = 0$: this is the main difference with the argument in [12].

Following the proof of [26, Theorem 4.1], in particular point (v) at pag. 44, we see that, H_σ is bijective, so it admits a proper inverse H_σ^{-1}, for any $\sigma \in \Sigma$.

Then we define the map $F^\sharp : \Sigma \to \Sigma$ as $F^\sharp(\sigma) = F \circ \sigma \circ H_\sigma^{-1}$, see [26, Theorem 4.1]. For $\mu = 0$, such a map is a contraction and its unique fixed point corresponds to the unperturbed centre manifold $\rho = 0$, i.e. $\Gamma \times P$ in the original coordinates.

Lemma 4.3 is obtained by using the ideas in the proof of Theorem 4.1 point f) in [26] (pp. 49–51). In fact we will show that $F^\sharp$ is a contraction for $\mu > 0$ too (small enough), and its unique fixed point $\sigma^\sharp$ parameterizes the integral manifold $V = V_{\mu,T}$, see also [12, Proposition 3.3].

The proof is developed in several steps, in which we rephrase the argument of [12, Proposition 3.3]: we enumerate them here for convenience of the reader, underlining the changes and remanding to [12, Proposition 3.3] for details.

We recall that μ is a small positive constant in the whole argument: when $\mu = 0$ we obtain the classical AH autonomous perturbation.

Step 1. *$H_\sigma = \Pi \circ F \circ \sigma : S^1 \times P \to S^1 \times P$ is bijective for any fixed σ and we denote by H_σ^{-1} its inverse.*

Proof. The function H_σ is clearly invertible in its P component due to the skew-product nature of the flow of (3.15) Further, integrating (3.15), we get

$$\Pi_\theta[F_\mu(s(\theta, p), \theta, p)] = \theta + T[1 + \epsilon\varpi(\epsilon)] + 2\epsilon\varpi(\epsilon)O(\delta) + O(\epsilon\delta^2 + \epsilon^{3/2} + \epsilon^s\zeta(\mu))], \tag{4.9}$$

see [12] for details. Using (4.9) we find the following estimates for the derivatives of H_σ analogous to (3.3) in [12]:

$$\begin{pmatrix} \frac{\partial}{\partial\theta}\Pi_\theta H_\sigma & \frac{\partial}{\partial p}\Pi_\theta H_\sigma \\ \frac{\partial}{\partial\theta}\Pi_p H_\sigma & \frac{\partial}{\partial p}\Pi_p H_\sigma \end{pmatrix} = \begin{pmatrix} 1 + O(\delta\epsilon) & O(\epsilon^s\zeta(\mu)) \\ 0 & \mathcal{A}(\frac{T}{\mu}) \end{pmatrix}. \tag{4.10}$$

From assumption **H** we see that H_σ is invertible and its inverse, which is obtained simply reversing time, satisfies (4.10) too; so Step 1 is concluded. □

Now we rephrase [12, Lemma 3.6].

Lemma 4.4. *Let $\sigma'', \sigma' \in \Sigma$, then setting $c_1 = 3T\varpi(0) > 0$ we find*

$$\|H_{\sigma''} - H_{\sigma'}\| \leq c_1\epsilon\|\sigma'' - \sigma'\|, \qquad \|[H_{\sigma''}]^{-1} - [H_{\sigma'}]^{-1}\| \leq c_1\epsilon\|\sigma'' - \sigma'\|.$$

Proof. We remand the reader to [12] for details; by the way we observe that the key estimates are the following:

$$\begin{aligned} \frac{d}{dt}\frac{\partial\rho(t)}{\partial\rho(0)} &= [-2\epsilon + o(\epsilon) + O(\epsilon^s\zeta(\mu))]\frac{\partial\rho(t)}{\partial\rho(0)} + O(\epsilon^s\zeta(\mu) + \epsilon^{3/2}\delta)\frac{\partial\theta(t)}{\partial\rho(0)}, \\ \frac{d}{dt}\frac{\partial\theta(t)}{\partial\rho(0)} &= O(\epsilon^{3/2} + \epsilon^s\zeta(\mu))\frac{\partial\theta(t)}{\partial\rho(0)} + \left[2\epsilon\varpi(\epsilon)(1+\delta) + O(\epsilon^{3/2} + \epsilon^s\zeta(\mu))\right]\frac{\partial\rho(t)}{\partial\rho(0)} \end{aligned} \tag{4.11}$$

and that the Lemma then follows from the Gronwall-type result in [12, Lemma 3.5]. □

Step 2. *$F^\sharp := F \circ \sigma \circ H_\sigma^{-1}$ maps Σ into itself.* Observe first that by construction $F^\sharp$ is the identity in its θ, p coordinates (it simply moves the time backward with H_σ^{-1} and forward with F), for any $\sigma \in \Sigma$, but in general it does not fix the ρ coordinate.

Fix $\sigma \in \Sigma$ and let $\sigma^\sharp(\theta, p) = (s^\sharp(\theta, p), \theta, p) := F^\sharp(\sigma)$; since the set $|\rho| \leq \delta$ is positively invariant for the flow of (3.15), we see that $\|s^\sharp(\theta, p)\| \leq \delta$. Then we need to show that $\|\sigma^\sharp\|_{sl} \leq \delta$ as well, and Step 2 is proved. From a straightforward repetition of the argument of Step 2 in [12, Lemma 3.6] we find the following estimates:

$$\left\|\frac{\partial s^\sharp(\theta, p)}{\partial\theta}\right\| \leq \delta\left(1 - \frac{\epsilon T}{2}\right), \qquad \left\|\frac{\partial s^\sharp(\theta, p)}{\partial p}\right\| \leq \delta\left(1 - \frac{\epsilon T}{2}\right). \tag{4.12}$$

The inequalities in (4.12) are obtained simply by replacing the parameter μ of the proof [12, Lemma 3.6] by $\epsilon^s\zeta(\mu)$ and using the fact that $\epsilon^s\zeta(\mu) = o(\epsilon)$. So we immediately see that $\|\sigma^\sharp\|_{sl} \leq \delta$.

Then reasoning again as in [12, Proposition 3.3, Step 3] we get the following.

Step 3. *$F^\sharp$ is a contraction in Σ of factor $K = 1 - \frac{\epsilon T}{2}$.* Hence we see that $F^\sharp$ has a unique fixed point in Σ, which we denote by $\sigma^F(\theta, p) = (s^F(\theta, p), \theta, p)$. Obviously $\sigma^F(\theta, p)$ is invariant for the action of F. Let us denote by $v_{\mu,T}(\theta, p) := s^F(\theta, p)$. We have already shown that $\|v_{\mu,T}\| = O(\delta)$, $\|\frac{\partial v_{\mu,T}}{\partial\theta,\partial p}\|_\infty \leq \delta$. Further the set $V_{\mu,T}$ defined in (4.8) (i.e. the image of $\sigma^F(\theta, p)$) is invariant for forward and backward iterates of $F = F_{\mu,T}$.

We just need to show that $v_{\mu,T} \in C^r$ then Lemma 4.3 is proved. For this purpose we need to show the following estimates for any $\sigma \in \Sigma$

$$\begin{aligned} \|\Pi_\rho[F(\rho_2, \theta, p)] - \Pi_\rho[F(\rho_1, \theta, p)]\| &\leq K\|\rho_2 - \rho_1\|, \\ \|\Pi[H_\sigma^{-1}(\theta_2, p_2) - H_\sigma^{-1}(\theta_1, p_1)]\| &\leq \alpha\|(\theta_2, p_2) - (\theta_1, p_1)\|. \end{aligned} \tag{4.13}$$

The first inequality in (4.13) is obtained for $K = 1 - \frac{\epsilon T}{2}$ and is proved as in [12, Proposition 3.3, Step 4]. Further, using (4.10) and assumption **H** we see that there is $c > 0$ such that the second inequality in (4.13) holds with $\alpha = 1 + c\delta\epsilon$. Then, for any $r \geq 1$, we can choose $r\delta > 0$ (hence $\mu > 0$, see (4.6)) small enough so that

$$K\alpha^r = \left(1 - \frac{\epsilon T}{2}\right)(1 + c\delta\epsilon)^r = 1 - \epsilon\left(\frac{T}{2} - cr\delta\right) + o(\epsilon\delta) < 1 - \frac{\epsilon T}{3}.$$

This way we have shown that $K\alpha^r < 1$ for any $0 < \epsilon < \epsilon_0$ and any $0 < \mu < \mu_0$. Hence F_μ *is a fiber contraction of sharpness r*. So we are in the position to apply the argument of point d) in [26, Theorem 4.1], which is in fact based on the fiber contraction theorem for C^r maps, i.e. in [26, Theorem 3.5]. This proves that the fixed point σ^F of $F^\sharp$ is actually C^r in its θ and p variables, so the proof of Lemma 4.3 is concluded.

Proof of Theorem 3.14. Theorem 3.14 now easily follows passing from discretized time to continuous time. The proof is a straightforward repetition of [12, Theorem 3.9] which is rephrased from [20, Theorem 2.16]. The key lies in the fact that all the arguments are uniform with respect to $T > 0$. □

5 Some remarks on the dynamics on M_μ

Let $m_0 = (p, Q_0) \in M_\mu$, so that $m_0 \in \bar{W}_\mu(p)$; we denote by $\psi_t(m_0) = (\phi_{t/\mu}(p), y(t, p; Q_0))$. Notice that $\psi_t(m_0) \in M_\mu$ for any $t \in \mathbb{R}$, hence $(\{\psi_t\}, M_\mu)$ defines a flow. In fact $(\{\psi_t\}, M_\mu)$ is often referred to as *the circle extension of* $(\{\phi_{t/\mu}\}, P)$, since it is obtained by adding an extra variable which takes values in S^1 (in fact in a set homeomorphic to S^1).

In this section we briefly review some facts concerning the dynamics in $(\{\psi_t\}, M_\mu)$; they have already been stated and proved in [12, §4] for the analogous regular perturbation problem. In fact the same results hold in this fast oscillation context with no changes: we summarise them here for convenience of the reader. The main difference with respect to the regular perturbation setting of [12, §4] is that it becomes ineludible the problem of avoiding resonances between the frequencies of $(\{\phi_{t/\mu}\}, P)$ and the "rotation" around M_μ, whose frequency is close to 2π. So we will have intermittency phenomena: sequences of values $\mu_j \searrow 0$ for which the flow in M_μ is simply the results of the addition of the rotation number (see below) to the frequencies of $(\{\phi_{t/\mu}\}, P)$, and other values for which we have mixing and weakly mixing flow $(\{\psi_t\}, M_\mu)$. In fact the frequency modulus of $(\{\phi_{t/\mu}\}, P)$ diverge while the rotation number stay close to 2π. We emphasize that similar results have already been observed in [21] for systems with rapidly varying coefficients and subject to saddle-node or transcritical bifurcation.

We assume that the hypotheses of Theorem 3.14 are satisfied so that M_μ is C^r in p as well. This will allow us to introduce Diliberto map and to identify the flow $(\{\psi_t\}, M_\mu)$ as a suspension flow of the base flow $(\{\phi_t\}, P)$.

Namely, by construction, for any $p \in P$ the manifold M_μ intersects the y_1 positive semi-axis (or more precisely $L_1 = P \times \{(y_1, 0) \mid y_1 > 0\}$) in a unique point $m_0(p) = (p, [\Gamma(0) + \sqrt{\epsilon}v(p, 0)\mathrm{e}^{i0}])$. Let $\mathfrak{T}(p)$ be the first retour map of $m_0(p) = (p, Q_0(p))$, i.e. the least positive time such that $\psi_t(m_0(p))$ is in L_1. Hence $\psi_{\mathfrak{T}(p)}(p, Q_0(p)) = (\phi_{\mathfrak{T}(p)}(p), Q_1(p))$ where $Q_1(p) = \Gamma(2\pi) + \sqrt{\epsilon}v(\phi_{\frac{\mathfrak{T}(p)}{\mu}}(p), 2\pi)\mathrm{e}^{i2\pi}$.

Then, from [12, Proposition 4.3] we see that the Diliberto map $K : P \to P$ defined as $K(p) = \phi_{\frac{\mathfrak{T}(p)}{\mu}}(p)$ is a homemorphism.

Now consider the product space $P \times \mathbb{R}$, subject to the equivalence relation $\sim$ defined as follows:

$$(p, s + \mathfrak{T}(p)) \sim (K(p), s) \qquad (p \in P, s \in \mathbb{R}).$$

One defines a flow $\{\sigma_t\}$ on the quotient space $\Sigma = P \times \mathbb{R}/\sim$ by setting

$$\sigma_t[p, s] = [p, s + t]$$

where $[p,s]$ denotes the equivalence class (i.e., elements of Σ) which contains $(p,s) \in P \times \mathbb{R}$. Then $\Sigma = \Sigma_K^{\mathfrak{T}}$ is a compact metrizable space, and $\{\sigma_t\}$ is a flow on Σ, which is usually referred to as *the suspension flow of K with roof function* $\mathfrak{T}$.

From [12, Proposition 4.4] we see that the circle extension $(\{\psi_t\}, M_\mu)$ of $(\{\phi_{t/\mu}\}, P)$ can be identified with $(\Sigma, \{\sigma_t\})$, i.e. we have the following.

Remark 5.1. The flow of ψ_t on M_μ is isomorphic to the flow of $\{\sigma_t\}$ on Σ.

Remark 5.1 can now be used to study the dynamics of $(\{\psi_t\}, M_\mu)$. We sum up some results borrowed again from [12, §4]. We consider $\mu \in]0, \mu_0]$ fixed, and we look for sufficient conditions to have information on the dynamics of $(M_\mu, \{\psi_t\})$: this will implicitly result, in particular, in asking for non-resonance conditions, which means choosing particular values of $\mu_j \searrow 0$.

Assume that $(\{\phi_t\}, P)$ is uniquely ergodic, so that $(\{\phi_{t/\mu}\}, P)$ is uniquely ergodic too. Let $\bar{Q}_0 = \rho_0 e^{i\theta_0}$ be such that $m_0 = (p, \bar{Q}_0) \in M_\mu$, i.e. $\rho_0 = \Gamma(0) + \sqrt{\epsilon} v(p, \theta_0)$; denote by $\theta(t, p, \theta_0)$ the solution of the second equation in (3.9). Then the limit

$$\lim_{|t| \to \infty} \frac{\theta(t)}{t} := \rho_\mu \tag{5.1}$$

exists and is uniform in $(p, \theta_0) \in P \times S^1$. Let us fix $0 < \mu \le \mu_0$. In the difficult case in which P is a generic compact metric space and $(P, \{\phi_t\})$ is just a minimal almost periodic flow, the flow $(M_\mu, \{\psi_t\})$ is a Furstenberg extension of $(P, \{\phi_t\})$, see [12, Proposition 4.12]; we remand the interested reader to the last pages in [12] for details, see also [14].

We specialize now to the case where P is a d dimensional torus and $(\{\phi_t\}, P)$ is quasi-periodic with frequencies ω_i, $i = 1, \ldots, d$. We also pick up μ so that $\mu\rho_\mu \ne \omega_i - 2k\pi$ for any i and any $k \in \mathbb{Z}$. Then one would expect that the flow on $(\{\psi_t\}, M_\mu)$ is quasi periodic with frequencies $\{\omega_i/\mu \mid i = 1, \ldots, d\} \cup \{\rho_\mu\}$. This is not always the case; in fact the flow of $(\{\psi_t\}, M_\mu)$ could even be weakly mixing, due to resonances (or better to the failure of Diophantine conditions).

A useful tool to ensure the persistence of almost periodic behaviors is the bounded mean motion property, see [19]. Let $m = (p, r, \theta) \in M_\mu$, where $r = r(p, \theta)$, and let $\psi_t(m) = (\phi_{t/\mu}(p), r(t)e^{i\theta(t)}) \in M_\mu$.

Definition 5.2. We say that the flow $\{\psi_t\}$ on M_μ has the bounded mean motion property (bmm for short), if there exists a fixed constant c such that, for each $m \in M_\mu$ we have

$$|\theta(t) - \rho_\mu t| \le c \qquad (t \in \mathbb{R},\ p \in P). \tag{5.2}$$

Let us consider μ fixed. Assume that $(\{\phi_t\}, P)$ is minimal and almost periodic, and that $(\{\psi_t\}, M_\mu)$ has the bmm property. Then each minimal subset of M_μ is almost automorphic and its frequency modulus is generated by the frequency modulus of $(\{\phi_{t/\mu}\}, P)$ and ρ_μ, see [19, Theorem 8.3], or [12, Introduction] for definitions.

Assume further that $(\{\phi_t\}, P)$ is quasi-periodic with frequencies ω_i, $i = 1, \ldots, d$, and that $\mu\rho_\mu \ne \omega_i$ modulus 2π for any i. Then either $(M, \{\psi_t\})$ is quasi periodic with frequencies ω_i/μ and ρ_μ (the easy case whose existence we conjectured few lines above) or it is a "Cantorus", see [19], and it laminates in almost periodic minimal flows.

We emphasize that if we are considering a periodic perturbation, i.e. $(\{\phi_t\}, P)$ is just a rigid rotation in S^1, then $(\{\phi_t\}, P)$ always has bmm, see [4]. But this is not the case if the perturbation is quasi periodic, or it has more general recurrence properties.

A possible way to reveal the presence (or the absence) of bmm for $(M, \{\psi_t\})$ is to conjugate such a flow to some simple map on the d-Torus P. The bmm holds if the flow is conjugated with an irrational rotation R_{ρ_μ} of the d-Torus P, and in this case M_μ is either a $d+1$-Torus or a Cantorus. To obtain such a conjugation with an irrational rotation R_{ρ_μ} one might profit of some results of KAM theory as explained in [12, Proposition 4.10], which rely on [15,16]. For this purpose some Diophantine conditions and C^r smoothness of $(M, \{\psi_t\})$, with r large are essential.

If such a conjugation fails the flow might be "rigid" [8], weakly mixing [9] or mixing [8].

We emphasize that in any case we need to compare the frequencies ω_i/μ of $(\{\phi_{t/\mu}\}, P)$ with the rotation number ρ_μ modulus 2π so in general we should expect for intermittency phenomena. I.e. the bmm property is probably satisfied for at most a sequence of values $\mu_j \to 0$. So as μ decreases we should expect for some rare values for which the flow on M_μ is quasi periodic, and a majority of values in which it is just almost automorphic, weakly mixing or mixing.

Appendix A Concerning the change of variables by Bellman et al.

In this appendix we make a brief digression about the methods developed in [1,2]. The main idea is the following: in a system with fast varying coefficients it is possible to construct a change of variables which reveals a shift in the coefficients of the averaged system, which may result in a displacement of the bifurcation value. Such a phenomenon may be responsible of gain or loss of stability of the equilibria, see [1,2]. In [13], we have already discussed the case of a Van der Pol oscillator with rapidly varying coefficients, using the methods developed in [1,2]. In fact the discussion can be generalized so to embrace the general case of an AH bifurcation pattern. Again we just give the main ideas remanding the interested reader to [13, §3.4] for details, see also [2]. We consider an autonomous system which undergoes to an AH bifurcation pattern, and we assume that the coefficients of its linear part are subject to a rapidly varying non-autonomous perturbation.

So we consider

$$\frac{dx}{dt} = \begin{pmatrix} \epsilon + h_1\left(\frac{t}{\mu}\right) & -1 \\ 1 & \epsilon + h_2\left(\frac{t}{\mu}\right) \end{pmatrix} x - \begin{pmatrix} 1 & \varpi(\epsilon) \\ -\varpi(\epsilon) & 1 \end{pmatrix} |x|^2 x + W(x,\epsilon) \tag{A.1}$$

where the functions $h_i(t)$ have a bounded primitive with 0 mean value, for $i = 1,2$, i.e. $H_i(\tau) = \int_0^\tau h_i(t)dt$ such that $\lim_{\tau\to\infty} \frac{1}{\tau} H_i(\tau) = 0$. The method consists in applying a first change of variables, i.e. we set

$$\begin{aligned} x_i(\tau) &= y_i(\tau)\mathrm{e}^{\mu H_i(\tau)}, \qquad \tau = \frac{t}{\mu}, \\ a(\tau) &= \mathrm{e}^{\mu[H_1(\tau)-H_2(\tau)]}, \qquad b(\tau) = [a(\tau)]^{-1} \end{aligned} \tag{A.2}$$

so that we pass from (A.1) to

$$\frac{1}{\mu}\frac{dy}{d\tau} = \begin{pmatrix} \epsilon & -b(\tau) \\ a(\tau) & \epsilon \end{pmatrix} y - \begin{pmatrix} 1 & \varpi(\epsilon)b(\tau) \\ -\varpi(\epsilon)a(\tau) & 1 \end{pmatrix} K(y,\tau)y + \tilde{W}(y,\tau,\epsilon) \tag{A.3}$$

where $K(y,\tau) = [y_1 \mathrm{e}^{\mu H_1(\tau)}]^2 + [y_2 \mathrm{e}^{\mu H_2(\tau)}]^2$, and $\tilde{W}(y,\tau,\epsilon) = O(|y|^5)$ uniformly in τ and ϵ. The key observation is that, even if the functions H_i have 0 average, using Jensen's inequality we see that

$$\bar{a} = \lim_{\tau \to +\infty} \frac{1}{\tau} \int_0^\tau a(\sigma) d\sigma = 1 + \bar{A}\mu, \qquad \bar{b} = \lim_{\tau \to +\infty} \frac{1}{\tau} \int_0^\tau b(\sigma) d\sigma = 1 + \bar{B}\mu \tag{A.4}$$

where $\bar{A} > 0$, $\bar{B} > 0$.

Therefore, if we consider the averaged system where we replace a, b by $\bar{a}$, $\bar{b}$ and we pass to polar coordinates, **we observe an increment in the rate of rotation of order $\mu(\bar{A} + \bar{B})/2$ even if the perturbation $h_i(t)$ of the coefficients have 0 average**. Namely if we set $\bar{r} = [\bar{a}y_1^2 + \bar{b}y_2^2]^{1/2}$, and $\bar{\theta} = \arctan\left[\frac{\sqrt{\bar{b}}y_2}{\sqrt{\bar{a}}y_1}\right]$ we get

$$\begin{aligned} \frac{d\bar{r}}{dt} &= \mu\Big\{\epsilon\bar{r} - \bar{r}^3(1 + \mu C_1(\bar{\theta})) + \bar{r}^5 C_2(\bar{\theta}, \epsilon)\Big\}, \\ \frac{d\bar{\theta}}{dt} &= \mu\sqrt{\bar{a}\bar{b}}\Big\{1 + \omega(\epsilon)\bar{r}^2 C_3(\bar{r}, \bar{\theta}) + \bar{r}^4 C_4(\bar{r}, \bar{\theta}, \epsilon)\Big\} \end{aligned} \tag{A.5}$$

where the functions C_i are uniformly bounded in their respective variables.

From this computation we also see that the attracting invariant manifold, if exists, "tend to assume a more elliptic like shape", i.e. it is a small τ dependent deformation of the ellipse $\bar{a}y_1^2 + \bar{b}y_2^2 = \epsilon$, that is $\bar{r} = \sqrt{\epsilon}$.

Following [2] we point out that if we replace the functions $h_i(\frac{t}{\mu})$ in (A.1) by the large and fast varying functions $\frac{\alpha}{\mu}h_i(t/\mu)$ (where α is a constant) we obtain that $\bar{a} = 1 + \alpha\bar{A}$, $\bar{b} = 1 + \alpha\bar{B}$ in (A.4), i.e. **we have a macroscopic change in the speed of rotation close to the origin (of order $O(1)$)**. However, if α is not small enough, the whole structure of the bifurcation pattern might be washed away when the whole non-autonomous perturbation problem is considered (and not just its averaged system as in [1,2]): in fact we need $\alpha = o(\epsilon)$ to apply our techniques and to obtain the results in Sections 3.1, 3.2. This might depend on the method of proof we used, but we believe that some smallness condition on α is most probably needed.

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