PERIODIC SOLUTIONS OF NEUTRAL DUFFING EQUATIONS

Z. Q. Zhang, Z. C. Wang and J. S. Yu

Dept. of Appl. Math., Hunan Univ., 410082 Changsha, China

Abstract. We consider the following neutral delay Duffing equation

$$ax''(t) + bx'(t) + cx(t) + g(x(t-\tau_1),\ x'(t-\tau_2),\ x''(t-\tau_3)) = p(t) = p(t+2\pi),$$

where a, b and c are constants, τ_i, $i = 1, 2, 3$, are nonnegative constants, $g : R \times R \times R \to R$ is continuous, and $p(t)$ is a continuous 2π-periodic function. In this paper, combining the Brouwer degree theory with a continuation theorem based on Mawhin's coincidence degree, we obtain a sufficient condition for the existence of 2π-periodic solution of above equation.

Key words: Periodic solution, Duffing equation, Brouwer degree, coincidence degree.

1991 AMS Subject Classification: 34K15

1. Introduction

On the existence problem of periodic solutions for the Duffing equations

$$x''(t) + g(x) = p(t) = p(t+2\pi), \tag{1.1}$$

so far there has been a wide literature since the interest in studying Eq.(1.1) comes from different sources. Under the conditions which exclude the resonance cases,

The Project Supported by NNSF of China(No:19971026, 19831030)

many results have been obtained [1,2,3,4]. At resonance, many authors have paid much attention to the problem in recent years. [5] and [6] resolved the existence problem of 2π-periodic solutions of Eq.(1.1) under some different conditions, respectively.

On the other hand, a few papers have appeared[7,8,9,10,11,12] which dealt with the existence problem of periodic solutions to the delay Duffing equations such as

$$x''(t) + g(x(t-\tau)) = p(t) = p(t+2\pi). \tag{1.2}$$

Under some conditions which exclude the resonance cases, some results have been obtained[13,14,15].

Next, [17] discussed the Duffing equations of the form

$$x''(t) + m^2x(t) + g(x(t-\tau)) = p(t) = p(t+2\pi), \tag{1.3}$$

where m is a positive integer, and proved the existence of 2π-periodic solutions of Eq.(1.3) under some conditions.

Jack Hale [21] and [22] put forward the Euler's equations which are of the form

$$x''(t) = f(t, x(t), x(t-r), x'(t), x'(t-r), x''(t-r)),$$

where r is a positive constant.

Motivated by above papers, in the present paper, we consider the neutral Duffing equations of the form

$$ax''(t)+bx'(t)+cx(t)+g(x(t-\tau_1),\ x'(t-\tau_2),\ x''(t-\tau_3)) = p(t) = p(t+2\pi), \tag{1.4}$$

where a, b, c are constants, τ_1, τ_2, τ_3 are nonnegative constants, $g : R\times R \times R \to R$ is continuous, and p(t) is a continuous 2π-periodic function.

To the best of our knowledge, in this direction, few papers can be found in the literature. In this paper, combining the Brouwer degree theory with a continuation theorem based on Mawhin's coincidence degree[16], we obtain a sufficient condition for the existence of 2π-periodic solution of Eq.(1.4).

2. Existence of a Periodic Solution

In order to obtain the existence of a periodic solution of Eq. (1.4), we first make the following preparations.

Let X and Z be two Banach spaces. Consider an operator equation

$$Lx = \lambda Nx,$$

where L: Dom $L \cap X \to Z$ is a linear operator and $\lambda \in [0, 1]$ a parameter. Let P and Q denote two projectors such that

$$P : \mathrm{D}om\, L \cap X \to \mathrm{K}er\, L \quad \text{and} \quad Q : Z \to Z/\mathrm{I}m\, L.$$

In the sequel, we will use the following result of Mawhin[16].

LEMMA 2.1. *Let X and Z be two Banach spaces and L a Fredholm mapping of index 0. Assume that $N : \overline{\Omega} \to Z$ is L-compact on $\overline{\Omega}$ with Ω open bounded in X. Furthermore suppose*

(a). *For each $\lambda \in (0, 1)$, $x \in \partial\Omega \cap$ $Dom\, L$*

$$Lx \neq \lambda Nx.$$

(b). *For each $x \in \partial\Omega \cap$ $Ker\, L$,*

$$QNx \neq 0$$

and

$$\deg\{QN,\ \Omega\cap\ KerL,\ 0\}\neq 0.$$

Then $Lx = Nx$ has at least one solution in $\overline{\Omega}$.

Recall that a linear mapping L: Dom $L \subset X \to Z$ with Ker $L = L^{-1}(0)$ and Im $L = L(\mathrm{Dom}L)$, will be called a Fredholm mapping if the following two conditions hold:

(i). Ker L has a finite dimension;

(ii). Im L is closed and has a finite codimension.

Recalled also that the codimension of Im L is the dimension of $Z/\mathrm{Im}\ L$, i.e., the dimension of the cokernel coker L of L.

When L is a Fredholm mapping, its (Fredholm) index is the integer

$$\mathrm{I}nd\ L = \dim \mathrm{Ker}\ L - \mathrm{codim}\,\mathrm{Im}\, L.$$

We shall say that a mapping N is L-compact on Ω if the mapping QN : $\bar{\Omega} \to Z$ is continuous, $QN(\bar{\Omega})$ is bounded, and $K_P(I-Q)N : \bar{\Omega} \to X$ is compact, i.e., it is continuous and $K_P(I-Q)N(\bar{\Omega})$ is relatively compact, where K_P: $\mathrm{Im}\, L \to$Dom $L\cap$Ker P is a inverse of the restriction L_P of L to Dom $L\cap$Ker P, so that $LK_P = I$ and $K_P L = I - P$.

THEOREM 2.1. *Assume that there exist a positive constant M and three non-negative constants $\beta_1,\ \beta_2, \beta_3$ such that*

$$|g(x_1,x_2,x_3)| \leq M + \beta_1|x_1| + \beta_2|x_2| + \beta_3|x_3| \ \ for\ \forall (x_1,x_2,x_3)\in R^3 \tag{2.1}$$

and

$$|abc| - |bc|\beta_3 - |ac|\beta_2 - (2|ab| + 2\pi|ac|)\beta_1 > \beta_3\sqrt{(|ac| - \beta_3|c| - \beta_1|a|)|c|(|c| - \beta_1)}. \tag{2.2}$$

Then Eq.(1.4) has at least one 2π-periodic solution.

Proof. In order to use Lemma 2.1 for Eq.(1.4), we take $X = \{x(t) \in C^2(R, R) : x(t + 2\pi) = x(t)\}$ and $Z = \{z(t) \in C(R, R) : z(t + 2\pi) = z(t)\}$, and denote $|x|_0 = \max_{t\in[0,2\pi]} |x(t)|$ and $|x|_2 = \max\{|x|_0, |x'|_0, |x''|_0\}$. Then X and Z are Banach spaces when they are endowed with norms $|\cdot|_2$ and $|\cdot|_0$, respectively.

Set

$$Lx = ax''(t), \quad Nx = -bx'(t) - cx(t) - g(x(t - \tau_1), x'(t - \tau_2), x''(t - \tau_3)) + p(t),$$

$$Px = \frac{1}{2\pi}\int_0^{2\pi} x(t)dt, \quad x \in X, \;\; Qz = \frac{1}{2\pi}\int_0^{2\pi} z(t)dt, \quad z \in Z.$$

Since $\operatorname{Ker} L = R$ and $\operatorname{Im} L = \{x \in Z : \int_0^{2\pi} x(t)dt = 0\}$, $\operatorname{Im} L$ is closed and $\dim \operatorname{Ker} L = \dim Z/\operatorname{Im} L = 1$. Therefore, L is a Fredholm mapping of index 0.

Corresponding to the operator equation

$$Lx = \lambda Nx, \quad \lambda \in (0, 1),$$

we have

$$ax''(t) + \lambda bx'(t) + \lambda cx(t) + \lambda g(x(t - \tau_1), x'(t - \tau_2), x''(t - \tau_3)) = \lambda p(t). \qquad (2.3)$$

Let $x(t) \in X$ is a solution of Eq.(2.3) for a certain $\lambda \in (0, 1)$. Integrating (2.3) from 0 to 2π, we have

$$\int_0^{2\pi} cx(t)dt = \int_0^{2\pi} [p(t) - g(x(t - \tau_1), x'(t - \tau_2), x''(t - \tau_3))]dt,$$

from which, it implies that there exists a $t^* \in (0, 2\pi)$ such that

$$2\pi cx(t^*) = \int_0^{2\pi} [p(t) - g(x(t - \tau_1), x'(t - \tau_2), x''(t - \tau_3))]dt.$$

Let $m = \max_{t\in[0,2\pi]} |p(t)|$. Then

$$\begin{aligned}2\pi|cx(t^*)| &\le 2\pi(m+M) + \beta_1\int_0^{2\pi}|x(t-\tau_1)|dt\\ &\quad + \beta_2\int_0^{2\pi}|x'(t-\tau_2)|dt + \beta_3\int_0^{2\pi}|x''(t-\tau_3)|dt\\ &= 2\pi(m+M) + \beta_1\int_0^{2\pi}|x(t)|dt + \beta_2\int_0^{2\pi}|x'(t)|dt + \beta_3\int_0^{2\pi}|x''(t)|dt.\end{aligned}$$

Since for $\forall t \in [0, 2\pi]$,

$$\begin{aligned}x(t) &= x(t^*) + \int_{t^*}^{t} x'(s)ds,\\ |x(t)| &\le |x(t^*)| + \int_0^{2\pi}|x'(s)|ds\\ &\le \frac{1}{\sqrt{2\pi}|c|}\Bigg[\sqrt{2\pi}(m+M) + \beta_1\left(\int_0^{2\pi}|x(t)|^2dt\right)^{\frac{1}{2}}\\ &\quad + (2\pi|c|+\beta_2)\left(\int_0^{2\pi}|x'(t)|^2dt\right)^{\frac{1}{2}} + \beta_3\left(\int_0^{2\pi}|x''(t)|^2dt\right)^{\frac{1}{2}}\Bigg].\end{aligned}$$

Thus

$$\begin{aligned}|c|\left(\int_0^{2\pi}|x(t)|^2dt\right)^{\frac{1}{2}} &\le \sqrt{2\pi}|c|\max_{t\in[0,2\pi]}|x(t)|\\ &\le \sqrt{2\pi}(m+M) + \beta_1\left(\int_0^{2\pi}|x(t)|^2dt\right)^{\frac{1}{2}}\\ &\quad + \beta_3\left(\int_0^{2\pi}|x''(t)|^2dt\right)^{\frac{1}{2}}\\ &\quad + (2\pi|c|+\beta_2)\left(\int_0^{2\pi}|x'(t)|^2dt\right)^{\frac{1}{2}}\end{aligned}$$

from which, it follows that

$$\begin{aligned}(|c|-\beta_1)\left(\int_0^{2\pi}|x(t)|^2dt\right)^{\frac{1}{2}} &\le \sqrt{2\pi}(m+M) + (2\pi|c|+\beta_2)\left(\int_0^{2\pi}|x'(t)|^2dt\right)^{\frac{1}{2}}\\ &\quad +\beta_3\left(\int_0^{2\pi}|x''(t)|^2dt\right)^{\frac{1}{2}}. \qquad (2.4)\end{aligned}$$

Multipling (2.3) by $x''(t)$ and integrating from 0 to 2π, we get

$$a\int_0^{2\pi}|x''(t)|^2dt-\lambda c\int_0^{2\pi}|x'(t)|^2dt$$
$$+\lambda\int_0^{2\pi}x''(t)[g(x(t-\tau_1),x'(t-\tau_2),x''(t-\tau_3))-p(t)]dt=0,$$

from which, it implies that

$$|a|\int_0^{2\pi}|x''(t)|^2dt$$
$$\le|c|\int_0^{2\pi}|x'(t)|^2dt+\int_0^{2\pi}|x''(t)|\Bigg[m+M$$
$$+\beta_1|x(t-\tau_1)|+\beta_2|x'(t-\tau_2)|+\beta_3|x''(t-\tau_3)|\Bigg]dt$$
$$\le|c|\int_0^{2\pi}|x'(t)|^2dt+\left(\int_0^{2\pi}|x''(t)|^2dt\right)^{\frac{1}{2}}\Bigg[\sqrt{2\pi}(m+M)$$
$$+\beta_1\left(\int_0^{2\pi}|x(t)|^2dt\right)^{\frac{1}{2}}+\beta_2\left(\int_0^{2\pi}|x'(t)|^2dt\right)^{\frac{1}{2}}+\beta_3\left(\int_0^{2\pi}|x''(t)|^2dt\right)^{\frac{1}{2}}\Bigg].$$

Therefore,

$$(|a|-\beta_3)\int_0^{2\pi}|x''(t)|^2dt\le|c|\int_0^{2\pi}|x'(t)|^2dt+\left(\int_0^{2\pi}|x''(t)|^2dt\right)^{\frac{1}{2}}\Bigg[\sqrt{2\pi}(m+M)$$
$$+\beta_1\left(\int_0^{2\pi}|x(t)|^2dt\right)^{\frac{1}{2}}+\beta_2\left(\int_0^{2\pi}|x'(t)|^2dt\right)^{\frac{1}{2}}\Bigg]. \quad (2.5)$$

From (2.4) and (2.5), we have

$$(|c|-\beta_1)(|a|-\beta_3)\int_0^{2\pi}|x''(t)|^2dt$$
$$\le|c|(|c|-\beta_1)\int_0^{2\pi}|x'(t)|^2dt$$
$$+\left(\int_0^{2\pi}|x''(t)|^2dt\right)^{\frac{1}{2}}\Bigg[\sqrt{2\pi}|c|(m+M)$$
$$+\beta_1\beta_3\left(\int_0^{2\pi}|x''(t)|^2dt\right)^{\frac{1}{2}}+(2\pi\beta_1+\beta_2)|c|\left(\int_0^{2\pi}|x'(t)|^2\right)\Bigg],$$

from which, it follows that

$$
\begin{aligned}
&(|ac| - \beta_3|c| - \beta_1|a|)\int_0^{2\pi}|x''(t)|^2dt\\
&\leq |c|(|c| - \beta_1)\int_0^{2\pi}|x'(t)|^2dt\\
&\quad + \left(\int_0^{2\pi}|x''(t)|^2dt\right)^{\frac{1}{2}}\left[\sqrt{2\pi}|c|(m+M) + |c|(2\pi\beta_1+\beta_2)\left(\int_0^{2\pi}|x'(t)|^2dt\right)^{\frac{1}{2}}\right].
\end{aligned}
$$

Thus

$$
\begin{aligned}
&2(|ac| - \beta_3|c| - \beta_1|a|)\left(\int_0^{2\pi}|x''(t)|^2dt\right)^{\frac{1}{2}}\\
&\leq \sqrt{2\pi}|c|(m+M) + |c|(2\pi\beta_1+\beta_2)\left(\int_0^{2\pi}|x'(t)|^2dt\right)^{\frac{1}{2}}\\
&\quad + \left\{\left[\sqrt{2\pi}|c|(m+M) + |c|(2\pi\beta_1+\beta_2)\left(\int_0^{2\pi}|x'(t)|^2dt\right)^{\frac{1}{2}}\right]^2\right.\\
&\quad \left. + 4(|ac| - \beta_3|c| - \beta_1|a|)|c|(|c| - \beta_1)\int_0^{2\pi}|x'(t)|^2dt\right\}^{\frac{1}{2}}.
\end{aligned}
\tag{2.6}
$$

Using inequality $(a+b)^{\frac{1}{2}} \leq a^{\frac{1}{2}} + b^{\frac{1}{2}}$, for $a \geq 0$ and $b \geq 0$,we have

$$
\begin{aligned}
&\left\{\left[\sqrt{2\pi}|c|(m+M) + |c|(2\pi\beta_1+\beta_2)\left(\int_0^{2\pi}|x'(t)|^2dt\right)^{\frac{1}{2}}\right]^2\right.\\
&\quad \left. + 4(|ac| - \beta_3|c| - \beta_1|a|)|c|(|c| - \beta_1)\int_0^{2\pi}|x'(t)|^2dt\right\}^{\frac{1}{2}}\\
&\leq \sqrt{2\pi}|c|(m+M) + |c|(2\pi\beta_1+\beta_2)\left(\int_0^{2\pi}|x'(t)|^2dt\right)^{\frac{1}{2}}\\
&\quad + 2\sqrt{(|ac| - \beta_3|c| - \beta_1|a|)|c|(|c| - \beta_1)}\left(\int_0^{2\pi}|x'(t)|^2dt\right)^{\frac{1}{2}}.
\end{aligned}
\tag{2.7}
$$

By (2.6) and (2.7), we have

$$
\begin{aligned}
&(|ac| - \beta_3|c| - \beta_1|a|)\left(\int_0^{2\pi} |x''(t)|^2 dt\right)^{\frac{1}{2}} \\
&\le \sqrt{2\pi}|c|(m+M) + \Bigg[|c|(2\pi\beta_1 + \beta_2) \\
&\quad + \sqrt{(|ac| - \beta_3|c| - \beta_1|a|)|c|(|c| - \beta_1)}\Bigg]\left(\int_0^{2\pi} |x'(t)|^2 dt\right)^{\frac{1}{2}}.
\end{aligned} \tag{2.8}
$$

Multipling (2.3) by $x'(t)$ and integrating from 0 to 2π, we obtain

$$
b\int_0^{2\pi} |x'(t)|^2 dt + \int_0^{2\pi} x'(t)[g(x(t-\tau_1), x'(t-\tau_2), x''(t-\tau_3)) - p(t)]dt = 0,
$$

from which, it implies that

$$
\begin{aligned}
|b|\int_0^{2\pi} |x'(t)|^2 dt &\le \left(\int_0^{2\pi} |x'(t)|^2 dt\right)^{\frac{1}{2}} \Bigg[\sqrt{2\pi}(m+M) + \beta_1\left(\int_0^{2\pi} |x(t-\tau_1)|^2 dt\right)^{\frac{1}{2}} \\
&\quad + \beta_2\left(\int_0^{2\pi} |x'(t-\tau_2)|^2 dt\right)^{\frac{1}{2}} + \beta_3\left(\int_0^{2\pi} |x''(t-\tau_3)|^2 dt\right)^{\frac{1}{2}}\Bigg] \\
&= \left(\int_0^{2\pi} |x'(t)|^2 dt\right)^{\frac{1}{2}} \Bigg[\sqrt{2\pi}(m+M) + \beta_1\left(\int_0^{2\pi} |x(t)|^2 dt\right)^{\frac{1}{2}} \\
&\quad + \beta_2\left(\int_0^{2\pi} |x'(t)|^2 dt\right)^{\frac{1}{2}} + \beta_3\left(\int_0^{2\pi} |x''(t)|^2 dt\right)^{\frac{1}{2}}\Bigg].
\end{aligned}
$$

Thus

$$
\begin{aligned}
(|b| - \beta_2)\left(\int_0^{2\pi} |x'(t)|^2 dt\right)^{\frac{1}{2}} &\le \sqrt{2\pi}(m+M) + \beta_1\left(\int_0^{2\pi} |x(t)|^2 dt\right)^{\frac{1}{2}} \\
&\quad + \beta_3\left(\int_0^{2\pi} |x''(t)|^2 dt\right)^{\frac{1}{2}},
\end{aligned} \tag{2.9}
$$

from which, together with (2.4), it implies that

$$
\begin{aligned}
&(|c| - \beta_1)(|b| - \beta_2)\left(\int_0^{2\pi} |x'(t)|^2 dt\right)^{\frac{1}{2}} \\
&\le \sqrt{2\pi}|c|(m+M) + \beta_3|c|\left(\int_0^{2\pi} |x''(t)|^2 dt\right)^{\frac{1}{2}} \\
&\quad + (2\pi\beta_1|c| + \beta_1\beta_2)\left(\int_0^{2\pi} |x'(t)|^2 dt\right)^{\frac{1}{2}}.
\end{aligned} \tag{2.10}
$$

In view of (2.8) and (2.10), we can obtain

$$
\begin{aligned}
&(|c|-\beta_1)(|b|-\beta_2)(|ac|-\beta_3|c|-\beta_1|a|)\left(\int_0^{2\pi}|x'(t)|^2dt\right)^{\frac{1}{2}}\\
&\le \sqrt{2\pi}|c|(m+M)(|ac|-\beta_3|c|-\beta_1|a|)+\sqrt{2\pi}\beta_3c^2(m+M)\\
&\quad+\Bigg\{(2\pi|c|\beta_1+\beta_1\beta_2)(|ac|-\beta_3|c|-\beta_1|a|)+\beta_3|c|\Bigg[|c|(2\pi\beta_1+\beta_2)\\
&\quad+\sqrt{(|ac|-\beta_3|c|-\beta_1|a|)|c|(|c|-\beta_1)}\Bigg]\Bigg\}\left(\int_0^{2\pi}|x'(t)|^2dt\right)^{\frac{1}{2}},
\end{aligned}
$$

from which, together with (2.2), it implies that there exists a positive constant R_1 such that

$$\int_0^{2\pi}|x'(t)|^2dt\le R_1. \tag{2.11}$$

By (2.6) and (2.11), there exists a positive constant R_2 such that

$$\int_0^{2\pi}|x''(t)|^2dt\le R_2. \tag{2.12}$$

From (2.4), (2.11) and (2.12), there exists a positive constant R_3 such that

$$\int_0^{2\pi}|x(t)|^2dt\le R_3. \tag{2.13}$$

Therefore, there exist three positive constants R_1^*, R_2^* and R_3^* such that $\forall t\in[0,2\pi]$,

$$|x(t)|\le R_1^* \quad |x'(t)|\le R_2^*, \quad |x''(t)|\le R_3^*.$$

Let $A=\max\{R_1^*,R_2^*,R_3^*,(m+M)/(|c|-\beta_1)\}$ and take $\Omega=\{x(t)\in X: |x|_2< A\}$. We now will show that N is L-compact on $\bar{\Omega}$. For any $x\in\bar{\Omega}$,

$$
\begin{aligned}
|QNx|_0&\le\frac{1}{2\pi}\int_0^{2\pi}[|b|R_2^*+|c|R_1^*+m+M+\beta_1R_1^*+\beta_2R_2^*+\beta_3R_3^*]dt\\
&=M_1,
\end{aligned}
$$

where $M_1 = |b|R_2^* + |c|R_1^* + m + M + \beta_1 R_1^* + \beta_2 R_2^* + \beta_3 R_3^*$. Hence, $QN(\bar{\Omega})$ is a bounded set in R. Obviously, $QNx : \bar{\Omega} \to Z$ is continuous. For $\forall z \in \mathrm{Im}\, L \cap Z$,

$$(K_P z)(t) = \int_0^t ds \int_0^s z(u)du - \frac{1}{2\pi} \int_0^{2\pi} dt \int_0^t ds \int_0^s z(u)du$$

is continuous with respect to z, and

$$|K_P z|_0 \leq \frac{8}{3}\pi^2 \max_{t\in[0,2\pi]} |z(t)|,$$

$$\begin{aligned}|K_P(I-Q)Nx|_0 &\leq \frac{8}{3}\pi^2|Nx|_0 + \frac{8}{3}\pi^2|QNx|_0 \\ &\leq \frac{16}{3}\pi^2|Nx|_0 \\ &\leq \frac{16}{3}\pi^2 M_1.\end{aligned}$$

For $\forall x \in \Omega$, we have

$$\begin{aligned}\left|\frac{d}{dt}(K_P(I-Q)Nx)\right|_0 &\leq \int_0^t |[(I-Q)Nx](t)|_0 dt \\ &\leq 2\pi|[(I-Q)Nx](t)|_0 \\ &\leq 4\pi\, |Nx|_0 \leq 4\pi M_1.\end{aligned}$$

Thus, the set $\{K_P(I-Q)Nx | x \in \bar{\Omega}\}$ is equicontinuous and uniformly bounded. Consequently, N is L-compact. This satisfies condition (a) in Lemma 2.1.

When $x \in \partial\Omega \cap KerL = \partial\Omega \cap R$, x is a constant with $|x| = A$. Then

$$\begin{aligned}QNx &= \frac{1}{2\pi}\int_0^{2\pi} [-bx'(t) - cx(t) - g(x(t-\tau_1), x'(t-\tau_2), x''(t-\tau_3)) + p(t)]dt \\ &= -cx - g(x,0,0) + \frac{1}{2\pi}\int_0^{2\pi} p(t)dt.\end{aligned}$$

Thus

$$\begin{aligned}|QNx|_0 &\geq |c|\left(|x| - \frac{|g(x,0,0)| + m}{|c|}\right) \\ &\geq |c|\left(A - \frac{m + M + \beta_1 A}{|c|}\right) > 0.\end{aligned}$$

Therefore, $QNx \neq 0, x \in \partial\Omega \cap R$.

Set for $0 \leq \mu \leq 1$

$$\phi(x,\mu) = \mu x(t) + (1-\mu)\Bigg[x(t) + g(x(t-\tau_1), x'(t-\tau_2), x''(t-\tau_3)) - \frac{1}{2\pi}\int_0^{2\pi} p(t)dt\Bigg].$$

When $x \in \partial\Omega \cap \operatorname{Ker} L$ and $\mu \in [0,1]$, x is a constant with $|x| = A$. Without loss of generality, we suppose $x = A$. Now we consider two possible cases: (1) $x = A, c > 0$; (2) $x = A, c < 0$.

(1). When $x = A$ and $c > 0$,

$$\begin{aligned}\phi(x,\mu) &= cA + (1-\mu)\left[g(A,0,0) - \frac{1}{2\pi}\int_0^{2\pi} p(t)dt\right] \\ &\geq c\left[A - \frac{1-\mu}{c}\left(|g(A,0,0)| + \frac{1}{2\pi}\int_0^{2\pi} |p(t)|dt\right)\right] \\ &\geq c\left(A - \frac{m+M+\beta_1 A}{c}\right) > 0;\end{aligned}$$

(2). When $x = A$ and $c < 0$,

$$\phi(x,\mu) \leq c\left(A - \frac{m+M+\beta_1 A}{|c|}\right) < 0.$$

Thus when $x = A, \phi(x,\mu) \neq 0$. Therefore,

$$\begin{aligned}\deg(QN, \Omega \cap KerL, 0) &= \deg\Bigg\{-cx(t) - g(x(t-\tau_1), x'(t-\tau_2), x''(t-\tau_3)) \\ &\qquad + \frac{1}{2\pi}\int_0^{2\pi} p(t)dt, \Omega \cap \operatorname{Ker} L, 0\Bigg\} \\ &= \deg(-cx, \Omega \cap \operatorname{Ker} L, 0) \neq 0.\end{aligned}$$

By now we know that Ω verifies all the requirements in Lemma 2.1. This completes the proof of Theorem 2.1.

Example The second order neutral delay differential equation

$$10x''(t) + 100x'(t) + 5x(t) + \frac{1 + \frac{1}{2}x(t-1) + \frac{1}{2}x'(t-2) + \frac{1}{100}x''(t-3)}{1 + x^2(t-1)}$$

$$= sint, \tag{2.14}$$

satisfies all conditions in Theorem 2.1. Therefore, Eq.(2.14) has at least one 2π-periodic solution.

Acknowledgement

The authors would like to thank the referee for helpful suggestions.

References

1. W. Y. Ding, *Fixed points of twist mappings and periodic solutions of ordinary differential equations*, Acta. Math. Sinica **25** (1981), 227-235.
2. D. E. Leach, *On Poincare's perturbation theorem and a theorem of W. S. Loud*, J. Differential Equations **7** (1970), 34-53.
3. R. Reissig, *Contractive mappings and periodically perturbed, non-conservative systems*, Atti Accad Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. **58** (1975), 696-702.
4. T. R.Ding, R. Iannacci and F. Zanolin, *On periodic solutions of sublinear Duffing equations*, J. Math. Anal. Appl. **158** (1991), 316-332.
5. T. R. Ding, *Nonlinear oscillations at a point of resonance*, Sci. Sinica(Chinese), Series A **1** (1982), 1-13.
6. T. R. Ding, An infinite class of periodic solutions of periodically perturbed Duffing equation at resonance, Proc. Amer. Math. Soc. **86** (1982), 47-54.
7. T. R. Ding and F. Zanolin, *Time-maps for solvability of periodically perturbed nonlinear Duffing equations*, Nonlinear Analysis (TMA) **17(7)** (1991), 635-653.
8. D. B. Qian, *Time-maps and Duffing equations across resonance points*, Sci. Sinica (Chinese), Series A **23(5)** (1993), 471-479.
9. A. C. Lazer and D. E. Leach, *Bounded perturbations of forced harmonic oscillations at resonance*, Ann. Mat. Pura. Appl. **82** (1969), 49-68.
10. L. Césari, *Nonlinear problems across a point of resonance for non-self-adjoint system*, Nonlinear Analysis (A Collection of Papers in Honor of Erich H.Rothe), edited by L.Césari et al., Academic Press, New York, 1978, pp.43-67..
11. W. Layton, *Periodic solutions of nonlinear delay equations*, J. Math. Anal. Appl. **77** (1980), 198-204.

12. R. Iannacci and M. N. Nkashama, *On periodic solutions of forced second order differential equations with a deviating argument*, Lecture Notes in Math. 1151, Springer-Verlag, 1984, 224-232..

13. J. Mawhin, J. R. Ward Jr, *Nonuniform nonresonance conditions at the two first eigenvalues for periodic solutions of forced Lienard and Duffing equations*, Rocky Mountain J. Math. **12(4)** (1982), 643-654.

14. X. K. Huang, *2π-periodic solutions of conservative systms with a deviating argument*, J. Sys. Sci. & Math. Scis.(Chinese), **9(4)** (1989), 298-308.

15. X. K. Huang and Z. G. Xiang, *2π-periodic solutions of Duffing equations with a deviating argument*, Chinese Sci. Bull. (Chinese) **39(3)** (1994), 201-203.

16. R. E. Gaines and J. L. Mawhin, *Coincidence Degree, and Nonlinear Differential Equations*, Springer-Verlag, Berlin, 1977..

17. Ma. Shiwang, Wang Zhicheng and Yu Jianshe, *Coincidence degree and periodic solutions of Duffing equations*, Nonlinear Analysis(TMA) **34** (1998), 443-460.

18. J. Mawhin, *Equivalence theorems for nonlinear operator equations and coincidence degree theory for some mappings in locally convex topological vector spaces*, J. Differential Equations **12** (1972), 610-636.

19. K. Deimling, *Nonlinear Functional Analysis*, Spring-Verlag, New York, 1985.

20. J. K. Hale, *Ordinary Differential Equations*, Wiley-Interscience, New York, 1969.

21. Jack Hale, *Theory of Functional Differential Equations*, Springer-Verlag, New York, 1977.

22. T. A. Burton and Tetsuo Furumochi, *Periodic solution of a neutral integro-differential equation*, Funkcialaj Ekvacioj **41** (1998), 327-336.