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# Stability and Hopf bifurcation of a diffusive Gompertz population model with nonlocal delay effect

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**Abstract.** In this paper, we investigate the dynamics of a diffusive Gompertz population model with nonlocal delay effect and Dirichlet boundary condition. The stability of the positive spatially nonhomogeneous steady-state solutions and the existence of Hopf bifurcations with the change of the time delay are discussed by analyzing the distribution of eigenvalues of the infinitesimal generator associated with the linearized system. Then we derive the stability and bifurcation direction of Hopf bifurcating periodic orbits by using the normal form theory and the center manifold reduction. Finally, we give some numerical simulations.

**Keywords:** reaction–diffusion, nonlocal delay, Hopf bifurcations, stability.

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## 1 Introduction

The Gompertz equation is one of the models that are often used to describe the dynamics of the populations, including cellular populations of tumour growth, see [18, 26, 28–30, 37]. The basic Gompertz model has the following form

$$\dot{V}(t) = -rV(t)\ln\frac{V(t)}{K}, \quad V(0) = V_0,$$

where $V$ is simply the number of cells/individuals and $K$ is the plateau number of cells/individuals. It was proposed by Benjamin Gompertz in 1825 for the first time (see [18]). Since Laird et al. [30] showed that the Gompertz model could describe the normal growth of an organism such as the guinea pig over an incredible 10000-fold range of the growth in [26], the Gompertz equation is often used in the formulation of equations describing the population dynamics and to describe the inner growth of tumour. In order to better describe the investigated

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phenomena, the time delays are often introduced into models [1–4, 7, 12–17, 31, 33, 34, 36]. Literature [35] introduced the discrete time delay to the classical Gompertz model in different ways and obtained the following four models with delays:

$$\dot{V}(t) = -rV(t)\ln\frac{V(t-\tau)}{K};$$
$$\dot{V}(t) = -rV(t-\tau)\ln\frac{V(t-\tau)}{K};$$
$$\dot{V}(t) = -rV(t-\tau_1)\ln\frac{V(t-\tau_2)}{K};$$

and it also introduced another model with two delays in which it separated two right-hand side terms describing two different processes, namely, the term $r\ln KV(t)$ (with $K \neq 1$) describing the growth of the population and the term $-rV(t)\ln V(t)$ describing the competition between individuals, and by using such biological interpretation, it proposed the model with two delays :

$$\dot{V}(t) = r\ln KV(t-\tau_1) - rV(t-\tau_2)\ln V(t-\tau_2),$$

where $\tau_1$ and $\tau_2$ reflect the delay of growth and competition, respectively. In [35], it showed that the model's dynamics depend crucially on the place where the delay/delays are included. As the placement of delays in the models reflects the delays of different biological processes to their stimuli, so this conclusion is not surprising from the biological point of view. The mathematical and numerical analysis presented in it could help researchers who want to incorporate the Gompertz equation with delays into their models to choose the most appropriate version of the equation.

Moreover, in mathematical biology, many models of population dynamics can be described by the delayed reaction–diffusion equations [6, 8, 9, 20]. In recent years, some researchers [27, 32, 39, 41] have worked on the following reaction–diffusion equations with delay effect:

$$\begin{cases} \dfrac{\partial u}{\partial t} = d\Delta u + uf(u(x,t-\tau), v(x,t-\tau)), \\ \dfrac{\partial v}{\partial t} = d\Delta v + vg(u(x,t-\tau), v(x,t-\tau)). \end{cases}$$

In a reaction–diffusion model with time-delay effect, the individuals which located at $x$ in previous times may not be at the same point in space presently. So the diffusion and time delay are always not independent of each other for a delayed reaction–diffusion model (see References [5, 10, 11, 19, 21, 22, 24, 40]). Thus, it is more reasonable to consider the diffusive type model with nonlocal delay. For instance, Britton [5] introduced the following model:

$$\frac{\partial u(x,t)}{\partial t} = d\Delta u(x,t) + \lambda u(x,t)(1+\alpha u - (1+\alpha) g**u),$$

where

$$g**u = \int_{-\infty}^{t}\int_{\Omega} g(x,y,t-s)u(y,s)dyds,$$

and analyzed the traveling waves on unbounded domain. Then Gourley and Britton [19] proposed a predator–prey system with spatiotemporal delay. In [10], Chen and Yu analyzed the following reaction–diffusion equation with spatiotemporal delay and homogeneous Dirichlet boundary condition:

$$\begin{cases} \dfrac{\partial u}{\partial t} = d\dfrac{\partial^2 u}{\partial x^2} + \lambda uF(u, \displaystyle\int_0^{\infty}\int_0^{\pi} G(x,y,s)f(s)u(y,t-s)dyds), \quad x \in (0,\pi),\ t > 0, \\ u(x,t) = 0, \quad x = 0,\pi,\ t > 0, \end{cases}$$

where

$$G(x,y,t) = \frac{2}{\pi} \sum_{k=1}^{\infty} e^{-dk^2 t} \sin kx \sin ky,$$

and $f(t)$ is the delay kernel, satisfying $f(t) \geq 0$, for $t \geq 0$, and $\int_0^\infty f(t)dt = 1$. It is shown that a positive spatially nonhomogeneous equilibrium can bifurcate from the trivial equilibrium. Moreover, the stability of the bifurcated positive equilibrium was investigated. And they also proved that, for the given spatiotemporal delay, the bifurcated equilibrium is stable under some conditions, and Hopf bifurcation cannot occur. Chen and Yu [11] studied the following general form:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = d\Delta u + \lambda u(x,t) F\left(u(x,t), \displaystyle\int_\Omega K(x,y)u(y,t-\tau)dy\right), & x \in \Omega,\ t > 0, \\ u(x,t) = 0, \quad x \in \partial\Omega,\ t > 0. \end{cases}$$

Guo and Yan [24] investigated the following diffusive Lotka–Volterra type population model with nonlocal delay effect:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = d\Delta u + \lambda u[1 - (A_{11} * u)(x,t-\tau) - (A_{12} * v)(x,t-\tau)], \\ \dfrac{\partial v(x,t)}{\partial t} = d\Delta v + \lambda v[1 - (A_{21} * u)(x,t-\tau) - (A_{22} * v)(x,t-\tau)], \end{cases}$$

for all $x \in \Omega$ and $t > 0$, where $A_{ij}$, $i,j = 1,2$, are kernel functions and

$$(A_{ij} * f)(x,t) = \int_\Omega A_{ij}(x,y) f(y,t) dy, \qquad i,j = 1,2.$$

The existence and multiplicity of spatially nonhomogeneous steady-state solutions are obtained by using Lyapunov–Schmidt reduction. Through analyzing the distribution of eigenvalues of the infinitesimal generator associated with the linearized system, we show the stability of spatially nonhomogeneous steady-state solutions and the existence of Hopf bifurcation with the changes of the time delay. The stability and bifurcation direction of Hopf bifurcating periodic orbits are derived by the normal form theory and the center manifold reduction.

In this paper, we investigate the following diffusive Gompertz population model with nonlocal delay effect:

$$\begin{cases} \dfrac{\partial w(x,t)}{\partial t} = d\Delta w(x,t) \\ \qquad\qquad + \lambda w(x,t)\left(1 - \rho(\lambda) \displaystyle\int_\Omega K(x,y) w(y,t-\tau) \ln w(y,t-\tau) dy\right), \quad x \in \Omega,\ t > 0, \\ w(x,t) = 0, \quad x \in \partial\Omega,\ t > 0, \end{cases}$$

where $w(x,t)$ is the population density at time $t$ and location $x$, $d > 0$ is the diffusion coefficient, $\tau \geq 0$ is the time delay, $\lambda > 0$ is a scaling constant, $\Omega$ is a connected bounded open domain in $\mathbb{R}^n$ $(n \geq 1)$, with a smooth boundary $\partial\Omega$, and Dirichlet boundary condition is imposed so the exterior environment is hostile, $\rho(\lambda)$ is the function of $\lambda$, $K(x,y)$ is a kernel function which describes the dispersal behavior of the population. The nonlocal growth rate per capita incorporates the possible dispersal of the individuals during the maturation period, hence it is a more realistic model.

We first introduce some notations. Denote $\mathbb{X} = H^2(\Omega) \cap H_0^1(\Omega)$, $\mathbb{Y} = L^2(\Omega)$, where $H_0^1(\Omega) = \{u \in H^1(\Omega) \mid u(x) = 0, x \in \partial\Omega\}$. For a space $Z$, we also define the complexification of $Z$ to be $Z_{\mathbb{C}} \triangleq Z \bigoplus iZ = \{x_1 + ix_2 \mid x_1, x_2 \in Z\}$. Denote by $C([-\tau, 0], \mathbb{Y})$ the Banach space of continuous mappings from $[-\tau, 0]$ into $\mathbb{Y}$ equipped with the supremum norm $\|\phi\| = \sup_{-\tau \le \theta \le 0}\{\|\phi(\theta)\|_{\mathbb{Y}}\}$ for $\phi \in C([-\tau, 0], \mathbb{Y})$. For a linear operator $\mathcal{L} : Z_1 \to Z_2$, we denote the domain of $\mathcal{L}$ by $\mathcal{D}(\mathcal{L})$. For the complex-valued Hilbert space $\mathbb{Y}_{\mathbb{C}}^2$, we use the standard inner product $\langle u, v\rangle = \int_\Omega \bar{u}^T(x)v(x)dx$.

Let $\lambda_*$ be the principal eigenvalue of the linear operator $-d\Delta$ subject to the homogeneous Dirichlet boundary condition $w = 0$ on $\partial\Omega$, and let $\phi$ be the corresponding eigenfunction of $\lambda_*$ such that $\phi(x) > 0$ for all $x \in \Omega$.

Throughout the paper, we assume that the kernel function $K(x, y)$ is a continuous and non-negative function on $\bar{\Omega} \times \bar{\Omega}$, and $\int_\Omega K(x, y)\varphi(y)dy > 0$ for all positive continuous functions $\varphi$ on $\Omega$, and $\rho(\lambda) = \lambda - \lambda_*$. When $\rho(\lambda) = \lambda - \lambda_*$, the above model becomes

$$\begin{cases} \dfrac{\partial w(x,t)}{\partial t} = d\Delta w(x,t) + \lambda w(x,t) \\ \qquad\qquad \times \left(1 - (\lambda - \lambda_*) \displaystyle\int_\Omega K(x,y) w(y, t-\tau) \ln w(y, t-\tau) dy\right), \quad x \in \Omega,\ t > 0, \\ w(x,t) = 0, \quad x \in \partial\Omega,\ t > 0. \end{cases} \tag{1.1}$$

We consider system (1.1) with the following initial condition:

$$w(x, s) = \eta(x, s), \qquad x \in \Omega,\ s \in [-\tau, 0], \tag{1.2}$$

where $\eta \in C([-\tau, 0], \mathbb{Y})$. From [25], we know that the operator $d\Delta$ generates an analytic strongly positive semigroup $T(t)$ on $\mathbb{Y}$ with the domain $\mathcal{D}(d\Delta) = \mathbb{X}$.

The rest of the paper is organized as follows. In Section 2, we study the existence of the positive spatially nonhomogeneous equilibrium of system (1.1). In Section 3, we consider the eigenvalue problems. In Section 4, we show the stability of the bifurcated positive equilibrium and the occurrence of Hopf bifurcation. In Section 5, the direction of the Hopf bifurcation is given by using normal form theorem and the center manifold theorem. Some numerical simulations are given in Section 6.

## 2 The existence of the positive spatially nonhomogeneous equilibrium

In this section, we study the existence of the spatially nonhomogeneous positive steady state solutions of system (1.1), which satisfies the following boundary value problem:

$$\begin{cases} d\Delta w(x) + \lambda w(x) \left(1 - (\lambda - \lambda_*) \displaystyle\int_\Omega K(x,y) w(y) \ln w(y) dy\right) = 0, \quad x \in \Omega, \\ w(x) = 0, \quad x \in \partial\Omega. \end{cases} \tag{2.1}$$

Firstly, we have the following decompositions:

$$\mathbb{X} = \mathcal{N}(d\Delta + \lambda_*) \oplus \mathbb{X}_1,$$
$$\mathbb{Y} = \mathcal{N}(d\Delta + \lambda_*) \oplus \mathbb{Y}_1,$$

where

$$\mathcal{N}(d\Delta+\lambda_*) = \text{span}\{\phi\},$$

$$\mathbb{X}_1 = \left\{\psi \in \mathbb{X} \mid \int_\Omega \phi(x)\psi(x)dx = 0\right\},$$
$$\mathbb{Y}_1 = \left\{\psi \in \mathbb{Y} \mid \int_\Omega \phi(x)\psi(x)dx = 0\right\}.$$

Then we can obtain the following theorem about the existence of the positive equilibrium solutions of Eq. (2.1) by using the implicit function theorem.

**Theorem 2.1.** *There exist $\lambda^* > \lambda_*$ and a continuously differential mapping $\lambda \to (\xi_\lambda, \beta_\lambda)$ from $[\lambda_*, \lambda^*]$ to $\mathbb{X}_1 \times \mathbb{R}^+$, such that (1.1) has an equilibrium solution*

$$w_\lambda = \beta_\lambda[\phi + (\lambda-\lambda_*)\xi_\lambda], \tag{2.2}$$

*where $\beta_{\lambda_*} > 0$ satisfies*

$$\lambda_*\beta \int_\Omega\int_\Omega K(x,y)\phi^2(x)\phi(y)\ln(\beta\phi(y))dxdy = \int_\Omega \phi^2(x)dx, \tag{2.3}$$

*and $\xi_{\lambda_*}$ is the unique solution of the equation*

$$(d\Delta+\lambda_*)\xi + \phi\left[1-\lambda_*\beta_{\lambda_*}\int_\Omega K(x,y)\phi(y)\ln(\beta_{\lambda_*}\phi(y))dy\right] = 0. \tag{2.4}$$

*Proof.* Since $d\Delta+\lambda_*$ is bijective from $\mathbb{X}_1$ to $\mathbb{Y}_1$ and $\phi\left[1-\lambda_*\beta_{\lambda_*}\int_\Omega K(x,y)\phi(y)(\ln\beta_{\lambda_*}\phi(y))dy\right] \in \mathbb{Y}_1$, we have $\xi_{\lambda_*}$ is well defined.

Next, we prove $w_\lambda$ is the solution to (2.1). Define $g : \mathbb{X}_1 \times \mathbb{R} \times \mathbb{R} \to \mathbb{Y}$ by

$$\begin{aligned} g(\xi,\beta,\lambda) = {}& (d\Delta+\lambda_*)\xi + \phi + (\lambda-\lambda_*)\xi \\ & - \lambda\beta[\phi+(\lambda-\lambda_*)\xi]\int_\Omega K(x,y)[\phi+(\lambda-\lambda_*)\xi]\ln\beta[\phi+(\lambda-\lambda_*)\xi]dy. \end{aligned}$$

From Eqs. (2.3) and (2.4), we see that $g(\xi_{\lambda_*}, \beta_{\lambda_*}, \lambda_*) = 0$, and

$$D_{(\xi,\beta)}g(\xi_{\lambda_*}, \beta_{\lambda_*}, \lambda_*)[\gamma,\epsilon] = (d\Delta+\lambda_*)\gamma - \lambda_*\phi\int_\Omega K(x,y)\phi(y)(\ln\beta_{\lambda_*}\phi(y)+1)dy\epsilon.$$

Here $D_{(\xi,\beta)}g(\xi_{\lambda_*}, \beta_{\lambda_*}, \lambda_*)[\gamma,\epsilon]$ is the *Fréchet* derivative of $g$ with respect to $(\xi,\beta)$.
As

$$\int_\Omega\int_\Omega K(x,y)\phi^2(x)\phi(y)(\ln\beta_{\lambda_*}\phi(y)+1)dxdy \neq 0,$$

we have

$$\phi\int_\Omega K(x,y)\phi(y)(\ln\beta_{\lambda_*}\phi(y)+1)dy \notin \mathbb{Y}_1.$$

So $D_{(\xi,\beta)}g(\xi_{\lambda_*}, \beta_{\lambda_*}, \lambda_*)$ is bijective from $\mathbb{X}_1 \times \mathbb{R}$ to $\mathbb{Y}$. Then from the implicit function theorem, there exist a $\lambda^* > \lambda_*$ and a continuously differentiable mapping $\lambda \mapsto (\xi_\lambda, \beta_\lambda) \in \mathbb{X}_1 \times \mathbb{R}^+$ such that

$$g(\xi_\lambda, \beta_\lambda, \lambda) = 0, \qquad \lambda \in [\lambda_*, \lambda^*],$$

which implies that $w_\lambda$ solves Eq. (2.1). $\square$

## 3 Eigenvalue problems

Let $\lambda \in (\lambda_*, \lambda^*]$, and $w_\lambda$ be the positive equilibrium solution of (2.1) obtained in Theorem 2.1. Linearizing system (2.1) at $w_\lambda$, we have

$$\begin{cases} \dfrac{\partial v(x,t)}{\partial t} = d\Delta v(x,t) + \lambda \left(1 - (\lambda - \lambda_*) \displaystyle\int_\Omega K(x,y) w_\lambda(y) \ln w_\lambda(y) dy\right) v(x,t) \\ \qquad\qquad - \lambda(\lambda - \lambda_*) w_\lambda(x) \displaystyle\int_\Omega K(x,y)(\ln w_\lambda(y) + 1) v(y, t-\tau) dy, \quad x \in \Omega,\ t > 0, \\ v(x,t) = 0, \quad x \in \partial\Omega,\ t > 0, \end{cases} \tag{3.1}$$

Define a linear operator $B(\lambda) : \mathcal{D}(B(\lambda)) \to \mathbb{Y}$ by

$$B(\lambda) = d\Delta + \lambda \left(1 - (\lambda - \lambda_*) \int_\Omega K(x,y) w_\lambda(y) \ln w_\lambda(y) dy\right),$$

with domain $\mathcal{D}(B(\lambda)) = \mathbb{X}$. From [38], the semigroup induced by the solutions of (3.1) has the infinitesimal generator $B_\tau(\lambda)$ given by

$$B_\tau(\lambda)\varphi = \dot{\varphi}, \tag{3.2}$$

where

$$\mathcal{D}(B_\tau(\lambda)) = \Big\{ \varphi \in C_{\mathbb{C}} \cap C^1_{\mathbb{C}} : \varphi(0) \in \mathbb{X}_{\mathbb{C}},$$
$$\dot{\varphi} = B(\lambda)\varphi(0) - \lambda(\lambda - \lambda_*) w_\lambda \int_\Omega K(\cdot, y)(\ln w_\lambda(y) + 1)\varphi(-\tau)(y) dy \Big\},$$

and $C^1_{\mathbb{C}} = C^1([-\tau, 0], \mathbb{Y}_{\mathbb{C}})$.

The spectral set of $B_\tau(\lambda)$ is

$$\sigma(B_\tau(\lambda)) = \{\mu \in \mathbb{C} : \Delta(\lambda, \mu, \tau)\psi = 0, \text{ for some } \psi \in \mathbb{X}_{\mathbb{C}\setminus\{0\}}\},$$

where

$$\Delta(\lambda, \mu, \tau)\psi := B(\lambda)\psi - \lambda(\lambda - \lambda_*) w_\lambda \int_\Omega K(\cdot, y)(\ln w_\lambda(y) + 1)\psi(y) dy e^{-\mu\tau} - \mu\psi.$$

Then $B_\tau(\lambda)$ has a purely imaginary eigenvalue $\mu = i\omega (\omega \neq 0)$ for some $\tau \geq 0$ if and only if

$$B(\lambda)\psi - \lambda(\lambda - \lambda_*) w_\lambda \int_\Omega K(\cdot, y)(\ln w_\lambda(y) + 1)\psi(y) dy e^{-i\theta} - i\omega\psi = 0. \tag{3.3}$$

is solvable for some $\omega > 0$, $\psi \neq 0$ and $\theta \in [0, 2\pi)$. So if there exists a pair $(\omega, \theta)$ such that (3.3) has a solution $\psi$, then

$$\Delta(\lambda, i\omega, \tau_n)\psi = 0, \qquad \tau_n = \frac{\theta + 2n\pi}{\omega}, \qquad n = 0, 1, 2, \ldots$$

Next, we will show that for $\lambda \in (\lambda_*, \lambda^*]$, there exists a unique pair $(\omega, \theta)$ which solves (3.3). Assume that $(\omega, \theta, \psi)$ is a solution of (3.3) with $\psi(\neq 0) \in \mathbb{X}_{\mathbb{C}}$. Ignoring a scalar factor, $\psi$ can be represented as

$$\begin{aligned} \psi &= \alpha\phi + (\lambda - \lambda_*) z, \qquad \langle \phi, z \rangle = 0,\ \alpha \geq 0, \\ \|\psi\|^2_{\mathbb{Y}_{\mathbb{C}}} &= \alpha^2 \|\phi\|^2_{\mathbb{Y}_{\mathbb{C}}} + (\lambda - \lambda_*)^2 \|z\|^2_{\mathbb{Y}_{\mathbb{C}}} = \|\phi\|^2_{\mathbb{Y}_{\mathbb{C}}}. \end{aligned} \tag{3.4}$$

Substituting (2.2), (3.4) and $\omega = (\lambda - \lambda_*)h$ into (3.3), we obtain the following equation equivalent to (3.3):

$$\begin{cases} f_1(z,\alpha,h,\theta,\lambda) := (d\Delta + \lambda_*)z \\ \qquad\qquad + [\alpha\phi + (\lambda-\lambda_*)z]\left(1 - \lambda \int_\Omega K(\cdot,y) w_\lambda(y) \ln w_\lambda(y) dy - ih\right) \\ \qquad\qquad - \lambda\beta_\lambda[\phi + (\lambda-\lambda_*)\xi_\lambda] \\ \qquad\qquad \times \int_\Omega K(\cdot,y)[\alpha\phi + (\lambda-\lambda_*)z](\ln\beta_\lambda[\phi+(\lambda-\lambda_*)\xi_\lambda]+1)dy e^{-i\theta}, \\ f_2(z,\alpha,\lambda) := (\alpha^2-1)\|\phi\|^2_{\mathbb{Y}_\mathbb{C}} + (\lambda-\lambda_*)^2\|z\|^2_{\mathbb{Y}_\mathbb{C}}. \end{cases} \tag{3.5}$$

Note that when $\lambda = \lambda_*$,

$$f_2(z,\alpha,\lambda) = 0 \Leftrightarrow \alpha = \alpha_{\lambda_*} = 1.$$

We have

$$\begin{aligned} f_1(z,\alpha_{\lambda_*},h,\theta,\lambda_*) = (d\Delta+\lambda_*)z + \phi\left(1 - \lambda_*\beta_{\lambda_*}\int_\Omega K(\cdot,y)\phi(y)\ln\beta_{\lambda_*}\phi(y)dy - ih\right) \\ - \lambda_*\beta_{\lambda_*}\phi\int_\Omega K(\cdot,y)\phi(y)(\ln\beta_{\lambda_*}\phi(y)+1)dy e^{-i\theta}. \end{aligned}$$

Hence

$$f_1(z,\alpha_{\lambda_*},h,\theta,\lambda_*) = 0,$$

is solvable for some value of $z \in (\mathbb{X}_1)_\mathbb{C}$, $h \geq 0$ and $\theta \in [0,2\pi)$ if and only if there exists a pair $(h,\theta)$ with $h \geq 0$ and $\theta \in [0,2\pi)$ satisfying

$$\begin{cases} \lambda_*\beta_{\lambda_*}\int_\Omega\int_\Omega K(x,y)\phi^2(x)\phi(y)(\ln\beta_{\lambda_*}\phi(y)+1)dxdy\cos\theta \\ \qquad = \int_\Omega \phi^2(x)dx - \lambda_*\beta_{\lambda_*}\int_\Omega\int_\Omega K(x,y)\phi^2(x)\phi(y)\ln\beta_{\lambda_*}\phi(y)dy, \\ h\int_\Omega\phi^2(x)dx = \lambda_*\beta_{\lambda_*}\int_\Omega\int_\Omega K(x,y)\phi^2(x)\phi(y)(\ln\beta_{\lambda_*}\phi(y)+1)dxdy\sin\theta. \end{cases}$$

Solving the above equation, we have

$$\theta_{\lambda_*} = \arccos\frac{\int_\Omega\phi^2(x)dx - \lambda_*\beta_{\lambda_*}\int_\Omega\int_\Omega K(x,y)\phi^2(x)\phi(y)\ln\beta_{\lambda_*}\phi(y)dy}{\lambda_*\beta_{\lambda_*}\int_\Omega\int_\Omega K(x,y)\phi^2(x)\phi(y)(\ln\beta_{\lambda_*}\phi(y)+1)dxdy},$$

$$h_{\lambda_*} = \frac{\lambda_*\beta_{\lambda_*}\int_\Omega\int_\Omega K(x,y)\phi^2(x)\phi(y)(\ln\beta_{\lambda_*}\phi(y)+1)dxdy\sin\theta_{\lambda_*}}{\int_\Omega\phi^2(x)dx},$$

and $z_{\lambda_*} \in (\mathbb{X}_1)_\mathbb{C}$ is the unique solution of the following equation

$$\begin{aligned} (d\Delta+\lambda_*)z + \phi\left(1 - \lambda_*\beta_{\lambda_*}\int_\Omega K(\cdot,y)\phi(y)\ln\beta_{\lambda_*}\phi(y)dy - ih_{\lambda_*}\right) \\ - \lambda_*\beta_{\lambda_*}\phi\int_\Omega K(\cdot,y)\phi(y)(\ln\beta_{\lambda_*}\phi(y)+1)dy e^{-i\theta_{\lambda_*}} = 0. \end{aligned}$$

Define $F : (\mathbb{X}_1)_\mathbb{C} \times \mathbb{R}^3 \times \mathbb{R} \to \mathbb{Y}_\mathbb{C} \times \mathbb{R}$ by $F = (f_1, f_2)$. Then we have the following theorem on the solvability of $F = 0$.

**Theorem 3.1.** *There exists a continuously differentiable mapping $\lambda \mapsto (z_\lambda, \alpha_\lambda, h_\lambda, \theta_\lambda)$ from $[\lambda_*, \lambda^*]$ to $\mathbb{X}_{\mathbb{C}} \times \mathbb{R}^3$ such that $F(z_\lambda, \alpha_\lambda, h_\lambda, \theta_\lambda, \lambda) = 0$. Moreover,*

$$\begin{cases} F(z, \alpha, h, \theta, \lambda) = 0, \\ \alpha, h \geq 0, \ \theta \in [0, 2\pi). \end{cases}$$

*has a unique solution $(z_\lambda, \alpha_\lambda, h_\lambda, \theta_\lambda)$.*

The proof is similar to Theorem 2.5 of [8] and we omit it here.

To summarise, we have the following result about the eigenvalue problem.

**Corollary 3.2.** *For $\lambda \in (\lambda_*, \lambda^*]$, the eigenvalue problem*

$$\Delta(\lambda, i\omega, \tau)\psi = 0, \qquad \omega \geq 0, \ \tau \geq 0, \ \psi(\neq 0) \in \mathbb{X}_{\mathbb{C}},$$

*has a solution if and only if*

$$\omega = \omega_\lambda = (\lambda - \lambda_*)h_\lambda, \qquad \tau = \tau_n = \frac{\theta_\lambda + 2n\pi}{\omega_\lambda}, \qquad n = 0, 1, 2, \ldots, \tag{3.6}$$

*and*

$$\psi = c\psi_\lambda, \qquad \psi_\lambda = \alpha_\lambda \phi + (\lambda - \lambda_*) z_\lambda,$$

*where $c$ is a nonzero constant, and $z_\lambda$, $\alpha_\lambda$, $h_\lambda$, $\theta_\lambda$ are defined as in Theorem 3.1.*

Next, we consider the adjoint operator of $B_\tau(\lambda)$ for later application. Similar as in [8], we see that the adjoint operator is

$$\tilde{\Delta}(\lambda, i\omega, \tau)\tilde{\psi} = B(\lambda)\tilde{\psi} - \lambda(\lambda - \lambda_*) \int_\Omega K(y, \cdot) w_\lambda(y)(\ln w_\lambda(x) + 1)\tilde{\psi}(y) dy e^{i\omega\tau} + i\omega\tilde{\psi},$$

which satisfies

$$\langle \tilde{\psi}, \Delta(\lambda, i\omega, \tau)\psi \rangle = \langle \tilde{\Delta}(\lambda, i\omega, \tau)\tilde{\psi}, \psi \rangle,$$

Its point spectrum is the same as that of $\Delta(\lambda, i\omega, \tau)$:

$$\sigma_p(\Delta(\lambda, i\omega, \tau)) = \sigma_p(\tilde{\Delta}(\lambda, i\omega, \tau)).$$

We conclude that if the corresponding adjoint equation

$$B(\lambda)\tilde{\psi} - \lambda(\lambda - \lambda_*) \int_\Omega K(y, \cdot) w_\lambda(y)(\ln w_\lambda(x) + 1)\tilde{\psi}(y) dy e^{i\tilde{\theta}} + i\tilde{\omega}\tilde{\psi} = 0, \quad \tilde{\psi}(\neq 0) \in \mathbb{X}_{\mathbb{C}}, \tag{3.7}$$

is solvable for some value of $\tilde{\omega} > 0$, $\tilde{\theta} \in [0, 2\pi)$, then

$$\tilde{\Delta}(\lambda, i\tilde{\omega}, \tilde{\tau}_n)\tilde{\psi} = 0, \qquad \tilde{\tau}_n = \frac{\tilde{\theta} + 2n\pi}{\tilde{\omega}}, \qquad n = 0, 1, 2, \ldots$$

Similarly, for $\lambda \in (\lambda_*, \lambda^*]$, there is a unique $(\tilde{\omega}, \tilde{\theta}, \tilde{\psi})$ which is the solution to (3.7), $\tilde{\psi}(\neq 0) \in \mathbb{X}_{\mathbb{C}}$. $\tilde{\psi}$ can be represented as

$$\begin{aligned} &\tilde{\psi} = \tilde{\alpha}\phi + (\lambda - \lambda_*)\tilde{z}, \quad \langle \phi, \tilde{z} \rangle = 0, \quad \tilde{\alpha} \geq 0, \\ &\|\tilde{\psi}\|^2_{\mathbb{Y}_C} = \tilde{\alpha}^2 \|\phi\|^2_{\mathbb{Y}_C} + (\lambda - \lambda_*)^2 \|\tilde{z}\|^2_{\mathbb{Y}_C} = \|\phi\|^2_{\mathbb{Y}_C}. \end{aligned} \tag{3.8}$$

Substituting (3.8) and $\tilde{\omega} = (\lambda - \lambda_*)\tilde{h}$ into (3.7), we obtain the following equation equivalent to (3.7):

$$\begin{cases} \tilde{f}_1(\tilde{z}, \tilde{\alpha}, \tilde{h}, \tilde{\theta}, \lambda) := (d\Delta + \lambda_*)\tilde{z} + [\tilde{\alpha}\phi + (\lambda - \lambda_*)\tilde{z}] \\ \qquad \times \left(1 - \lambda \int_\Omega K(x,y) w_\lambda(y) \ln w_\lambda(x) dy + i\tilde{h}\right) \\ \qquad - \lambda\beta_\lambda \int_\Omega K(y,x)[\phi(y) + (\lambda - \lambda_*)\xi_\lambda(y)][\tilde{\alpha}\phi + (\lambda - \lambda_*)\tilde{z}] \\ \qquad \times (\ln \beta_\lambda[\phi(x) + (\lambda - \lambda_*)\xi_\lambda(x)] + 1) dy e^{i\tilde{\theta}}, \\ \tilde{f}_2(\tilde{z}, \tilde{\alpha}, \lambda) := (\tilde{\alpha}^2 - 1)\|\phi\|^2_{\mathbb{Y}_\mathbb{C}} + (\lambda - \lambda_*)^2 \|\tilde{z}\|^2_{\mathbb{Y}_\mathbb{C}}. \end{cases} \tag{3.9}$$

Similarly to (3.5), we obtain

$$\tilde{\alpha}_{\lambda_*} = 1,$$

$$\tilde{\theta}_{\lambda_*} = \arccos \frac{\int_\Omega \phi^2(x)dx - \lambda_*\beta_{\lambda_*} \int_\Omega \int_\Omega K(x,y)\phi^2(x)\phi(y) \ln \beta_{\lambda_*}\phi(y) dy}{\lambda_*\beta_{\lambda_*} \int_\Omega \int_\Omega K(y,x)\phi(x)\phi^2(y)(\ln \beta_{\lambda_*}\phi(x) + 1) dx dy},$$

$$\tilde{h}_{\lambda_*} = \frac{\lambda_*\beta_{\lambda_*} \int_\Omega \int_\Omega K(y,x)\phi(x)\phi^2(y)(\ln \beta_{\lambda_*}\phi(x) + 1) dx dy \sin \tilde{\theta}_{\lambda_*}}{\int_\Omega \phi^2(x) dx},$$

and $\tilde{z}_{\lambda_*} \in (\mathbb{X}_1)_\mathbb{C}$ is the unique solution of the following equation

$$(d\Delta + \lambda_*)\tilde{z} + \phi\left(1 - \lambda_*\beta_{\lambda_*} \int_\Omega K(\cdot, y)\phi(y) \ln \beta_{\lambda_*}\phi(y) dy + i\tilde{h}_{\lambda_*}\right) \\ - \lambda_*\beta_{\lambda_*}\phi \int_\Omega K(y, \cdot)\phi(y)(\ln \beta_{\lambda_*}\phi(x) + 1) dy e^{-i\theta_{\lambda_*}} = 0.$$

Define $\tilde{F} : (\mathbb{X}_1)_\mathbb{C} \times \mathbb{R}^3 \times \mathbb{R} \to \mathbb{Y}_\mathbb{C} \times \mathbb{R}$ by $\tilde{F} = (\tilde{f}_1, \tilde{f}_2)$. Then we have the following result which can be proved similarly as in Theorem 3.1 and Corollary 3.2.

**Theorem 3.3.**

(1) *There exists a continuously differentiable mapping $\lambda \mapsto (\tilde{z}_\lambda, \tilde{\alpha}_\lambda, \tilde{h}_\lambda, \tilde{\theta}_\lambda)$ from $[\lambda_*, \lambda^*]$ to $\mathbb{X}_\mathbb{C} \times \mathbb{R}^3$ such that $\tilde{F}(\tilde{z}_\lambda, \tilde{\alpha}_\lambda, \tilde{h}_\lambda, \tilde{\theta}_\lambda, \lambda) = 0$. Moreover,*

$$\begin{cases} \tilde{F}(z, \alpha, h, \theta, \lambda) = 0, \\ \alpha, h \geq 0, \ \theta \in [0, 2\pi). \end{cases}$$

*has a unique solution $(\tilde{z}_\lambda, \tilde{\alpha}_\lambda, \tilde{h}_\lambda, \tilde{\theta}_\lambda)$.*

(2) *For $\lambda \in (\lambda_*, \lambda^*]$, the eigenvalue problem*

$$\tilde{\Delta}(\lambda, i\tilde{\omega}, \tilde{\tau})\tilde{\psi} = 0, \qquad \tilde{\omega} \geq 0, \ \tilde{\tau} \geq 0, \ \tilde{\psi}(\neq 0) \in \mathbb{X}_\mathbb{C},$$

*has a solution if and only if*

$$\tilde{\omega} = \tilde{\omega}_\lambda = (\lambda - \lambda_*)\tilde{h}_\lambda, \qquad \tilde{\tau} = \tilde{\tau}_n = \frac{\tilde{\theta}_\lambda + 2n\pi}{\tilde{\omega}_\lambda}, \qquad n = 0, 1, 2, \ldots \tag{3.10}$$

*and*

$$\tilde{\psi} = c\tilde{\psi}_\lambda, \qquad \tilde{\psi}_\lambda = \tilde{\alpha}_\lambda\phi + (\lambda - \lambda_*)\tilde{z}_\lambda,$$

*where $c$ is a nonzero constant.*

**Remark 3.4.** From the above discussion, we can see that $h_\lambda = \tilde{h}_\lambda$ and $\theta_\lambda = \tilde{\theta}_\lambda$, and consequently $\omega_\lambda = \tilde{\omega}_\lambda$ and $\tau_n = \tilde{\tau}_n$. Therefore, in the following, we will only use $(h_\lambda, \theta_\lambda, \omega_\lambda, \tau_n)$ and not the ones with tilde. But the corresponding eigenfunctions of $\Delta(\lambda, i\omega_\lambda, \tau_n)$ may be different from the ones for the adjoint operator $\tilde{\Delta}(\lambda, i\omega_\lambda, \tau_n)$.

# 4 Stability and Hopf bifurcations

In this section, we first give the stability of the positive equilibrium $w_\lambda$ of (1.1) when $\tau = 0$ and then discuss the existence of Hopf bifurcations.

**Proposition 4.1.** *For each $\lambda \in (\lambda_*, \lambda^*]$, all the eigenvalues of $B_\tau(\lambda)$ have negative real parts when $\tau = 0$, therefore the positive equilibrium $w_\lambda$ of (1.1) is locally asymptotically stable when $\tau = 0$.*

*Proof.* Otherwise, there exists a sequence $\{\lambda_n\}_{n=1}^{\infty}$, such that $\lambda_n > \lambda_*$ for $n \geq 1$, $\lim_{n\to\infty} \lambda_n = \lambda_*$, and for $n \geq 1$, the corresponding eigenvalue problem

$$\begin{cases} B(\lambda_n)\psi - \lambda_n(\lambda_n - \lambda_*)w_{\lambda_n} \int_\Omega K(\cdot, y)(\ln w_{\lambda_n}(y) + 1)\psi(y)dy = \mu\psi, & x \in \Omega, \\ \psi(x) = 0, & x \in \partial\Omega, \end{cases} \tag{4.1}$$

has an eigenvalue $\mu_{\lambda_n} \geq 0$, $\mathrm{Re}\mu_{\lambda_n} \geq 0$ and the eigenfunction $\psi_{\lambda_n}$, $\|\psi_{\lambda_n}\| = 1$.

For $n \geq 1$, we write $\psi_{\lambda_n}$ as $\psi_{\lambda_n} = c_{\lambda_n} w_{\lambda_n} + \phi_{\lambda_n}$ , where $c_{\lambda_n} \in \mathbb{C}$ and $c_{\lambda_n} = \langle w_{\lambda_n}, \psi_{\lambda_n}\rangle / \langle w_{\lambda_n}, w_{\lambda_n}\rangle$. $w_{\lambda_n}$ is the positive solution of (1.1) when $\lambda = \lambda_n$, and $\phi_{\lambda_n} \in \mathbb{X}_{\mathbb{C}}$ satisfies $\langle \phi_{\lambda_n}, w_{\lambda_n}\rangle = 0$. If $\phi_{\lambda_n} \equiv 0$, then we substitute $\psi_{\lambda_n} = c_{\lambda_n} w_{\lambda_n}$ and $\mu = \mu_{\lambda_n}$ into the first equation of (4.1) and obtain

$$-\mu_{\lambda_n} w_{\lambda_n} = \lambda_n(\lambda_n - \lambda_*)w_{\lambda_n} \int_\Omega K(\cdot, y)(\ln w_{\lambda_n}(y) + 1)w_{\lambda_n}(y)dy,$$

which is a contradiction. Hence $\phi_{\lambda_n} \not\equiv 0$ for each $n \geq 1$. Since

$$\langle B(\lambda_n)\phi_{\lambda_n}, w_{\lambda_n}\rangle = \langle \phi_{\lambda_n}, B(\lambda_n)w_{\lambda_n}\rangle, \qquad B(\lambda_n)w_{\lambda_n} = 0,$$

multiplying the first equation of (4.1) by $\psi_{\lambda_n} = c_{\lambda_n} w_{\lambda_n} + \phi_{\lambda_n}$ when $\mu = \mu_{\lambda_n}$, we can get

$$\langle B(\lambda_n)\phi_n, \phi_{\lambda_n}\rangle = \lambda_n(\lambda_n - \lambda_*)\left\langle w_{\lambda_n} \int_\Omega K(\cdot, y)(\ln w_{\lambda_n}(y) + 1)\psi_{\lambda_n}(y)dy,\ \psi_{\lambda_n}\right\rangle + \mu_{\lambda_n}. \tag{4.2}$$

As $w_{\lambda_n}$ is the principal eigenfunction of $B_{\lambda_n}$ with principal eigenvalue 0, so $\langle B_{\lambda_n}\phi_{\lambda_n}, \phi_{\lambda_n}\rangle < 0$. Then

$$0 \leq \mathrm{Re}(\mu_{\lambda_n}) \leq \mathrm{Re}\left[ - \lambda_n(\lambda_n - \lambda_*)\left\langle w_{\lambda_n} \int_\Omega K(\cdot, y)(\ln w_{\lambda_n}(y) + 1)\psi_{\lambda_n}(y)dy,\ \psi_{\lambda_n}\right\rangle\right] \to 0,$$

as $n \to \infty$, hence $\lim_{n\to\infty} \mathrm{Re}(\mu_{\lambda_n}) = 0$.

From (4.2), we have

$$|\mathrm{Im}(\mu_{\lambda_n})| = \left|\mathrm{Im}\left[ - \lambda_n(\lambda_n - \lambda_*)\left\langle w_{\lambda_n} \int_\Omega K(\cdot, y)(\ln w_{\lambda_n}(y) + 1)\psi_{\lambda_n}(y)dy,\ \psi_{\lambda_n}\right\rangle\right]\right| \to 0,$$

as $n \to \infty$. Similar to the proof of Lemma 2.3 of [8], we get

$$|\lambda_2(\lambda_n)| \cdot \|\phi_{\lambda_n}\|^2_{\mathbb{Y}_{\mathbb{C}}} \leq |\langle B(\lambda_n)\phi_{\lambda_n}, \phi_{\lambda_n}\rangle|, \tag{4.3}$$

where $\lambda_2(\lambda_n)$ is the second eigenvalue of $B(\lambda_n)$.
Then

$$|\lambda_2(\lambda_n)|\cdot\|\phi_{\lambda_n}\|^2 \le \left|\lambda_n(\lambda_n-\lambda_*)\left\langle w_{\lambda_n}\int_\Omega K(\cdot,y)(\ln w_{\lambda_n}(y)+1)\psi_{\lambda_n}(y)dy,\ \psi_{\lambda_n}\right\rangle\right| + |\mu_{\lambda_n}|.$$

Since

$$\lambda_2(\lambda_n) = \lambda_2 - \lambda_* > 0,$$

so $\lim_{n\to\infty}\|\phi_{\lambda_n}\|_{\mathbb{Y}_\mathbb{C}} = 0$.

Denote $E_{\lambda_n} = \lambda_n(\lambda_n-\lambda_*)\langle w_{\lambda_n}\int_\Omega K(\cdot,y)(\ln w_{\lambda_n}(y)+1)\psi_{\lambda_n}(y)dy, \psi_{\lambda_n}\rangle$, then

$$\begin{aligned}
E_{\lambda_n} &= \lambda_n(\lambda_n-\lambda_*)|c_{\lambda_n}|^2\left\langle w_{\lambda_n}\int_\Omega K(\cdot,y)(\ln w_{\lambda_n}(y)+1)w_{\lambda_n}(y)dy,\ w_{\lambda_n}\right\rangle\\
&\quad+\lambda_n(\lambda_n-\lambda_*)c_{\lambda_n}\left\langle w_{\lambda_n}\int_\Omega K(\cdot,y)(\ln w_{\lambda_n}(y)+1)w_{\lambda_n}(y)dy,\ \phi_{\lambda_n}\right\rangle\\
&\quad+\lambda_n(\lambda_n-\lambda_*)c_{\lambda_n}\left\langle w_{\lambda_n}\int_\Omega K(\cdot,y)(\ln w_{\lambda_n}(y)+1)\phi_{\lambda_n}(y)dy,\ w_{\lambda_n}\right\rangle\\
&\quad+\lambda_n(\lambda_n-\lambda_*)\left\langle w_{\lambda_n}\int_\Omega K(\cdot,y)(\ln w_{\lambda_n}(y)+1)\phi_{\lambda_n}(y)dy,\ \phi_{\lambda_n}\right\rangle.
\end{aligned}$$

Since

$$\begin{aligned}
&\lim_{n\to\infty}\left\langle w_{\lambda_n}\int_\Omega K(\cdot,y)(\ln w_{\lambda_n}(y)+1)w_{\lambda_n}(y)dy,\ w_{\lambda_n}\right\rangle\\
&\quad=\beta_{\lambda_*}^3\int_\Omega\int_\Omega K(x,y)\phi^2(x)\phi(y)(\ln\beta_{\lambda_*}\phi(y)+1)dxdy>0,
\end{aligned}$$

and $\lim_{n\to\infty}\|\phi_{\lambda_n}\|_{\mathbb{Y}_\mathbb{C}}=0$, then there exists $N_*\in\mathbb{N}$ such that for each $n\ge N_*$, $\mathrm{Re}(E_{\lambda_n})>0$, which implies that

$$\mathrm{Re}(\mu_{\lambda_n}) = \langle B(\lambda_n)\phi_{\lambda_n},\phi_{\lambda_n}\rangle - \mathrm{Re}(E_{\lambda_n}) < 0.$$

This is a contradiction with $\mathrm{Re}(\mu_{\lambda_n})\ge 0$ for $n\ge 1$. So all the eigenvalues of $B_\tau(\lambda)$ have negative real parts when $\tau=0$. □

**Theorem 4.2.** *Assume that $\lambda\in(\lambda_*,\lambda^*]$, then $\mu=i\omega_\lambda$ is a simple eigenvalue of $B_{\tau_n}$ for $n=0,1,2,\dots$*

*Proof.* Suppose that there exists $\phi_1\in\mathcal{D}(B_{\tau_n})\cap\mathcal{D}([B_{\tau_n}]^2)$ such that $[B_{\tau_n}(\lambda)-i\omega_\lambda]^2\phi_1=0$, then

$$[B_{\tau_n}(\lambda)-i\omega_\lambda]\phi_1\in\mathcal{N}[B_{\tau_n}(\lambda)-i\omega_\lambda]=\mathrm{Span}\{e^{i\omega_\lambda\cdot}\psi_\lambda\}.$$

So there exists a constant a such that

$$[B_{\tau_n}(\lambda)-i\omega_\lambda]\phi_1 = ae^{i\omega_\lambda\cdot}\psi_\lambda.$$

Hence

$$\begin{aligned}
&\dot\phi_1(\theta) = i\omega_\lambda\phi_1(\theta)+ae^{i\omega_\lambda\theta}\psi_\lambda,\quad \theta\in[-\tau_n,0],\\
&\dot\phi_1(0) = B(\lambda)\phi_1(0)-\lambda(\lambda-\lambda_*)w_\lambda\int_\Omega K(\cdot,y)(\ln w_\lambda(y)+1)\phi_1(-\tau_n)(y)dy.
\end{aligned}\tag{4.4}$$

From the first equation of (4.4), we have

$$\begin{aligned}&\phi_1(\theta) = \phi_1(0)e^{i\omega_\lambda\theta} + a\theta e^{i\omega_\lambda\theta}\psi_\lambda,\\&\dot{\phi}_1(0) = i\omega_\lambda\phi_1(0) + a\psi_\lambda.\end{aligned} \tag{4.5}$$

Then from Eqs. (4.4) and (4.5), we can obtain

$$\begin{aligned}\Delta(\lambda, i\omega_\lambda, \tau_n)\phi_1(0) &= [B(\lambda) - i\omega_\lambda]\phi_1(0) - \lambda(\lambda-\lambda_*)w_\lambda \int_\Omega K(\cdot,y)(\ln w_\lambda(y)+1)\phi_1(0)(y)dye^{-i\theta_\lambda}\\&= a\psi_\lambda + \lambda(\lambda-\lambda_*)w_\lambda\int_\Omega K(\cdot,y)(\ln w_\lambda(y)+1)\phi_1(-\tau_n)(y)dy\\&\quad - \lambda(\lambda-\lambda_*)w_\lambda\int_\Omega K(\cdot,y)(\ln w_\lambda(y)+1)\phi_1(0)(y)dye^{-i\theta_\lambda}.\end{aligned}$$

Since $\phi_1(-\tau_n) = \phi_1(0)e^{-i\omega_\lambda\tau_n} - a\tau_n e^{-i\omega_\lambda\tau_n}$, then we have

$$\Delta(\lambda, i\omega, \tau_n)\phi_1(0) = a\Big(\psi_{\lambda_n} - \lambda(\lambda-\lambda_*)\tau_n w_\lambda\int_\Omega K(\cdot,y)(\ln w_\lambda(y)+1)\psi_\lambda(y)dye^{-i\theta_\lambda}\Big).$$

Hence

$$\begin{aligned}0 &= \langle\tilde{\Delta}(\lambda, i\tilde{\omega}, \tilde{\tau}_n)\tilde{\psi}_\lambda, \phi_1(0)\rangle\\&= \langle\tilde{\Delta}(\lambda, i\omega, \tau_n)\tilde{\psi}_\lambda, \phi_1(0)\rangle\\&= \langle\tilde{\psi}_\lambda, \Delta(\lambda, i\omega, \tau_n)\phi_1(0)\rangle\\&= a\Big(\int_\Omega \bar{\tilde{\psi}}_\lambda(y)\psi_\lambda(y)dy\\&\qquad - \lambda(\lambda-\lambda_*)\tau_n w_\lambda\int_\Omega\int_\Omega K(x,y)w_{\lambda_n}(x)(\ln w_\lambda(y)+1)\bar{\tilde{\psi}}_\lambda(x)\psi_\lambda(y)dxdye^{-i\theta_\lambda}\Big).\end{aligned}$$

When $\lambda \to \lambda_*$,

$$\begin{aligned}&\int_\Omega \bar{\tilde{\psi}}_\lambda(y)\psi_\lambda(y)dy - \lambda(\lambda-\lambda_*)\tau_n w_\lambda\int_\Omega\int_\Omega K(x,y)w_{\lambda_n}(x)(\ln w_\lambda(y)+1)\bar{\tilde{\psi}}_\lambda(x)\psi_\lambda(y)dxdye^{-i\theta_\lambda}\\&\to \int_\Omega \phi^2(y)dy > 0.\end{aligned}$$

So $a = 0$. Thus $\phi_1 \in \mathcal{N}[B_{\tau_n}(\lambda) - i\omega_\lambda]$.

By induction, we obtain

$$\mathcal{N}[B_{\tau_n}(\lambda) - i\omega_\lambda]^j = \mathcal{N}[B_{\tau_n}(\lambda) - i\omega_\lambda], \qquad j = 1,2,\dots,\ n = 0,1,2,\dots$$

Therefore, $\mu = i\omega_\lambda$ is a simple eigenvalue of $B_{\tau_n}$ for $n = 0,1,2,\dots$ $\square$

Since $\mu = i\omega_\lambda$ is a simple eigenvalue of $B_{\tau_n}$ , then from the implicit function theorem, there are a neighborhood $O_{1n} \times O_{2n} \times O_{3n} \subset \mathbb{R} \times \mathbb{C} \times \mathbb{X}_\mathbb{C}$ of $(\tau_n, i\omega_\lambda, \psi_\lambda)$ and a continuously differential function $(\mu, \psi) : O_{1n} \to O_{2n} \times O_{3n}$ such that for each $\tau \in O_{1n}$, the only eigenvalue of $B_\tau(\lambda)$ in $O_{2n}$ is $\mu(\tau)$, and

$$\begin{aligned}&\mu(\tau_n) = i\omega_\lambda, \qquad \psi(\tau_n) = \psi_\lambda,\\ \Delta(\lambda, \mu(\tau), \tau) &= [B(\lambda) - \mu(\tau)]\psi(\tau)\\&\quad - \lambda(\lambda-\lambda_*)w_\lambda\int_\Omega K(\cdot,y)(\ln w_\lambda(y)+1)\psi(\tau)(y)dye^{-\mu(\tau)\tau}\\&= 0, \qquad \tau \in O_{1n}.\end{aligned} \tag{4.6}$$

Then we have the following transversality condition.

**Theorem 4.3.** *Suppose that* $\lambda \in (\lambda_*, \lambda^*]$ *and* $\mu(\tau)$ *is the eigenvalue of* $B_\tau(\lambda)$*, then*

$$\frac{d\,\mathrm{Re}(\mu(\tau_n))}{d\tau} > 0, \qquad n = 0, 1, 2, \dots$$

*Proof.* Differentiating (4.6) with respect to $\tau$ at $\tau = \tau_n$, we obtain

$$\begin{aligned}&\left[-\psi_\lambda + \lambda(\lambda-\lambda_*)\tau_n w_\lambda \int_\Omega K(\cdot,y)(\ln w_\lambda(y)+1)\psi_\lambda(y)dye^{-i\theta_\lambda}\right]\frac{d\mu(\tau_n)}{d\tau}\\ &\quad + \Delta(\lambda, i\omega_\lambda, \tau_n)\frac{d\psi(\tau_n)}{d\tau} + \lambda(\lambda-\lambda_*)i\omega_\lambda w_\lambda \int_\Omega K(\cdot,y)(\ln w_\lambda(y)+1)\psi_\lambda(y)dye^{-i\theta_\lambda} = 0.\end{aligned}$$

Multiplying the above equation by $\bar{\tilde{\psi}}_\lambda(x)$ and integrating on $\Omega$, we have

$$\begin{aligned}\frac{d\mu(\tau_n)}{d\tau} &= \frac{\lambda(\lambda-\lambda_*)i\omega_\lambda \int_\Omega\int_\Omega K(x,y)w_\lambda(x)(\ln w_\lambda(y)+1)\psi_\lambda(y)\bar{\tilde{\psi}}_\lambda(x)dxdye^{-i\theta_\lambda}}{\int_\Omega \psi_\lambda(x)\bar{\tilde{\psi}}_\lambda(x)dx - \lambda(\lambda-\lambda_*)\tau_n\int_\Omega\int_\Omega K(x,y)w_\lambda(x)(\ln w_\lambda(y)+1)\psi_\lambda(y)\bar{\tilde{\psi}}_\lambda(x)dxdye^{-i\theta_\lambda}}\\ &= \frac{\lambda(\lambda-\lambda_*)\omega_\lambda \int_\Omega\int_\Omega K(x,y)w_\lambda(x)(\ln w_\lambda(y)+1)\psi_\lambda(y)\bar{\tilde{\psi}}_\lambda(x)dxdy[\sin\theta_\lambda + i\cos\theta_\lambda]}{\int_\Omega \psi_\lambda(x)\bar{\tilde{\psi}}_\lambda(x)dx - \lambda(\lambda-\lambda_*)\tau_n\int_\Omega\int_\Omega K(x,y)w_\lambda(x)(\ln w_\lambda(y)+1)\psi_\lambda(y)\bar{\tilde{\psi}}_\lambda(x)dxdy(\cos\theta_\lambda - i\sin\theta_\lambda)}.\end{aligned}$$

$$\begin{aligned}&\mathrm{Re}\frac{d\mu(\tau_n)}{d\tau}\\ &= \frac{\lambda(\lambda-\lambda_*)\omega_\lambda \int_\Omega\int_\Omega K(x,y)w_\lambda(x)(\ln w_\lambda(y)+1)\psi_\lambda(y)\bar{\tilde{\psi}}_\lambda(x)dxdy\int_\Omega \psi_\lambda(x)\bar{\tilde{\psi}}_\lambda(x)dx\sin\theta_\lambda}{M_\lambda},\end{aligned}$$

where

$$\begin{aligned}M_\lambda = \Bigg|&\int_\Omega \psi_\lambda(x)\bar{\tilde{\psi}}_\lambda(x)dx\\ &- \lambda(\lambda-\lambda_*)\tau_n\int_\Omega\int_\Omega K(x,y)w_\lambda(x)(\ln w_\lambda(y)+1)\psi_\lambda(y)\bar{\tilde{\psi}}_\lambda(x)dxdy\cos\theta_\lambda\Bigg|^2\\ &+ \left|\lambda(\lambda-\lambda_*)\tau_n\int_\Omega\int_\Omega K(x,y)w_\lambda(x)(\ln w_\lambda(y)+1)\psi_\lambda(y)\bar{\tilde{\psi}}_\lambda(x)dxdy\sin\theta_\lambda\right|^2.\end{aligned}$$

When $\lambda \to \lambda_*$,

$$\psi_\lambda \to \phi, \qquad \tilde{\psi}_\lambda \to \phi, \qquad w_\lambda \to \beta_{\lambda_*}\phi.$$

So

$$\begin{aligned}&\int_\Omega\int_\Omega K(x,y)w_\lambda(x)(\ln w_\lambda(y)+1)\psi_\lambda(y)\bar{\tilde{\psi}}_\lambda(x)dxdy\int_\Omega \psi_\lambda(x)\bar{\tilde{\psi}}_\lambda(x)dx\sin\theta_\lambda\\ &\to \beta_{\lambda_*}\int_\Omega\int_\Omega K(x,y)\phi^2(x)\phi(y)(\ln\beta_{\lambda_*}\phi(y)+1)dxdy\int_\Omega \phi^2(x)dx\sin\theta_{\lambda_*} > 0.\end{aligned}$$

Therefore, when $\lambda \in (\lambda_*, \lambda^*]$, $\frac{d\mathrm{Re}(\mu(\tau_n))}{d\tau} > 0, \qquad n = 0, 1, 2, \dots$ □

From the above conclusions, we have the following theorem.

**Theorem 4.4.** *For* $\lambda \in (\lambda_*, \lambda^*]$*, the infinitesimal generator* $B_\tau(\lambda)$ *has exactly* $2(n+1)$ *eigenvalues with positive real parts when* $\tau \in (\tau_n, \tau_{n+1}]$*,* $n = 0, 1, 2, \dots$

Then we obtain the following theorem.

**Theorem 4.5.** *For* $\lambda \in (\lambda_*, \lambda^*]$*, the positive equilibrium solution* $w_\lambda$ *of* (1.1) *is locally asymptotically stable when* $\tau \in [0, \tau_0)$ *and is unstable when* $\tau \in (\tau_0, \infty)$*. Moreover, a Hopf bifurcation occurs at* $\tau = \tau_n$ $(n = 0, 1, 2, \dots)$*, that is, a branch of spatially nonhomogeneous periodic orbits of* (1.1) *emerges from* $(\tau_n, w_\lambda)$*.*

# 5 The direction of the Hopf bifurcation

Let $W(t) = w(\cdot,t) - w_\lambda$, $\tau = \tau_n + \gamma$, then $\gamma = 0$ is the Hopf bifurcation value of system (1.1). Let $t \to \frac{t}{\tau}$, then system (1.1) can be written in the following form

$$\frac{dW(t)}{dt} = \tau_n(d\Delta W(t) + L_0(W(t))) + J(W_t,\gamma) \tag{5.1}$$

where $W_t \in C$,

$$\begin{aligned} L_0(\psi) = {} & \lambda(1 - (\lambda - \lambda_*)\int_\Omega K(\cdot,y)w_\lambda(y)\ln w_\lambda(y)dy)\psi(0) \\ & - \lambda(\lambda - \lambda_*)w_\lambda \int_\Omega K(\cdot,y)(\ln w_\lambda(y) + 1)\psi(-1)(y)dy, \end{aligned}$$

$$\begin{aligned} J(\psi,\gamma) = {} & \gamma d\Delta\psi(0) + \gamma L_0(\psi) \\ & - (\gamma + \tau_n)\lambda(\lambda - \lambda_*)\psi(0)\int_\Omega K(\cdot,y)(\ln w_\lambda(y) + 1)\psi(-1)(y)dy + O(3), \end{aligned}$$

for $\psi \in C$, $C = C([-1,0],\mathbb{Y})$.

Denote $\mathcal{B}_{\tau_n}$ to be the infinitesimal generator of the linearized equation

$$\frac{dW(t)}{dt} = \tau_n(d\Delta W(t) + L_0(W(t))).$$

Then

$$\mathcal{B}_{\tau_n}\psi = \dot\psi,$$

$$\begin{aligned} \mathcal{D}(\mathcal{B}_{\tau_n}) = \Big\{ & \psi \in \mathcal{C} \cap \mathcal{C}^1 : \psi(0) \in \mathbb{X}_\mathbb{C}, \\ & \dot\psi(0) = \tau_n B(\lambda)\psi(0) - \lambda(\lambda - \lambda_*)\tau_n w_\lambda \int_\Omega K(\cdot,y)(\ln w_\lambda(y) + 1)\psi(-1)(y)dy \Big\}, \end{aligned}$$

where $C^1 = C^1([-1,0],\mathbb{Y}_\mathbb{C})$.

Hence (5.1) can be written in the following abstract form

$$\frac{dW_t}{dt} = \mathcal{B}_{\tau_n}W_t + X_0 J(W_t,\gamma), \tag{5.2}$$

where

$$X_0(\theta) = \begin{cases} 0, & \theta \in [-1,0), \\ I, & \theta = 0. \end{cases}$$

We know that $\mathcal{B}_{\tau_n}$ has only one pair of purely imaginary eigenvalues $\pm i\omega_\lambda\tau_n$ which are simple. The corresponding eigenfunction with respect to $i\omega_\lambda\tau_n$ (or $-i\omega_\lambda\tau_n$) is $\psi_\lambda(x)e^{i\omega_\lambda\tau_n\theta}$ (or $\bar\psi_\lambda(x)e^{-i\omega_\lambda\tau_n\theta}$) for $\theta \in [-1,0]$. We introduce the formal duality $\ll \cdot,\cdot \gg$ by

$$\begin{aligned} \ll \tilde\psi,\psi \gg = {} & \langle \tilde\psi(0),\psi(0)\rangle \\ & - \lambda(\lambda - \lambda_*)\tau_n \int_{-1}^0 \Big\langle \tilde\psi(s+1),\ w_\lambda \int_\Omega K(\cdot,y)(\ln w_\lambda(y) + 1)\psi(s)(y)dy \Big\rangle ds, \end{aligned} \tag{5.3}$$

for $\psi \in C$ and $\tilde\psi \in C^*_\mathbb{C} := C([0,1],\mathbb{Y}_\mathbb{C})$.

Define an operator $\mathcal{B}^*_{\tau_n} : \mathcal{D}(\mathcal{B}^*_{\tau_n}) \to C^*_{\mathbb{C}}$, $\mathcal{B}^*_{\tau_n}\tilde{\psi}(s) = -\dot{\tilde{\psi}}(s)$, and

$$\mathcal{D}(\mathcal{B}^*_{\tau_n}) = \Big\{\tilde{\psi} \in C^*_{\mathbb{C}} \cap (C^*_{\mathbb{C}})^1 : \tilde{\psi}(0) \in \mathbb{X}_{\mathbb{C}},$$
$$\dot{\tilde{\psi}}(0) = \tau_n B(\lambda)\tilde{\psi}(0) - \lambda(\lambda-\lambda_*)\tau_n \int_\Omega K(y,\cdot)w_\lambda(y)(\ln w_\lambda(y)+1)\tilde{\psi}(1)(y)dy\Big\},$$

where $(C^*_{\mathbb{C}})^1 = C^1([0,1], \mathbb{Y}_{\mathbb{C}})$.

Then $\mathcal{B}^*_{\tau_n}$ and $\mathcal{B}_{\tau_n}$ satisfy

$$\ll \mathcal{B}^*_{\tau_n}\tilde{\psi}, \psi \gg = \ll \tilde{\psi}, \mathcal{B}_{\tau_n}\psi \gg, \quad \text{for } \psi \in \mathcal{D}(\mathcal{B}_{\tau_n}),\ \tilde{\psi} \in \mathcal{D}(\mathcal{B}^*_{\tau_n}).$$

The operator $\mathcal{B}^*_{\tau_n}$ has only one pair of purely imaginary eigenvalues $\pm i\omega_\lambda\tau_n$ which are simple, and the corresponding eigenfunction with respect to $i\omega_\lambda\tau_n$ (or $-i\omega_\lambda\tau_n$) is $\bar{\tilde{\psi}}_\lambda(x)e^{-i\omega_\lambda\tau_n s}$ (or $\tilde{\psi}_\lambda(x)e^{i\omega_\lambda\tau_n s}$) for $s \in [0,1]$. So $\mathcal{B}^*_{\tau_n}$ and $\mathcal{B}_{\tau_n}$ are adjoint operators under the bilinear form (5.3).

The center subspace of (5.1) is $P = \text{span}\{p(\theta), \bar{p}(\theta)\}$, where $p(\theta) = \psi_\lambda e^{i\omega_\lambda\tau_n\theta}$ is the eigenfunction of $\mathcal{B}_{\tau_n}$ with respect to $i\omega_\lambda\tau_n$. Similarly, the formal adjoint subspace of $P$ with respect to the bilinear form (5.3) is $P^* = \text{span}\{q(s), \bar{q}(s)\}$, where $q(s) = \tilde{\psi}_\lambda e^{i\omega_\lambda\tau_n s}$ is the eigenfunction of $\mathcal{B}^*_{\tau_n}$ with respect to $-i\omega_\lambda\tau_n$. Then $C_{\mathbb{C}}$ can be decomposed as $C_{\mathbb{C}} = P \oplus Q$, where $Q = \{\psi \in C_{\mathbb{C}} : \ll \tilde{\psi}, \psi \gg = 0, \text{ for all } \tilde{\psi} \in P^*\}$.

Let $\Phi = (p(\theta), \bar{p}(\theta))$, $\Psi = D(q(s), \bar{q}(s))^T$, where

$$\bar{D} = \left[\int_\Omega \psi_\lambda(x)\bar{\tilde{\psi}}_\lambda(x) - \lambda(\lambda-\lambda_*)\tau_n \int_\Omega\int_\Omega K(x,y)w_\lambda(x)(\ln w_\lambda(y)+1)\bar{\tilde{\psi}}_\lambda(x)\psi_\lambda(y)e^{-i\omega_\lambda\tau_n}\right]^{-1},$$

then $\ll \Psi, \Phi \gg = Id_2$, where $Id_2$ is the identity matrix in $\mathbb{R}^{2\times 2}$.

Define $z(t) = \ll Dq(s), W_t \gg$, and denote

$$H(t,\theta) = W_t(\theta) - \Phi \cdot (z(t), \bar{z}(t))^T \tag{5.4}$$

then $H(t,\theta) \in Q$.

On the center manifold, we have $H(t,\theta) = H(z,\bar{z},\theta)$, where $H(z,\bar{z},\theta)$ can be expanded the power series of $z$ and $\bar{z}$,

$$H(z,\bar{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots, \tag{5.5}$$

where $z$ and $\bar{z}$ are local coordinates for the center manifold in the direction of $q$ and $\bar{q}$.

The flow of (5.1) on the center manifold can be written as:

$$\begin{aligned} \dot{z}(t) &= \frac{d}{dt} \ll Dq(s), W_t \gg \\ &= \ll Dq(s), \mathcal{B}_{\tau_n}W_t \gg + \ll Dq(s), X_0 J(W_t,0) \gg \\ &= \ll D\mathcal{B}^*_{\tau_n}q(s), W_t \gg + \langle Dq(0), J(W_t,0)\rangle \\ &= \ll -i\omega_\lambda\tau_n Dq(s), W_t \gg + \langle Dq(0), J(\Phi\cdot(z(t),\bar{z}(t))^T + H(z(t),\bar{z}(t),\theta),0)\rangle \\ &= i\omega_\lambda\tau_n z(t) + \langle Dq(0), J(\Phi\cdot(z(t),\bar{z}(t))^T + H(z(t),\bar{z}(t),\theta),0)\rangle. \end{aligned} \tag{5.6}$$

Denote

$$J(\Phi\cdot(z(t),\bar{z}(t))^T + H(z(t),\bar{z}(t),\theta),0)\rangle = J_{z^2}\frac{z^2}{2} + J_{z\bar{z}}z\bar{z} + J_{\bar{z}^2}\frac{\bar{z}^2}{2} + J_{z^2\bar{z}}\frac{z^2\bar{z}}{3} + \cdots$$

We rewrite (5.6) as

$$\dot{z}(t) = i\omega_\lambda \tau_n z(t) + g(z,\overline{z})(t), \tag{5.7}$$

where

$$\begin{aligned} g(z,\overline{z}) &= \langle Dq(0), J(\Phi \cdot (z(t),\overline{z}(t))^T + H(z(t),\overline{z}(t),\theta),0)\rangle \\ &= g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots, \end{aligned}$$

then

$$\begin{cases} g_{20} = & -2\lambda(\lambda-\lambda_*)\tau_n\bar{D}\int_\Omega\int_\Omega K(x,y)(\ln w_\lambda(y)+1)\psi_\lambda(x)\psi_\lambda(y)\bar{\overline{\psi}}_\lambda(x)dxdye^{-i\omega_\lambda\tau_n}, \\ g_{11} = & -\lambda(\lambda-\lambda_*)\tau_n\bar{D}\Big[\int_\Omega\int_\Omega K(x,y)(\ln w_\lambda(y)+1)\psi_\lambda(x)\bar{\psi}_\lambda(y)\bar{\overline{\psi}}_\lambda(x)dxdye^{i\omega_\lambda\tau_n} \\ & +\int_\Omega\int_\Omega K(x,y)(\ln w_\lambda(y)+1)\bar{\psi}_\lambda(x)\psi_\lambda(y)\bar{\overline{\psi}}_\lambda(x)dxdye^{-i\omega_\lambda\tau_n}\Big], \\ g_{02} = & -2\lambda(\lambda-\lambda_*)\tau_n\bar{D}\int_\Omega\int_\Omega K(x,y)(\ln w_\lambda(y)+1)\bar{\psi}_\lambda(x)\bar{\psi}_\lambda(y)\bar{\overline{\psi}}_\lambda(x)dxdye^{i\omega_\lambda\tau_n}, \\ g_{21} = & -\lambda(\lambda-\lambda_*)\tau_n\bar{D}\Big[2\int_\Omega\int_\Omega K(x,y)(\ln w_\lambda(y)+1)\psi_\lambda(x)\bar{\overline{\psi}}_\lambda(x)H_{11}(-1)(y)dxdy \\ & +\int_\Omega\int_\Omega K(x,y)(\ln w_\lambda(y)+1)\bar{\psi}_\lambda(y)\bar{\overline{\psi}}_\lambda(x)H_{20}(x)dxdye^{i\omega_\lambda\tau_n} \\ & +2\int_\Omega\int_\Omega K(x,y)(\ln w_\lambda(y)+1)\psi_\lambda(y)\bar{\overline{\psi}}_\lambda(x)H_{11}(0)(x)dxdye^{-i\omega_\lambda\tau_n}, \\ & +\int_\Omega\int_\Omega K(x,y)(\ln w_\lambda(y)+1)\bar{\psi}_\lambda(x)\bar{\overline{\psi}}_\lambda(x)H_{20}(-1)(y)dxdy\Big]. \end{cases} \tag{5.8}$$

Since there are $H_{20}(\theta)$, $H_{11}(\theta)$ in $g_{21}$, we still need to compute them.
From (5.4), we obtain

$$\begin{aligned} \dot{H} &= \dot{W}_t - \Phi(\dot{z},\dot{\bar{z}})^T \\ &= \mathcal{B}_{\tau_n}W_t + X_0J(W_t,0) - \Phi(\dot{z},\dot{\bar{z}})^T \\ &= \mathcal{B}_{\tau_n}(H(z,\bar{z},\theta)+\Phi(z,\bar{z})^T) - \Phi(\dot{z},\dot{\bar{z}})^T + X_0J(W_t,0) \\ &= \mathcal{B}_{\tau_n}H(z,\bar{z},\theta) + \mathcal{B}_{\tau_n}\Phi(z,\bar{z})^T - \Phi(\dot{z},\dot{\bar{z}})^T + X_0J(W_t,0) \\ &= \mathcal{B}_{\tau_n}H(z,\bar{z},\theta) - \Phi(g,\bar{g})^T + X_0J(H(z,\bar{z},\theta)+\Phi(z,\bar{z})^T,0) \end{aligned} \tag{5.9}$$

On the other hand, on the center manifold $C_0$, from (5.5) and (5.7), we have

$$\begin{aligned} \dot{H} &= H_z\dot{z} + H_{\overline{z}}\dot{\overline{z}} \\ &= [H_{20}(\theta)z + H_{11}(\theta)\overline{z}](i\omega_\lambda\tau_n z + g) \\ &\quad + [W_{11}(\theta)z + W_{02}(\theta)\overline{z}](-i\omega_\lambda\tau_n\overline{z} + \overline{g}) + \cdots \end{aligned} \tag{5.10}$$

Substituting the corresponding series into the above two equations and comparing the coefficients, we obtain

$$(2i\omega_\lambda\tau_n I - \mathcal{B}_{\tau_n})H_{20}(\theta) = \begin{cases} -g_{20}p(\theta) - \overline{g}_{02}\overline{p}(\theta), & \theta\in[-1,0), \\ -g_{20}p(0) - \overline{g}_{02}\overline{p}(0) + J_{z^2}, & \theta = 0. \end{cases} \tag{5.11}$$

$$-\mathcal{B}_{\tau_n} H_{11}(\theta) = \begin{cases} -g_{11}p(\theta) - \overline{g}_{11}\overline{p}(\theta), & \theta \in [-1,0), \\ -g_{11}p(0) - \overline{g}_{11}\overline{p}(0) + J_{z\bar{z}}, & \theta = 0. \end{cases} \tag{5.12}$$

When $\theta \in [-1,0)$, we have

$$H'_{20}(\theta) = 2i\omega_\lambda \tau_n H_{20}(\theta) + g_{20}p(\theta) + \overline{g}_{02}\overline{p}(\theta).$$

Solving this equation, we obtain

$$H_{20}(\theta) = \frac{ig_{20}}{\omega_\lambda \tau_n} p(\theta) + \frac{i\overline{g}_{02}}{3\omega_\lambda \tau_n}\overline{p}(\theta) + E_1 e^{2i\omega_\lambda \tau_n \theta}. \tag{5.13}$$

When $\theta = 0$,

$$\begin{aligned}(2i\omega_\lambda \tau_n I - \mathcal{B}_{\tau_n})E_1 e^{2i\omega_\lambda\tau_n\theta}|_{\theta=0} &= J_{z^2}(z,\bar{z}) \\ &= -2\lambda(\lambda-\lambda_*)\tau_n \psi_\lambda \int_\Omega K(\cdot,y)(\ln w_\lambda(y)+1)\psi_\lambda(y)dy e^{-i\omega_\lambda \tau_n},\end{aligned}$$

then

$$\Delta(\lambda, 2i\omega_\lambda, \tau_n)E_1 = 2\lambda(\lambda-\lambda_*)\psi_\lambda \int_\Omega K(\cdot,y)(\ln w_\lambda(y)+1)\psi_\lambda(y)dy e^{-i\omega_\lambda\tau_n}.$$

Since $2i\omega_\lambda$ is not the eigenvalue of $\mathcal{B}_{\tau_n}(\lambda)$, hence

$$E_1 = 2\lambda(\lambda-\lambda_*)\Delta^{-1}(\lambda, 2i\omega_\lambda, \tau_n)\left(\psi_\lambda \int_\Omega K(\cdot,y)(\ln w_\lambda(y)+1)\psi_\lambda(y)dy e^{-i\omega_\lambda\tau_n}\right).$$

Substituting $E_1$ into (5.13) gives $H_{20}(\theta)$.

Similarly, we have

$$H'_{11}(\theta) = g_{11}p(\theta) + \overline{g}_{11}\overline{p}(\theta),$$

and

$$H_{11}(\theta) = -\frac{ig_{11}}{\omega_\lambda\tau_n}p(\theta) + \frac{i\overline{g}_{11}}{\omega_\lambda\tau_n}\bar{p}(\theta) + E_2. \tag{5.14}$$

When $\theta = 0$,

$$\begin{aligned}-\mathcal{B}_{\tau_n}E_2 &= J_{z\bar{z}}(z,\bar{z}) \\ &= -\lambda(\lambda-\lambda_*)\tau_n\Big[\psi_\lambda\int_\Omega K(\cdot,y)(\ln w_\lambda(y)+1)\bar{\psi}_\lambda(y)dy e^{i\omega_\lambda\tau_n} \\ &\quad + \bar{\psi}_\lambda \int_\Omega K(\cdot,y)(\ln w_\lambda(y)+1)\psi_\lambda(y)dy e^{-i\omega_\lambda\tau_n}\Big].\end{aligned}$$

Then we obtain

$$\begin{aligned}E_2 = \lambda(\lambda-\lambda_*)\Delta^{-1}(\lambda,0,\tau_n)\Big[&\psi_\lambda \int_\Omega K(\cdot,y)(\ln w_\lambda(y)+1)\bar{\psi}_\lambda(y)dy e^{i\omega_\lambda\tau_n} \\ &+ \bar{\psi}_\lambda\int_\Omega K(\cdot,y)(\ln w_\lambda(y)+1)\psi_\lambda(y)dy e^{-i\omega_\lambda\tau_n}\Big].\end{aligned}$$

Substituting $E_2$ into (5.14) gives $H_{11}(\theta)$.

By now, $g_{20}$, $g_{11}$, $g_{02}$, $g_{21}$ are all obtained. And we can compute the following values (see [23]):

$$c_1(0) = \frac{i}{2\omega_\lambda\tau_n}\left[g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right] + \frac{g_{21}}{2},$$

$$\mu_2 = -\frac{\operatorname{Re} c_1(0)}{\mu'(\tau_n)},$$
$$\beta_2 = 2\operatorname{Re} c_1(0),$$
$$T_2 = -\frac{\operatorname{Im} c_1(0) + \mu_2 \operatorname{Im}(\mu'(\tau_n))}{\omega_\lambda}.$$

We know that $\mu_2$ determines the directions of the Hopf bifurcation: if $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $\tau > \tau_n$ ($\tau < \tau_n$); $\beta_2$ determines the stability of the bifurcating periodic solutions: the bifurcating periodic solutions are stable (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$); and $T_2$ determines the period of the bifurcating periodic solutions: the period increases (decreases) if $T_2 > 0$ ($T_2 < 0$).

## 6 Numerical simulations

In this section, we give an example:

$$\begin{cases} \dfrac{\partial w(x,t)}{\partial t} = \Delta w(x,t) + \lambda w(x,t) \\ \qquad\qquad \times \left(1 - (\lambda - \lambda_*) \displaystyle\int_\Omega K(x,y) w(y,t-\tau) \ln w(y,t-\tau) dy\right), \ x \in (0,\pi), \ t > 0, \\ w(x,t) = 0, \quad x = 0, \pi, \ t > 0. \end{cases} \tag{6.1}$$

We can see that $\lambda_* = 1$ is the principal eigenvalue of the linear operator $-\Delta$ subject to the Dirichlet boundary condition on the $\partial\Omega$ with $\Omega = (0,\pi)$ and the associated eigenvector is $\phi(x) = \sqrt{\frac{2}{\pi}} \sin x$. It follows from Theorem 2.1 that system (6.1) with $\lambda \in [1, \lambda^*]$ has a positive nonhomogeneous steady state solution $w_\lambda$.

We avoid the complex computation and only show two numerical simulations of system (1.1). The numerical simulations with a homogeneous kernel $K(x,y) = \sin y$ and a nonhomogeneous kernel $K(x,y) = (x-y)^2$ are shown in Figure 6.1 and in Figure 6.2, respectively. In each figure, $\lambda = 1.2$, $\Omega = (0,\pi)$, $d = 1$, and the initial value is $w(x,t) = 0.5 \sin^2 x$. In each case, the convergence to the spatially nonhomogeneous equilibrium $w_\lambda$ occurs when $\tau$ is less than the first Hopf bifurcation point $\tau_0$ and an oscillatory pattern emerges for $\tau > \tau_0$. Each simulation verifies the occurrence of spatially nonhomogeneous temporal oscillation and the spatial profiles of the periodic solutions are different due to the different dispersal kernels.

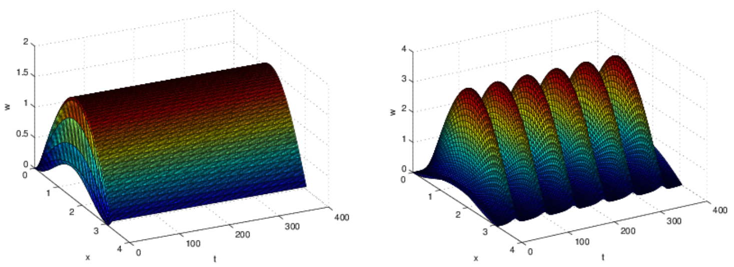

Figure 6.1: Spatially homogeneous kernel $K(x,y) = \sin y$. (Left): $\tau = 1.1$; (Right): $\tau = 3.0$.

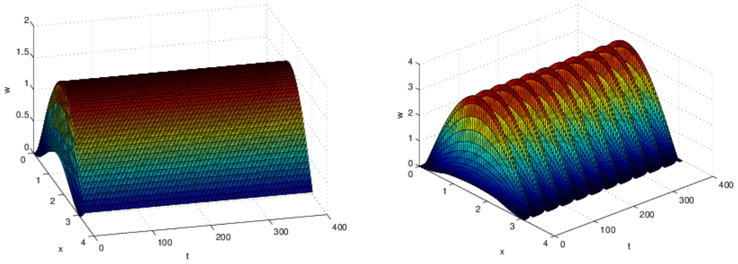

Figure 6.2: Spatially homogeneous kernel $K(x,y) = (x-y)^2$. (Left): $\tau = 0.2$; (Right): $\tau = 1.2$.

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