Electronic Journal of Qualitative Theory of Differential Equations
2017, No. **70**, 1–15; https://doi.org/10.14232/ejqtde.2017.1.70 www.math.u-szeged.hu/ejqtde/

Resonant semilinear Robin problems with a general potential

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Received 8 August 2017, appeared 27 October 2017
Communicated by Dimitri Mugnai

Abstract. We consider a semilinear Robin problem driven by the Laplacian plus an indefinite and unbounded potential. The reaction term is a Carathéodory function which is resonant with respect to any nonprincipal eigenvalue both at $\pm\infty$ and 0. Using a variant of the reduction method, we show that the problem has at least two nontrivial smooth solutions.

Keywords: resonant problem, reduction method, regularity theory, indefinite and unbounded potential, local linking.

2010 Mathematics Subject Classification: 35J20, 35J60.

1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following semilinear Robin problem:

$$\left\{ \begin{array}{ll} -\Delta u(z) + \xi(z)u(z) = f(z,u(z)) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} + \beta(z)u = 0 & \text{in } \partial\Omega\,. \end{array} \right\} \tag{1.1}$$

In this problem, the potential function $\xi(\cdot)$ is unbounded and indefinite (that is, sign-changing). So, in problem (1.1) the differential operator (on the left-hand side of the equation), is not coercive. The reaction term $f(z,x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \mapsto f(z,x)$ is measurable and for almost all $z \in \Omega$, $x \mapsto f(z,x)$ is continuous) and $f(z,\cdot)$ exhibits linear growth as $x \to \pm\infty$. In fact, we can have resonance with respect to any nonprincipal eigenvalue of $-\Delta + \xi(z)I$ with the Robin boundary condition. This general structure of the

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reaction term makes using variational methods problematic. To overcome these difficulties, we develop a variant of the so-called "reduction method", originally due to Amann [1] and Castro & Lazer [3]. However, in contrast to the aforementioned works, the particular features of our problem lead to a reduction on an infinite dimensional subspace and this is a source of additional technical difficulties. In the boundary condition, $\frac{\partial u}{\partial n}$ is the normal derivative defined by extension of the continuous linear map

$$u \mapsto \frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N} \quad \text{for all } u \in C^1(\overline{\Omega}),$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. The boundary coefficient $\beta \in W^{1,\infty}(\partial\Omega)$ satisfies $\beta(z) \geq 0$ for all $z \in \partial\Omega$. We can have $\beta \equiv 0$, which corresponds to the Neumann problem.

Recently, there have appeared existence and multiplicity results for semilinear elliptic problems with general potential. We mention the works of Hu & Papageorgiou [9], Kyritsi & Papageorgiou [10], Papageorgiou & Papalini [12], Qin, Tang & Zhang [17] (Dirichlet problems), Gasinski & Papageorgiou [6], Papageorgiou & Rădulescu [13,15] (Neumann problems), and for Robin problems the works of Shi & Li [18] (superlinear reaction), D'Aguì, Marano & Papageorgiou [4] (asymmetric reaction), Hu & Papageorgiou (logistic reaction), and Papageorgiou & Rădulescu [16] (reaction with zeros). In all the aforementioned works the conditions are in many respects more restrictive or different and consequently, the mathematical tools are different. It seems that our present paper is the first to use this variant of the reduction method on Robin problems.

2 Mathematical background

Let X be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Given $\varphi \in C^1(X, \mathbb{R})$, we say that φ satisfies the "Cerami condition" (the "C-condition" for short), if the following property holds

"Every sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and

$$(1 + \|u_n\|)\varphi'(u_n) \to 0 \text{ in } X^*,$$

admits a strongly convergent subsequence".

This is a compactness-type condition on the functional φ and is more general that the usual Palais–Smale condition. The two notions are equivalent when φ is bounded below (see Motreanu, Motreanu & Papageorgiou [11, p. 104]).

Our multiplicity result will use the following abstract "local linking" theorem of Brezis & Nirenberg [2].

Theorem 2.1. *Let X be a Banach space such that $X = Y \oplus V$ with $\dim Y < +\infty$. Assume that $\varphi \in C^1(X, \mathbb{R})$ satisfies the C-condition, and is bounded below, $\varphi(0) = 0$, $\inf_X \varphi < 0$, and there exists $\rho > 0$ such that*

$$\varphi(y) \leq 0 \quad \text{for all } y \in Y \text{ with } \|y\| \leq \rho,$$
$$\varphi(v) \geq 0 \quad \text{for all } v \in V \text{ with } \|v\| \leq \rho$$

(that is, φ has a local linking at $u = 0$ with respect to the direct sum $Y \oplus V$). Then φ has at least two nontrivial critical points.

Remark 2.2. The result is true even if one of the component subspaces Y or V is trivial. Moreover, if $\dim V = 0$, then we can allow Y to be infinite dimensional.

We will use the following spaces:

- the Sobolev space $H^1(\Omega)$;
- the Banach space $C^1(\overline{\Omega})$; and
- the "boundary" Lebesgue spaces $L^r(\partial\Omega)$ $1 \leq r \leq \infty$.

The Sobolev space $H^1(\Omega)$ is a Hilbert space with the following inner product

$$(u,v) = \int_\Omega uvdz + \int_\Omega (Du, Dv)_{\mathbb{R}^N} dz \quad \text{for all } u, v \in H^1(\Omega).$$

By $\|\cdot\|$ we denote the norm corresponding to this inner product, that is,

$$\|u\| = \left[\|u\|_2^2 + \|Du\|_2^2\right]^{1/2} \quad \text{for all } u, v \in H^1(\Omega).$$

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure denoted by $\sigma(\cdot)$. Using this measure on $\partial\Omega$, we can define in the usual way the Lebesgue spaces $L^r(\partial\Omega)$, $1 \leq r \leq \infty$. From the theory of Sobolev spaces we know that there exists a unique continuous linear map $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$, known as the "trace map", which satisfies

$$\gamma_0(u) = u|_{\partial\Omega} \quad \text{for all } u \in H^1(\Omega) \cap C(\overline{\Omega}).$$

So, the trace map assigns "boundary values" to any Sobolev function (not just to the regular ones). This map is compact into $L^r(\partial\Omega)$ for all $r \in \left[1, \frac{2(N-1)}{N-2}\right)$ if $N \geq 3$ and into $L^r(\partial\Omega)$ for all $r \geq 1$ if $N = 1, 2$. Also, we have

$$\ker \gamma_0 = H_0^1(\Omega) \quad \text{and} \quad \operatorname{im} \gamma_0 = H^{\frac{1}{2},2}(\partial\Omega).$$

In what follows, for the sake of notational simplicity, we shall drop the use the trace map γ_0. The restrictions of all Sobolev functions on $\partial\Omega$, are understood in the sense of traces.

Next, we recall some basic facts about the spectrum of the differential operator $-\Delta + \xi(z)I$ with the Robin boundary condition. So, we consider the following linear eigenvalue problem:

$$\left\{ \begin{array}{ll} -\Delta u(z) + \xi(z)u(z) = \hat{\lambda} u(z) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} + \beta(z)u = 0 & \text{on } \partial\Omega \end{array} \right\} \tag{2.1}$$

Our conditions on the data of (2.1) are the following:

- $\xi \in L^s(\Omega)$ with $s > N$; and
- $\beta \in W^{1,\infty}(\partial\Omega)$ with $\beta(z) \geq 0$ for all $z \in \partial\Omega$.

Let $\gamma : H^1(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\gamma(u) = \|Du\|_2^2 + \int_\Omega \xi(z)u^2 dz + \int_{\partial\Omega} \beta(z)u^2 d\sigma \quad \text{for all } u \in H^1(\Omega).$$

By D'Aguì, Marano & Papageorgiou [4], we know that there exists $\mu > 0$ such that

$$\gamma(u) + \mu\|u\|_2^2 \geq c_0\|u\|^2 \quad \text{for all } u \in H^1(\Omega), \text{ and some } c_0 > 0. \tag{2.2}$$

Using (2.2) and the spectral theorem for compact self-adjoint operators on a Hilbert space, we produce the spectrum $\sigma_0(\xi)$ of (2.1) such that $\sigma_0(\xi) = \{\hat{\lambda}_k\}_{k\geq 1}$ is a sequence of distinct eigenvalues with $\hat{\lambda}_k \to +\infty$ as $k \to +\infty$. By $E(\hat{\lambda}_k)$ (for all $k \in \mathbb{N}$), we denote the eigenspace corresponding to the eigenvalue $\hat{\lambda}_k$. We know that $E(\hat{\lambda}_k)$ is finite dimensional. Moreover, the regularity theory of Wang [19] implies that $E(\hat{\lambda}_k) \subseteq C^1(\overline{\Omega})$ for all $k \in \mathbb{N}$. The Sobolev space $H^1(\Omega)$ admits the following orthogonal direct sum decomposition

$$H^1(\Omega) = \overline{\bigoplus_{k\geq 1} E(\hat{\lambda}_k)}.$$

The elements of $\sigma_0(\xi)$ have the following properties:

- $\hat{\lambda}_1$ is simple (that is, $\dim E(\hat{\lambda}_1) = 1$);
- $\hat{\lambda}_1 = \inf\left[\frac{\gamma(u)}{\|u\|_2^2} : u \in H^1(\Omega), u \neq 0\right]$; and (2.3)
- $\hat{\lambda}_m = \inf\left[\frac{\gamma(u)}{\|u\|_2^2} : u \in \overline{\bigoplus_{k\geq m} E(\hat{\lambda}_k)}, u \neq 0\right]$
$= \sup\left[\frac{\gamma(u)}{\|u\|_2^2} : u \in \bigoplus_{k=1}^{m} E(\hat{\lambda}_k), u \neq 0\right]$ for $m \geq 2$. (2.4)

The infimum in (2.3) is realized on $E(\hat{\lambda}_1)$, while both the infimum and supremum in (2.4) are realized on $E(\hat{\lambda}_m)$. It follows that the elements of $E(\hat{\lambda}_1)$ have fixed signs, while those of $E(\hat{\lambda}_m)$ ($m \geq 2$) are nodal (sign-changing). The eigenspaces have the so-called "Unique Continuation Property" (UCP for short) which says that if $u \in E(\hat{\lambda}_k)$ and $u(\cdot)$ vanishes on a set of positive Lebesgue measure, then $u \equiv 0$. As a consequence of the UCP, we get the following useful inequalities (see D'Aguì, Marano & Papageorgiou [4]).

Lemma 2.3.

(a) If $\eta \in L^\infty(\Omega)$, $\eta(z) \geq \hat{\lambda}_m$ for almost all $z \in \Omega$, $m \in \mathbb{N}$ and $\eta \neq \hat{\lambda}_m$, then there exists $c_1 > 0$ such that

$$\gamma(u) - \int_\Omega \eta(z)u^2 dz \leq -c_1\|u\|^2 \quad \text{for all } u \in \bigoplus_{k=1}^{m} E(\hat{\lambda}_k).$$

(b) If $\eta \in L^\infty(\Omega)$, $\eta(z) \leq \hat{\lambda}_m$ for almost all $z \in \Omega$, $m \in \mathbb{N}$ and $\eta \neq \hat{\lambda}_m$ then there exists $c_2 > 0$ such that

$$\gamma(u) - \int_\Omega \eta(z)u^2 dz \geq c_2\|u\|^2 \quad \text{for all } u \in \overline{\bigoplus_{k\geq m} E(\hat{\lambda}_k)}.$$

Given $m \in \mathbb{N}$, let $H_- = \bigoplus_{k=1}^{m} E(\hat{\lambda}_k)$, $H^0 = E(\hat{\lambda}_{m+1})$, $H_+ = \overline{\bigoplus_{k\geq m+2} E(\hat{\lambda}_k)}$. We have the following orthogonal direct sum decomposition

$$H^1(\Omega) = H_- \oplus H^0 \oplus H_+.$$

So, every $u \in H^1(\Omega)$ admits a unique sum decomposition of the form

$$u = \bar{u} + u^0 + \hat{u} \quad \text{with} \quad \bar{u} \in H_-,\ u^0 \in H^0,\ \hat{u} \in H_+.$$

Also, we set

$$V = H^0 \oplus H_+.$$

Finally, let us fix our notation. By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$ and by $A \in \mathcal{L}(H^1(\Omega), H^1(\Omega)^*)$ the linear operator defined by

$$\langle A(u), h\rangle = \int_\Omega (Du, Dh)_{\mathbb{R}^N} dz \quad \text{for all } u, h \in H^1(\Omega)$$

(by $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair $(H^1(\Omega)^*, H^1(\Omega))$). Also, given a measurable function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ (for example a Carathéodory function), we set

$$N_f(u)(\cdot) = f(\cdot, u(\cdot)) \quad \text{for all } u \in H^1(\Omega)$$

(the Nemytski map corresponding to f). Evidently, $z \mapsto N_f(u)(z)$ is measurable. For $\varphi \in C^1(X, \mathbb{R})$, we set

$$K_\varphi = \{u \in X : \varphi'(u) = 0\}$$

(the critical set of φ).

3 Pair of nontrivial solutions

The hypotheses on the data of (1.1) are the following:

- $H(\xi)$: $\xi \in L^s(\Omega)$ with $s > N$; and
- $H(\beta)$: $\beta \in W^{1,\infty}(\partial\Omega)$ with $\beta(z) \geq 0$ for all $z \in \partial\Omega$.

Remark 3.1. We can have $\beta \equiv 0$ and this case corresponds to the Neumann problem.

$H(f)$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for almost all $z \in \Omega$ and

(i) $|f(z, x)| \leq a(z)(1 + |x|)$ for almost all $z \in \Omega$, and all $x \in \mathbb{R}$ with $a \in L^\infty(\Omega)_+$;

(ii) there exist $m \in \mathbb{N}$ and $\eta \in L^\infty(\Omega)$ such that

$$\begin{aligned} &\eta(z) \geq \hat{\lambda}_m \quad \text{for almost all } z \in \Omega, \eta \not\equiv \hat{\lambda}_m, \\ &(f(z, x) - f(z, x'))(x - x') \geq \eta(z)(x - x')^2 \quad \text{for almost all } z \in \Omega, \text{ and all } x, x' \in \mathbb{R}; \end{aligned}$$

(iii) if $F(z, x) = \int_0^x f(z, s)ds$, then $\limsup_{x\to\pm\infty} \frac{2F(z,x)}{x^2} \leq \hat{\lambda}_{m+1}$ and

$$\lim_{x\to\pm\infty} [f(z, x)x - 2F(z, x)] = +\infty \quad \text{uniformly for almost all } z \in \Omega;$$

(iv) there exist $l \in \mathbb{N}$, $l \geq m + 2$, a function $\vartheta \in L^\infty(\Omega)$ and $\delta > 0$ such that

$$\begin{aligned} &\vartheta(z) \leq \hat{\lambda}_l \quad \text{for almost all } z \in \Omega, \ \vartheta \not\equiv \hat{\lambda}_l, \\ &\hat{\lambda}_{l-1}x^2 \leq f(z, x)x \leq \vartheta(z)x^2 \quad \text{for almost all } z \in \Omega, \text{ and all } |x| \leq \delta. \end{aligned}$$

Let $\varphi : H^1(\Omega) \to \mathbb{R}$ be the energy (Euler) functional for problem (1.1) defined by

$$\varphi(u) = \frac{1}{2}\gamma(u) - \int_\Omega F(z, u)dz \quad \text{for all } u \in H^1(\Omega).$$

Evidently, $\varphi \in C^1(H^1(\Omega))$.

Recall that

$$H^1(\Omega) = H_- \oplus H^0 \oplus H_+$$

with $H_- = \overset{m}{\underset{k=1}{\oplus}} E(\hat{\lambda}_k)$. $H^0 = E(\hat{\lambda}_{m+1})$, $H_+ = \overline{\underset{k\geq m+2}{\oplus} E(\hat{\lambda}_k)}$ and

$$V = H^0 \oplus H_+.$$

The next proposition is crucial for the implementation of the reduction method.

Proposition 3.2. *If hypotheses $H(\xi)$, $H(\beta)$, $H(f)$ hold, then there exists a continuous map $\tau : V \to H_-$ such that*

$$\varphi(v + \tau(v)) = \max[\varphi(v + y) : y \in H_-] \quad \textit{for all } v \in V.$$

Proof. We fix $v \in V$ and consider the C^1-functional $\varphi_v : H^1(\Omega) \to \mathbb{R}$ defined by

$$\varphi_v(u) = \varphi(v + u) \quad \text{for all } u \in H^1(\Omega).$$

By $i_{H_-} : H_- \to H^1(\Omega)$ we denote the embedding of H_- into $H^1(\Omega)$. Let

$$\hat{\varphi}_v = \varphi_v \circ i_{H_-}.$$

By the chain rule, we have

$$\hat{\varphi}'_v = p_{H^*_-} \circ \varphi'_v, \tag{3.1}$$

with $p_{H^*_-}$ being the orthogonal projection of the Hilbert space $H^1(\Omega)$ onto H^*_-. By $\langle \cdot, \cdot \rangle_{H_-}$ we denote the duality brackets for the pair (H^*_-, H_-). For y, $y' \in H_-$, we have

$$\begin{aligned}
&\langle \hat{\varphi}'_v(y) - \hat{\varphi}'_v(y'), y - y' \rangle_{H_-} \\
&\quad = \langle \varphi'_v(y) - \varphi'_v(y'), y - y' \rangle \quad \text{(see (3.1))} \\
&\quad = \gamma(y - y') - \int_\Omega (f(z, v + y) - f(z, v + y'))(y - y')dz \\
&\quad \leq \gamma(y - y') - \int_\Omega \eta(z)(y - y')^2 dz \quad \text{(see hypothesis } H(f)\,(ii)) \\
&\quad \leq -c_1 \|y - y'\|^2 \quad \text{(see Lemma 2.3).}
\end{aligned} \tag{3.2}$$

This implies that

$$-\hat{\varphi}'_v \text{ is strongly monotone and therefore } -\hat{\varphi}_v \text{ is strictly convex.} \tag{3.3}$$

We have

$$\begin{aligned}
\langle \hat{\varphi}'_v(y), y \rangle_{H_-} &= \langle \varphi'_v(y), y \rangle \\
&= \langle \varphi'_v(y) - \varphi'_v(0), y \rangle + \langle \varphi'_v(0), y \rangle \\
&\leq -c_1 \|y\|^2 + c_3 \|y\| \quad \text{for some } c_3 > 0 \text{ (see (3.2))}, \\
\Rightarrow -\hat{\varphi}'_v &\quad \text{is coercive.}
\end{aligned} \tag{3.4}$$

The continuity and monotonicity of $-\hat{\varphi}'_v$ (see (3.3)), imply that

$$-\hat{\varphi}'_v \text{ is maximal monotone.} \tag{3.5}$$

However, a maximal monotone and coercive map is surjective (see, for example, Hu & Papageorgiou [8, p. 322]). So, we infer from (3.4) and (3.5) that there is a unique $y_0 \in H_-$ such that

$$\hat{\varphi}'_v(y_0) = 0 \quad \text{(see (3.3))}. \tag{3.6}$$

Moreover, y_0 is the unique maximizer of the function $\hat{\varphi}_v$. So, we can define the map $\tau : V \to H_-$ by setting $\tau(v) = y_0$. Then we have

$$\varphi(v + \tau(v)) = \max[\varphi(v + y) : y \in H_-], \tag{3.7}$$

$$\Rightarrow p_{H^*_-}\varphi'(v + \tau(v)) = 0 \quad \text{(see (3.6) and (3.1))}. \tag{3.8}$$

We need to show that the map $\tau : V \to H_-$ is continuous. To this end, let $v_n \to v$ in V. First, note that if $\bar{u} \in H_-$, then

$$\begin{aligned}
\varphi(\bar{u}) &= \frac{1}{2}\gamma(\bar{u}) - \int_\Omega F(z, \bar{u})dz \\
&\leq \frac{1}{2}\gamma(\bar{u}) - \frac{1}{2}\int_\Omega \eta(z)\bar{u}^2 dz \quad \text{(see hypothesis } H(f)\,(ii)) \\
&\leq -c_1\|\bar{u}\|^2 \quad \text{(see Lemma 2.3)}, \\
\Rightarrow \tau(0) &= 0.
\end{aligned}$$

Since $\varphi \in C^1(H^1(\Omega))$ and $\varphi'(u) = \gamma'(u) - N_f(u)$, it follows that φ' is bounded on bounded sets of $H^1(\Omega)$. Therefore

$$\|\varphi'(v_n)\|_* \leq c_4$$

with $c_4 > 0$ independent of $n \in \mathbb{N}$ (recall that $v_n \to v$ in $H^1(\Omega)$).

Then we have

$$\begin{aligned}
0 &= \langle \varphi'(v_n + \tau(v_n)), \tau(v_n)\rangle \quad \text{(see (3.8))} \\
&= \langle \varphi'(v_n + \tau(v_n)) - \varphi'(v_n + \tau(0)), \tau(v_n)\rangle + \langle \varphi'(v_n + \tau(0)), \tau(v_n)\rangle \\
&\leq -c_1\|\tau(v_n)\|^2 + c_4\|\tau(v_n)\|, \quad \text{for all } n \in \mathbb{N} \text{ (see (3.2))} \\
\Rightarrow \{\tau(v_n)\}_{n\geq 1} &\subseteq H_- \text{ is bounded.}
\end{aligned}$$

By passing to a suitable subsequence if necessary and using the finite dimensionality of H_-, we can infer that

$$\tau(v_n) \to \hat{y} \quad \text{in } H^1(\Omega),\ \hat{y} \in H_-. \tag{3.9}$$

We have

$$\begin{aligned}
&\varphi(v_n + \tau(v_n)) \leq \varphi(v_n + y) \text{ for all } y \in H_-, \text{ all } n \in \mathbb{N} \quad \text{(see (3.7))}, \\
\Rightarrow\ &\varphi(v + \hat{y}) \leq \varphi(v + y) \text{ for all } y \in H_- \quad \text{(see (3.9) and recall that } \varphi \text{ is continuous)}, \\
\Rightarrow\ &\hat{y} = \tau(v).
\end{aligned}$$

By the Urysohn convergence criterion (see, for example, Gasinski & Papageorgiou [7, p. 33]), we have for the original sequence

$$\begin{aligned}
\tau(v_n) &\to \tau(v) \quad \text{in } H_-, \\
\Rightarrow \tau(\cdot) &\text{ is continuous.}
\end{aligned}$$

□

Consider the functional $\tilde{\varphi} : V \to \mathbb{R}$ defined by

$$\tilde{\varphi}(v) = \varphi(v + \tau(v)) \quad \text{for all } v \in V.$$

Proposition 3.3. *If hypotheses $H(\xi)$, $H(\beta)$ and $H(f)$ hold, then $\tilde{\varphi} \in C^1(V,\mathbb{R})$ and $\tilde{\varphi}'(v) = p_{V^*}\varphi'(v+\tau(v))$ for all $v \in V$ (here, p_{V^*} denotes the orthogonal projection of the Hilbert space $H^1(\Omega)^*$ onto V^*).*

Proof. Let $v, h \in V$ and $t > 0$. We have

$$\begin{aligned} &\frac{1}{t}\left[\tilde{\varphi}(v+th) - \tilde{\varphi}(v)\right] \\ &\qquad \geq \frac{1}{t}\left[\varphi(v+th+\tau(v)) - \varphi(v+\tau(v))\right] \quad \text{(see (3.7))}, \\ &\Rightarrow \liminf_{t\to 0^+} \frac{1}{t}\left[\tilde{\varphi}(v+th) - \tilde{\varphi}(v)\right] \geq \left\langle \varphi'(v+\tau(v)), h\right\rangle. \end{aligned} \tag{3.10}$$

Also, we have

$$\begin{aligned} &\frac{1}{t}\left[\tilde{\varphi}(v+th) - \tilde{\varphi}(v)\right] \\ &\qquad \leq \frac{1}{t}\left[\varphi(v+th+\tau(v+th)) - \varphi(v+\tau(v+th))\right] \\ &\Rightarrow \limsup_{t\to 0^+} \frac{1}{t}\left[\tilde{\varphi}(v+th) - \tilde{\varphi}(v)\right] \leq \left\langle \varphi'(v+\tau(v)), h\right\rangle \end{aligned} \tag{3.11}$$

(recall that $\tau(\cdot)$ is continuous, see Proposition 3.2 and that $\varphi \in C^1(H^1(\Omega),\mathbb{R})$).

From (3.10) and (3.11) it follows that

$$\lim_{t\to 0^+} \frac{1}{t}\left[\tilde{\varphi}(v+th) - \tilde{\varphi}(v)\right] = \left\langle \varphi'(v+\tau(v)), h\right\rangle \quad \text{for all } v, h \in V. \tag{3.12}$$

Similarly we can show that

$$\lim_{t\to 0^-} \frac{1}{t}\left[\tilde{\varphi}(v+th) - \tilde{\varphi}(v)\right] = \left\langle \varphi'(v+\tau(v)), h\right\rangle \quad \text{for all } v, h \in V. \tag{3.13}$$

From (3.12) and (3.13) we conclude that

$$\tilde{\varphi} \in C^1(V,\mathbb{R}) \text{ and } \tilde{\varphi}'(v) = p_{V^*}\varphi'(v+\tau(v)) \text{ for all } v \in V. \qquad \square$$

Proposition 3.4. *If hypotheses $H(\xi), H(\beta), H(f)$ hold, then $v \in K_{\tilde{\varphi}}$ if and only if $v + \tau(v) \in K_{\varphi}$.*

Proof. $\Leftarrow$ Follows from Proposition 3.3.

$\Rightarrow$ Let $v \in K_{\tilde{\varphi}}$. Then

$$\begin{aligned} &0 = \tilde{\varphi}'(v) = p_{V^*}\varphi'(v+\tau(v)) \quad \text{(see Proposition 3.3)}, \\ &\Rightarrow \varphi'(v+\tau(v)) \in H^*_- \quad \text{(recall that } H^1(\Omega)^* = H^*_- \oplus V^*). \end{aligned} \tag{3.14}$$

On the other hand, from (3.8) we have

$$\begin{aligned} &p_{H^*_-}\varphi'(v+\tau(v)) = 0, \\ &\Rightarrow \varphi'(v+\tau(v)) \in V^*. \end{aligned} \tag{3.15}$$

But $H^*_- \cap V^* = \{0\}$. So, it follows from (3.14) and (3.15) that

$$\begin{aligned} &\varphi'(v+\tau(v)) = 0, \\ &\Rightarrow v + \tau(v) \in K_{\varphi}. \end{aligned}$$

$\square$

Proposition 3.5. *If hypotheses $H(\xi), H(\beta), H(f)$ hold, then $\tilde{\varphi}$ is coercive.*

Proof. Let $\psi = \varphi|_V$. Evidently, $\psi \in C^1(V, \mathbb{R})$ and by the chain rule we have

$$\psi' = p_{V^*} \circ \varphi'. \tag{3.16}$$

Claim 3.6. *ψ satisfies the C-condition.*

Let $\{v_n\}_{n\geq 1} \subseteq V$ be a sequence such that

$$|\psi(v_n)| \leq M_1 \quad \text{for some } M_1 > 0, \text{ and all } n \in \mathbb{N}, \tag{3.17}$$

$$(1 + \|v_n\|)\psi'(v_n) \to 0 \quad \text{in } V^* \text{ as } n \to \infty. \tag{3.18}$$

From (3.18) we have

$$\begin{aligned} &|\langle \psi'(v_n), h\rangle_V| \leq \frac{\epsilon_n \|h\|}{1 + \|v_n\|} \quad \text{for all } h \in V,\ n \in \mathbb{N}, \text{ with } \epsilon_n \to 0^+, \\ \Rightarrow\ &|\langle \varphi'(v_n), h\rangle| \leq \frac{\epsilon_n \|h\|}{1 + \|v_n\|} \quad \text{for all } h \in V,\ n \in \mathbb{N} \quad \text{(see (3.16))}. \end{aligned} \tag{3.19}$$

In (3.19) we choose $h = v_n \in V$ and obtain

$$\gamma(v_n) - \int_\Omega f(z, v_n) v_n dz \leq \epsilon_n \quad \text{for all } n \in \mathbb{N}. \tag{3.20}$$

We show that $\{v_n\}_{n\geq 1} \subseteq V$ is bounded. Arguing by contradiction, suppose that

$$\|v_n\| \to \infty. \tag{3.21}$$

Let $\hat{w}_n = \frac{v_n}{\|v_n\|}$, $n \in \mathbb{N}$. Then $\hat{w}_n \in V$, $\|\hat{w}_n\| = 1$ for all $n \in \mathbb{N}$. By passing to a suitable subsequence if necessary, we may assume that

$$\hat{w}_n \xrightarrow{w} \hat{w} \quad \text{in } H^1(\Omega) \quad \text{and} \quad \hat{w}_n \to \hat{w} \quad \text{in } L^2(\Omega) \text{ and } L^2(\partial\Omega). \tag{3.22}$$

Hypotheses $H(f)$ imply that

$$|f(z, x)| \leq c_5|x| \quad \text{for almost all } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ and some } c_5 > 0. \tag{3.23}$$

By (3.19) we have

$$\left|\langle \gamma'(\hat{w}_n), h\rangle - \int_\Omega \frac{N_f(v_n)}{||v_n||} h dz\right| \leq \frac{\epsilon_n \|h\|}{(1 + \|v_n\|)\|v_n\|} \quad \text{for all } n \in \mathbb{N},\ h \in H^1(\Omega). \tag{3.24}$$

From (3.23) and (3.22) we see that

$$\left\{\frac{N_f(v_n)}{\|v_n\|}\right\}_{n\geq 1} \subseteq L^2(\Omega) \quad \text{is bounded.} \tag{3.25}$$

So, if in (3.24) we choose $h = \hat{w}_n - \hat{w} \in H^1(\Omega)$, pass to the limit as $n \to \infty$ and use (3.21), (3.22) and (3.25), then

$$\begin{aligned} &\lim_{n\to\infty} \langle A(\hat{w}_n), \hat{w}_n - \hat{w}\rangle = 0, \\ \Rightarrow\ &\|D\hat{w}_n\|_2 \to \|D\hat{w}\|_2, \\ \Rightarrow\ &\hat{w}_n \to \hat{w} \quad \text{in } H^1(\Omega) \\ &\quad \text{(by the Kadec–Klee property, see Gasinski \& Papageorgiou [5, p. 911])}, \\ \Rightarrow\ &\|\hat{w}\| = 1 \text{ and so } \hat{w} \neq 0. \end{aligned}$$

Let $\Omega_0 = \{z \in \Omega : \hat{w}(z) \neq 0\}$. Then $|\Omega_0|_N > 0$ and

$$v_n(z) \to \pm\infty \quad \text{for almost all } z \in \Omega_0 \text{ (see (3.21))}.$$

Hypothesis $H(f)$ (iii) implies that

$$f(z, v_n(z))v_n(z) - 2F(z, v_n(z)) \to +\infty \quad \text{for almost all } z \in \Omega_0. \tag{3.26}$$

From (3.26) via Fatou's lemma (hypothesis $H(f)$ (iii) permits its use), we have

$$\int_{\Omega_0} [f(z, v_n)v_n - 2F(z, v_n)]dz \to +\infty. \tag{3.27}$$

Using hypothesis $H(f)$ (iii), we see that we can find $M_2 > 0$ such that

$$f(z, x)x - 2F(z, x) \geq 0 \quad \text{for almost all } z \in \Omega, \text{ and all } |x| \geq M_2. \tag{3.28}$$

So, we have

$$\begin{aligned}
&\int_{\Omega\setminus\Omega_0} [f(z, v_n)v_n - 2F(z, v_n)]dz \\
&\quad = \int_{(\Omega\setminus\Omega_0)\cap\{|v_n|\geq M_2\}} [f(z, v_n)v_n - 2F(z, v_n)]dz \\
&\qquad + \int_{(\Omega\setminus\Omega_0)\cap\{|v_n|< M_2\}} [f(z, v_n)v_n - 2F(z, v_n)]dz \\
&\quad \geq \int_{(\Omega\setminus\Omega_0)\cap\{|v_n|< M_2\}} [f(z, v_n)v_n - 2F(z, v_n)]dz \quad \text{(see (3.28))} \\
&\quad \geq -M_3 \quad \text{for some } M_3 > 0, \text{ and all } n \in \mathbb{N} \quad \text{(see hypothesis } H(f)\ (i)).
\end{aligned}$$

Then

$$\begin{aligned}
&\int_{\Omega} [f(z, v_n)v_n - 2F(z, v_n)]dz \\
&\qquad = \int_{\Omega_0} [f(z, v_n)v_n - 2F(z, v_n)]dz + \int_{\Omega\setminus\Omega_0} [f(z, v_n)v_n - 2F(z, v_n)]dz \\
&\qquad \geq \int_{\Omega_0} [f(z, v_n)v_n - 2F(z, v_n)]dz - M_3 \quad \text{for all } n \in \mathbb{N} \\
&\Rightarrow \int_{\Omega} [f(z, v_n)v_n - 2F(z, v_n)]dz \to +\infty \quad \text{as } n \to \infty \quad \text{(see (3.27))}.
\end{aligned} \tag{3.29}$$

From (3.19) with $h = v_n \in H^1(\Omega)$, we have

$$-\gamma(v_n) + \int_{\Omega} f(z, v_n)v_n dz \leq \epsilon_n \quad \text{for all } n \in \mathbb{N}. \tag{3.30}$$

Also, from (3.17) we have

$$\gamma(v_n) - \int_{\Omega} 2F(z, v_n)dz \leq 2M_1 \quad \text{for all } n \in \mathbb{N}. \tag{3.31}$$

We add (3.30) and (3.31) and obtain

$$\int_{\Omega} [f(z, v_n)v_n - 2F(z, v_n)]dz \leq M_4 \quad \text{for some } M_4 > 0, \text{ and all } n \in \mathbb{N}. \tag{3.32}$$

Comparing (3.29) and (3.32), we get a contradiction. This proves that $\{v_n\}_{n\geq 1} \subseteq V$ is bounded. So, we may assume that

$$v_n \xrightarrow{w} u \quad \text{in } H^1(\Omega) \quad \text{and} \quad v_n \to u \quad \text{in } L^2(\Omega) \text{ and } L^2(\partial\Omega). \tag{3.33}$$

In (3.19) we choose $h = v_n - u \in H^1(\Omega)$, pass to the limit as $n \to \infty$ and use (3.33). Then

$$\begin{aligned} &\lim_{n\to\infty} \langle A(v_n), v_n - u\rangle = 0, \\ &\Rightarrow v_n \to u \text{ in } H^1(\Omega) \quad \text{(as before via the Kadec–Klee property)}. \end{aligned}$$

This proves Claim 3.6.

Claim 3.7. $\hat{\lambda}_{m+1}x^2 - 2F(z,x) \to +\infty$ *as* $x \to +\infty$ *uniformly for almost all* $z \in \Omega$.

Hypothesis $H(f)\,(iii)$ implies that given any $\lambda > 0$, we can find $M_5 = M_5(\lambda) > 0$ such that

$$f(z,x)x - 2F(z,x) \geq \lambda \quad \text{for almost all } z \in \Omega, \text{ and all } |x| \geq M_5. \tag{3.34}$$

For almost all $z \in \Omega$, we have

$$\begin{aligned} &\frac{d}{dx}\left(\frac{F(z,x)}{|x|^2}\right) = \frac{f(z,x)x - 2F(z,x)}{|x|^2 x} \begin{cases} \geq \frac{\lambda}{x^2} & \text{if } x \geq M_5 \\ \leq \frac{\lambda}{|x|^2 x} & \text{if } x \leq -M_5 \end{cases} \quad \text{(see (3.34))}, \\ &\Rightarrow \frac{F(z,y)}{|y|^2} - \frac{F(z,v)}{|v|^2} \geq \frac{\lambda}{2}\left[\frac{1}{|v|^2} - \frac{1}{|y|^2}\right] \quad \text{for all } |y| \geq |v| \geq M_5. \end{aligned} \tag{3.35}$$

We let $|y| \to \infty$ and use hypothesis $H(f)\,(iii)$. Then

$$\hat{\lambda}_{m+1}|v|^2 - 2F(z,v) \geq \lambda \quad \text{for almost all } z \in \Omega, \text{ and all } |v| \geq M_5.$$

Since $\lambda > 0$ is arbitrary, we conclude that

$$\hat{\lambda}_{m+1}|v|^2 - 2F(z,v) \to +\infty \quad \text{as } v \to +\infty \text{ uniformly for almost all } z \in \Omega.$$

This proves Claim 3.7.

For every $v \in V$, we have

$$\begin{aligned} \psi(v) = \varphi(v) &= \frac{1}{2}\gamma(v) - \int_\Omega F(z,v)dz \\ &\geq \int_\Omega \left[\frac{1}{2}\hat{\lambda}_{m+1}v^2 - F(z,v)\right]dz \quad \text{(see (2.4))} \\ &\geq -c_6 \quad \text{for some } c_6 > 0 \quad \text{(see Claim 3.7 and hypothesis } H(f)\,(i)) \\ &\Rightarrow \psi \text{ is bounded below.} \end{aligned} \tag{3.36}$$

From (3.36) and Claim 3.6 it follows that

$$\psi \text{ is coercive}$$

(see Motreanu, Motreanu & Papageorgiou [11, p. 103]).

We have

$$\begin{aligned} &\psi(v) = \varphi(v) \leq \varphi(v + \tau(v)) = \tilde{\varphi}(v) \text{ for all } v \in V \quad \text{(see (3.7))}, \\ &\Rightarrow \tilde{\varphi} \text{ is coercive.} \end{aligned}$$

□

From Proposition 3.4, we deduce the following corollary.

Corollary 3.8. *If hypotheses $H(\xi), H(\beta), H(f)$ hold, then $\tilde{\varphi}$ is bounded below and satisfies the C-condition.*

Next, we show that $\tilde{\varphi}$ admits a local linking (see Theorem 2.1) with respect to the orthogonal direct sum decomposition $V = W \oplus \hat{E}$, where $W = \bigoplus_{i=m+1}^{l-1} E(\hat{\lambda}_i), \hat{E} = \overline{\bigoplus_{i \geq l} E(\lambda_i)}$.

Proposition 3.9. *If hypotheses $H(\xi), H(\beta), H(f)$ hold, then $\tilde{\varphi}$ has a local linking at $u = 0$ with respect to the decomposition*

$$V = W \oplus \hat{E}.$$

Proof. From hypotheses $H(f)\,(i), (iv)$, we see that given $r > 2$, we can find $c_7 = c_7(r) > 0$ such that

$$F(z,x) \leq \frac{\vartheta(z)}{2}x^2 + c_7|x|^r \quad \text{for almost all } z \in \Omega, \text{ and all } x \in \mathbb{R}. \tag{3.37}$$

For $\hat{v} \in \hat{E}$ we have

$$\begin{aligned}
\tilde{\varphi}(\hat{v}) &= \varphi(\hat{v} + \tau(\hat{v})) \\
&\geq \varphi(\hat{v}) \quad \text{(see Proposition 3.2)} \\
&= \frac{1}{2}\gamma(\hat{v}) - \int_\Omega F(z,\hat{v})dz \\
&\geq \frac{1}{2}\gamma(\hat{v}) - \frac{1}{2}\int_\Omega \vartheta(z)\hat{v}^2 dz - c_8\|\hat{v}\|^r \quad \text{for some } c_8 > 0 \quad \text{(see (3.37))} \\
&\geq c_9\|\hat{v}\|^2 - c_8\|\hat{v}\|^r \quad \text{for some } c_9 > 0 \quad \text{(see Lemma 2.3(b))}.
\end{aligned}$$

Since $r > 2$, we see that we can find $\rho_1 \in (0,1)$ small such that

$$\tilde{\varphi}(\hat{v}) > 0 \quad \text{for all } \hat{v} \in \hat{E} \text{ with } 0 < \|\hat{v}\| \leq \rho_1. \tag{3.38}$$

The space $Y = H_- \oplus W$ is finite dimensional and so all norms are equivalent. Hence we can find $\epsilon_0 > 0$ such that

$$y \in Y \quad \text{and} \quad \|y\| \leq \epsilon_0 \Rightarrow |y(z)| \leq \delta \quad \text{for all } z \in \overline{\Omega} \quad \text{(recall that } Y \subseteq C^1(\overline{\Omega})). \tag{3.39}$$

By Proposition 3.2 we know that $\tau(\cdot)$ is continuous. So, we can find $\rho_2 > 0$ such that

$$\tilde{u} \in W \text{ and } \|\tilde{u}\| \leq \rho_2 \Rightarrow \|\tilde{u} + \tau(\tilde{u})\| \leq \epsilon_0. \tag{3.40}$$

From (3.39) and (3.40) it follows that

$$\begin{aligned}
\tilde{\varphi}(\tilde{u}) &= \varphi(\tilde{u} + \tau(\tilde{u})) \\
&= \frac{1}{2}\gamma(\tilde{u} + \tau(\tilde{u})) - \int_\Omega F(z, \tilde{u} + \tau(\tilde{u}))dz \\
&\leq \frac{1}{2}\hat{\lambda}_{l-1}\|\tilde{u} + \tau(\tilde{u})\|_2^2 - \frac{1}{2}\hat{\lambda}_{l-1}\|\tilde{u} + \tau(\tilde{u})\|_2^2 \quad \text{(see hypothesis } H(f)\,(iv)) \\
&= 0.
\end{aligned}$$

So, we have that

$$\tilde{\varphi}(\tilde{u}) \leq 0 \quad \text{for all } \tilde{u} \in W \text{ with } \|\tilde{u}\| \leq \rho_2. \tag{3.41}$$

If $\rho = \min\{\rho_1, \rho_2\}$, then from (3.38) and (3.41) we conclude that φ has a local linking at $u = 0$ with respect to the decomposition $V = W \oplus \hat{E}$. □

Now we are ready for proving our multiplicity theorem.

Theorem 3.10. *If hypotheses $H(\xi), H(\beta), H(f)$ hold, then problem (1.1) admits at least two nontrivial solutions*

$$u_0, \hat{u} \in C^1(\overline{\Omega}).$$

Proof. From Proposition 3.9 we know that

$$\inf_V \tilde{\varphi} \leq 0.$$

If $\inf_V \tilde{\varphi} = 0$, then by Proposition 3.9 all $\tilde{u} \in W$ with $0 < \|\tilde{u}\| \leq \rho$ are nontrivial critical points of $\tilde{\varphi}$. Hence $\tilde{u} + \tau(\tilde{u})$ are nontrivial critical points of φ (see Proposition 3.4).

If $\inf_V \tilde{\varphi} < 0$, then we can apply Theorem 2.1 (see Corollary 3.8) and produce two nontrivial critical points $\tilde{u}_0$ and $\tilde{u}_*$ of $\tilde{\varphi}$. Then

$$u_0 = \tilde{u}_0 + \tau(\tilde{u}_0) \quad \text{and} \quad \hat{u} = \tilde{u}_* + \tau(\hat{u}_*)$$

are two nontrivial critical points of φ (see Proposition 3.4).

For $u = u_0$ or $u = \hat{u}$, we have

$$\begin{aligned} &-\Delta u(z) + \xi(z)u(z) = f(z, u(z)) \quad \text{for almost all } z \in \Omega, \\ &\frac{\partial u}{\partial n} + \beta(z)u = 0 \quad \text{on } \partial\Omega \quad \text{(see Papageorgiou \& Rădulescu [14,16])}. \end{aligned} \tag{3.42}$$

Evidently, hypotheses $H(f)$ imply that

$$|f(z,x)| \leq c_{10}|x| \quad \text{for almost all } x \in \mathbb{R}, \text{ and some } c_{10} > 0. \tag{3.43}$$

We set

$$b(z) = \begin{cases} \frac{f(z,u(z))}{u(z)} & \text{if } u(z) \neq 0 \\ 0 & \text{if } u(z) = 0. \end{cases}$$

From (3.43) it follows that $b \in L^\infty(\Omega)$. From (3.42) we have

$$\begin{cases} -\Delta u(z) = (b - \xi)(z)u(z) & \text{for almost all } z \in \Omega, \\ \frac{\partial u}{\partial n} + \beta(z)u = 0 & \text{on } \partial\Omega. \end{cases}$$

Note that $b - \xi \in L^s(\Omega)$, $s > N$ (see hypothesis $H(\xi)$). Then Lemmata 5.1 and 5.2 of Wang [19] imply that

$$\begin{aligned} &u \in W^{2,s}(\Omega), \\ &\Rightarrow u \in C^{1,\alpha}(\overline{\Omega}) \quad \text{with } \alpha = 1 - \frac{N}{s} > 0 \quad \text{(by the Sobolev embedding theorem)}. \end{aligned}$$

Therefore $u_0, \hat{u} \in C^1(\overline{\Omega})$. □

Acknowledgments

This research was supported by the Slovenian Research Agency grants P1-0292, J1-8131, and J1-7025. V. D. Rădulescu acknowledges the support through a grant of the Romanian Ministry of Research and Innovation, CNCS - UEFISCDI, project number PN-III-P4-ID-PCE-2016-0130, within PNCDI III.

References

[1] H. Amann, Saddle points and multiple solutions of differential equations, *Math. Z.* **169**(1979), No. 2, 127–166. MR0550724; https://doi.org/10.1007/BF01215273

[2] H. Brezis, L. Nirenberg, Remarks on finding critical points, *Comm. Pure Appl. Math.* **44**(1991), No. 8–9, 939–963. MR1127041; https://doi.org/10.1002/cpa.3160440808

[3] A. Castro, A. Lazer, Critical point theory and the number of solutions of a nonlinear Dirichlet problem, *Ann. Mat. Pura Appl. (4)* **120**(1979), 113–137. MR0551063; https://doi.org/10.1007/BF02411940

[4] G. D'Aguì, S. Marano, N. S. Papageorgiou, Multiple solutions to a Robin problem with indefinite weight and asymmetric reaction, *J. Math. Anal. Appl.* **433**(2016), No. 2, 1821–1845. MR3398793; https://doi.org/10.1016/j.jmaa.2015.08.065

[5] L. Gasinski, N. S. Papageorgiou, *Nonlinear analysis*, Series in Mathematical Analysis and Applications, Vol. 9, Chapman & Hall/CRC, Boca Raton, FL, 2006. MR2168068

[6] L. Gasinski, N. S. Papageorgiou, Multiplicity of solutions for Neumann problems with an indefinite and unbounded potential, *Commun. Pure Appl. Anal.* **12**(2013), No. 5, 1985–1999. MR3015667; https://doi.org/10.3934/cpaa.2013.12.1985

[7] L. Gasinski, N. S. Papageorgiou, *Exercises in analysis, Part 1*, Problem Books in Mathematics, Springer, Heidelberg, 2014. MR3307732

[8] S. Hu, N. S. Papageorgiou, *Handbook of multivalued analysis, Volume I: Theory*, Mathematics and its Applications, Vol. 419, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1997. MR1485775

[9] S. Hu, N. S. Papageorgiou, Positive solutions for Robin problems with general potential and logistic reaction, *Comm. Pure. Appl. Anal.* **15**(2016), No. 6, 2489–2507. MR3565951; https://doi.org/10.3934/cpaa.2016046

[10] S. Kyritsi, N. S. Papageorgiou, Multiple solutions for superlinear Dirichlet problems with an indefinite potential, *Ann. Mat. Pura Appl. (4)* **192**(2013), No. 2, 297–313. MR3035141; https://doi.org/10.1007/s10231-011-0224-z

[11] D. Motreanu, V. Motreanu, N. S. Papageorgiou, *Topological and Variational Methods with Applications to Nonlinear Boundary Value Problems*, Springer, New York, 2014. MR3136201

[12] N. S. Papageorgiou, F. Papalini, Seven solutions with sign information for sublinear equations with unbounded and indefinite potential and no symmetries, *Israel J. Math.* **201**(2014), No. 2, 761–796. MR3265302; https://doi.org/10.1007/s11856-014-1050-y

[13] N. S. Papageorgiou, V. D. Rădulescu, Semilinear Neumann problems with indefinite and unbounded potential and crossing nonlinearity, *Recent trends in nonlinear partial differential equations. II. Stationary problems*, 293–315, Contemp. Math., Vol. 595, Amer. Math. Soc., Providence, RI, 2013. MR3156380; https://doi.org/10.1090/conm/595

[14] N. S. Papageorgiou, V. D. Rădulescu, Multiple solutions with precise sign for nonlinear parametric Robin problems, *J. Differential Equations* **256**(2014), No. 7, 2449–2479. MR3160450; https://doi.org/10.1016/j.jde.2014.01.010

[15] N. S. Papageorgiou, V. D. Rădulescu, Multiplicity of solutions for resonant Neumann problems with an indefinite and unbounded potential, *Trans. Amer. Math. Soc.* **367**(2015), No. 12, 8723–8756. MR3403070; https://doi.org/10.1090/S0002-9947-2014-06518-5

[16] N. S. Papageorgiou, V. D. Rădulescu, Robin problems with indefinite, unbounded potential and reaction of arbitrary growth, *Revista Mat. Complut.* **29**(2016), No. 1, 91–126. MR3438028; https://doi.org/10.1007/s13163-015-0181-y

[17] D. Qin, X. Tang, J. Zhang, Multiple solutions for semilinear elliptic equations with sign-changing potential and nonlinearity, *Electron. J. Differential Equations* **2013**, No. 207, 1–9. MR3119061

[18] S. Shi, S. Li, Existence of solutions for a class of semilinear elliptic equations with the Robin boundary condition, *Nonlinear Anal.* **71**(2009), No. 7–8, 3292–3298. MR2532751; https://doi.org/10.1016/j.na.2009.01.209

[19] X. J. Wang, Neumann problems of semilinear elliptic equations involving critical Sobolev exponents, *J. Differential Equations* **93**(1991), No. 2, 283–310. MR1125221; https://doi.org/10.1016/0022-0396(91)90014-Z