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Existence of bounded solutions for second order neutral difference equations via measure of noncompactness

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Abstract. In this paper we analyze the existence of bounded solutions for a nonlinear second-order neutral difference equation, which is more general than other equations of this type studied recently. Moreover, our analysis is based under more general conditions than the required in other works, as we only assume the continuity of the involved functions. The main tool here are the so called measures of noncompactness, and more specifically, the celebrated Darbo fixed point theorem. Also we will state, in the specified sense, the stability of the solutions.

Keywords: difference equations, measures of noncompactness, fixed point, stability.

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1 Introduction

In what follows, as usual, we denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of the positive integers and real numbers, respectively. Also, $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$ and $\mathbb{N}_k := \{n \in \mathbb{N} : n \geq k\}$ with $k \in \mathbb{N}$.

As it is well known, difference equations serve as mathematical models in diverse areas of applied science and engineering, for concrete references see, for instance, the monographs [1,2]. We consider the following nonlinear second-order neutral difference equation

$$\Delta\left(r_n\left(\Delta(x_n + p_n x_{n-k})\right)^s\right) + a_n g(x_n) + b_n f(x_{n+1}) = 0 \quad \text{for all } n \in \mathbb{N}_k, \tag{1.1}$$

where $k \in \mathbb{N}$, $a, b, p : \mathbb{N}_0 \longrightarrow \mathbb{R}$, $r : \mathbb{N}_0 \longrightarrow \mathbb{R} \setminus \{0\}$, $f, g : \mathbb{R} \longrightarrow \mathbb{R}$, $s \leq 1$ ratio of odd positive integers, are given and satisfy some conditions that we will expose later. For a general background on difference equations theory, we refer to [1,2,10,12].

The above equation generalizes some well known and studied nonlinear second order difference equations, as the Sturm–Liouville difference equation $\Delta(r_n \Delta x_n) = a_n x_{n+1}$ or the Emden–Fowler difference equation (see, for instance, [11,13]) of the form

$$\Delta^2(x_n + p x_{n-k}) + a_n x_n^s = 0 \quad \text{for all } n \in \mathbb{N}_k,$$

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where $p > 0$, $p \neq 1$ and s is the ratio of two odd integers. Also, equation (1.1) has been analyzed in [9] for $g(x) := x^\alpha$, $\alpha \geq 1$ ratio of positive integers with odd denominator, and in [13], for the particular case $s := 1$ and $f \equiv 0$.

Although there are many methods to analyze, under suitable conditions, the existence of solutions for difference equations (see [1,2,15]), we focus here in those based on the so called *measures of noncompactness* (see, for instance, [9,13,14]). A key result in the cited works to prove the existence of solutions is the celebrated Darbo fixed point theorem (see Theorem 2.3), based in such measures.

On the other hand, in this paper we will show in Theorem 3.3 the existence of solutions for the equation (1.1) which, as we have pointed out above, is more general than the equations posed in others works. Moreover, the conditions on the functions f and g also will be more general, namely, only the continuity will be required.

We conclude the paper with a brief analysis of a special type of stability (see Definition 4.1) of the solutions of equation (1.1). We will prove in Proposition 4.2 that such stability is attained under only the continuity assumption on the functions f and g instead the Lipschitzian condition required in [9,13].

2 Measure of noncompactness and Darbo fixed point theorem

In whats follows, $(X, \|\cdot\|)$ will be an infinite-dimensional Banach space, and $\mathfrak{B}_X$ the class of non-empty and bounded subsets of X, while $\mathfrak{C}_X$ will be the class of its relatively compact sets. For a given $B \subset X$, we denote by $\bar{B}$ and $\operatorname{Conv}(B)$ the closure and the convex hull of B, respectively.

We will use the definition of measure of noncompactness given in [5].

Definition 2.1. A mapping $\mu : \mathfrak{B}_X \longrightarrow [0, +\infty)$ is said to be a measure of noncompactness, MNC, if satisfies the following properties:

(i) $\ker(\mu) := \{B \in \mathfrak{B}_X : \mu(B) = 0\} \neq \emptyset$ and $\ker(\mu) \subset \mathfrak{C}_X$;

(ii) $\mu(B) = \mu(\operatorname{Conv}(B)) = \mu(\bar{B})$, for all $B \in \mathfrak{B}_X$;

(iii) $\mu(B_1) \leq \mu(B_2)$, for all $B_1, B_2 \in \mathfrak{B}_X$ with $B_1 \subset B_2$;

(iv) $\mu(\lambda B_1 + (1-\lambda)B_2) \leq \lambda\mu(B_1) + (1-\lambda)\mu(B_2)$ for all $0 \leq \lambda \leq 1$ and $B_1, B_2 \in \mathfrak{B}_X$;

(v) if $(B_n)_{n\geq 0}$ is a decreasing sequence of closed sets in $\mathfrak{B}_X$ with $\lim_n \mu(B_n) = 0$, then $\cap_{n\geq 0} B_n \neq \emptyset$.

For a detailed exposition of the MNCs and their applications, we refer to [3–5] and references therein.

Example 2.2. Let $(\ell_\infty, \|\cdot\|_\infty)$ be the Banach space of all real bounded sequences $x : \mathbb{N}_0 \longrightarrow \mathbb{R}$ equipped with the standard supremum norm $\|x\|_\infty := \sup\{|x_n| : n \in \mathbb{N}_0\}$, for all $x := (x_n)_{n\geq 0} \in \ell_\infty$.

Given $B \in \mathfrak{B}_{\ell_\infty}$, for each $n \geq 0$, let $B_n := \{x_n : x \in B\}$ (i.e., the n-th terms of any sequence belonging to B). Then, the mapping $\mu : \mathfrak{B}_X \longrightarrow [0, +\infty)$ defined as

$$\mu(B) = \limsup_n \operatorname{Diam}(B_n),$$

is a MNC (see [5]), where $\operatorname{Diam}(B_n) := \sup\{|x_n - y_n| : x, y \in B\}$ is the diameter of the set B_n.

Next, we recall the Darbo fixed point theorem [7] which will be key in our main result.

Theorem 2.3. *Let $T : C \longrightarrow C$ be continuous, with $C \in \mathfrak{B}_X$ closed and convex, and μ a MNC. Assume that there is $0 \leq \eta < 1$ such that $\mu(T(B)) \leq \eta\mu(B)$ for each non-empty $B \subset C$. Then, T has a fixed point.*

3 Main result

The next result, which is true in a more general context, is due to Vanderbei [16] and generalizes the concept of Lipschitzian function.

Lemma 3.1. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous. Then, for each $\varepsilon > 0$ there is $L > 0$ such that $|f(x) - f(y)| \leq L|x - y| + \varepsilon$, for all $x, y \in [a, b]$.*

Remark 3.2. Clearly, any locally Lipschitzian function $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the above lemma in an interval $[a, b]$. However, the reciprocal of this fact does not hold in general. For instance, the function $f(x) := \sqrt{|x|}$ defined in $[-1, 1]$ is continuous but not Lipschitzian in this interval.

Let the following conditions hold.

(C1) The functions $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are continuous.

(C2) The sequence $p : \mathbb{N}_0 \longrightarrow \mathbb{R}$ satisfies

$$-1 < \liminf_n p_n \leq \limsup_n p_n < 1. \tag{3.1}$$

(C3) The sequences $a, b : \mathbb{N}_0 \longrightarrow \mathbb{R}$, $r : \mathbb{N}_0 \longrightarrow \mathbb{R} \setminus \{0\}$ satisfy

$$\sum_{n\geq 0} \left|\frac{1}{r_n}\right|^{\frac{1}{s}} \sum_{i\geq n} |a_i| < +\infty, \quad \sum_{n\geq 0} \left|\frac{1}{r_n}\right|^{\frac{1}{s}} \sum_{i\geq n} |b_i| < +\infty. \tag{3.2}$$

Some comments are necessary before continuing. Conditions (C2) and (C3) are often required to prove the existence of solutions for equation (1.1); see [9, 13]. However, condition (C1) on the functions f and g is more general than the required in the cited works. Specifically:

(I) In [9], $g(x) := x^\alpha$ where α is a ratio of positive integers with odd denominator and f is assumed to be locally Lipschitzian. Clearly, these conditions are particular cases (but not equivalent) of condition (C1).

(II) In addition to the continuity of g, the linear growth condition

$$|g(x)| \leq L|x| + M \quad \text{for all } x \in \mathbb{R}, \tag{3.3}$$

for some $L, M > 0$ is assumed in [13], and $f \equiv 0$. It is not very difficult to check that if $g : \mathbb{R} \longrightarrow \mathbb{R}$ is uniformly continuous, then (3.3) holds. However, a continuous function defined in $\mathbb{R}$ does not need to satisfy (3.3): we can take any $g : \mathbb{R} \longrightarrow \mathbb{R}$ continuous with $|g(x)|/|x| \to +\infty$ as $x \to +\infty$.

Now, we can state and prove our main result.

Theorem 3.3. *Assume conditions (C1)–(C3). Then, equation (1.1) has some bounded solution $x : \mathbb{N}_k \longrightarrow \mathbb{R}$.*

Proof. Let $(\ell_\infty, \|\cdot\|_\infty)$ and μ be as in Example 2.2, and fix $d > 0$. As, by condition (C1), f and g are continuous on the compact $[-d, d]$ we have

$$\begin{aligned} |f(x)| &\le M_f := \max\left\{|f(x)| : x \in [-d, d]\right\}, \\ |g(x)| &\le M_g := \max\left\{|g(x)| : x \in [-d, d]\right\}, \end{aligned} \tag{3.4}$$

for each $x \in [-d, d]$. Condition (C2) implies that there is $n_1 \in \mathbb{N}_0$ and $P \in [0, 1)$ such that

$$|p_n| \le P, \quad \text{for all } n \in \mathbb{N}_{n_1}. \tag{3.5}$$

From condition (C3) the series defined by the sequences $(a_n)_{n\ge 0}$ and $(b_n)_{n\ge 0}$ are convergent. Therefore, there is $n_2 \in \mathbb{N}_0$ such that

$$\sum_{i\ge n_2} |a_i| < 1 \quad \text{and} \quad \sum_{i\ge n_2} |b_i| < 1.$$

and, again noticing condition (C3), as $1/s \ge 1$ we have

$$\sum_{n\ge n_2} \left(\left|\frac{1}{r_n}\right| \sum_{i\ge n} |c_i| \right)^{\frac{1}{s}} \le \sum_{n\ge n_2} \left|\frac{1}{r_n}\right|^{\frac{1}{s}} \sum_{i\ge n} |c_i| < +\infty,$$

for $c_i := a_i, b_i$. So, taking

$$M := \frac{d - Pd}{2^{\frac{1}{s}-1} M_g^{1/s} + 2^{\frac{1}{s}-1} M_f^{1/s}},$$

there is $n_3 \in \mathbb{N}_0$ such that

$$\sum_{n\ge n_3} \left(\left|\frac{1}{r_n}\right| \sum_{i\ge n} |a_i| \right)^{\frac{1}{s}} \le M, \quad \sum_{n\ge n_3} \left(\left|\frac{1}{r_n}\right| \sum_{i\ge n} |b_i| \right)^{\frac{1}{s}} \le M. \tag{3.6}$$

Also, the function $r \mapsto r^{1/s}$ is locally Lipschitzian, therefore there is $L_d > 0$ such that

$$|x^{1/s} - y^{1/s}| \le L_d |x - y|, \tag{3.7}$$

for each $x, y \in [-d, d]$. Given any $\varepsilon_f, \varepsilon_g > 0$, let L_f and L_g be the positive constants provided by Lemma 3.1, that is,

$$|f(x) - f(y)| \le L_f |x - y| + \varepsilon_f, \quad |g(x) - g(y)| \le L_g |x - y| + \varepsilon_g \quad \text{for all } x, y \in [-d, d]. \tag{3.8}$$

Next, define the bounded, closed and convex set

$$C := \{x := (x_n)_{n\ge 0} \in \ell_\infty : |x_n| \le d', 0 \le n < n_4 \text{ and } |x_n| \le d, \forall n \in \mathbb{N}_{n_4}\},$$

where $d' > 0$ is an arbitrary number, $n_4 := k + \max\{n_1, n_2, n_3\}$, and let $T : C \longrightarrow \ell_\infty$ be the mapping

$$T(x)_n := \begin{cases} x_n, & 0 \le n < n_4. \\ -p_n x_{n-k} - \sum_{j\ge n} \left(\frac{1}{r_j} \sum_{i\ge j} \left(a_i g(x_i) + b_i f(x_{i+1})\right) \right)^{1/s}, & n \ge n_4. \end{cases}$$

By the above considerations T is well defined, that is to say, $T(x) \in \ell_\infty$ for each $x \in \ell_\infty$. In fact, as we will show below, $T(C) \subset C$.

To prove the theorem, firstly, we will show that T satisfies the conditions of Darbo fixed point theorem and therefore has a fixed point. For clarity, we divide the proof of this claim in two steps.

Step 1: T is continuous and $T(C) \subset C$.

The continuity of T, under our assumptions, is a routine checkup (see, for instance, [9,13]). So, we skip the details of the proof of this fact and will show that $T(C) \subset C$.

Let any $x \in C$ and $n \geq n_4$. As

$$|T(x)_n| \leq |p_n||x_{n-k}| + \sum_{j\geq n}\left(\left|\frac{1}{r_j}\right|\sum_{i\geq j}(|a_i||g(x_i)| + |b_i||f(x_{i+1})|)\right)^{1/s},$$

noticing the classical inequality $(a+b)^r \leq 2^{r-1}(a^r + b^r)$, with $r \geq 1$, $a, b > 0$, we have

$$|T(x)_n| \leq |p_n||x_{n-k}| + 2^{\frac{1}{s}-1}\sum_{j\geq n}\left[\left(\left|\frac{1}{r_j}\right|\sum_{i\geq j}|a_i||g(x_i)|\right)^{1/s} + \left(\left|\frac{1}{r_j}\right|\sum_{i\geq j}|b_i||f(x_{i+1})|\right)^{1/s}\right].$$

Next, from (3.4), (3.5) and (3.6), as $|x_i| \leq d$ we infer

$$\begin{aligned}
|T(x)_n| &\leq Pd + 2^{\frac{1}{s}-1}M_g^{1/s}\sum_{j\geq n}\left(\left|\frac{1}{r_j}\right|\sum_{i\geq j}|a_i|\right)^{1/s} + 2^{\frac{1}{s}-1}M_f^{1/s}\sum_{j\geq n}\left(\left|\frac{1}{r_j}\right|\sum_{i\geq j}|b_i|\right)^{1/s}\\
&\leq Pd + M(2^{\frac{1}{s}-1}M_g^{1/s} + 2^{\frac{1}{s}-1}M_f^{1/s})\\
&= Pd + d - Pd = d.
\end{aligned}$$

So, $T(x) \in C$ and therefore, by the arbitrariness of $x \in C$, $T(C) \subset C$ as claimed.

Step 2: Comparison of the measure of noncompactness.

Let $B \subset C$ be non-empty. Then, given $x, y \in B$ from (3.7) and (3.8), for each $n \geq n_4$ we have:

$$\begin{aligned}
&|T(x)_n - T(y)_n|\\
&\quad\leq P|x_{n-k} - y_{n-k}|\\
&\qquad+ \sum_{j\geq n}\left|\frac{1}{r_j}\right|^{\frac{1}{s}}\left|\left(\sum_{i\geq j}a_i g(x_i) + b_i f(x_{i+1})\right)^{\frac{1}{s}} - \left(\sum_{i\geq j}a_i g(y_i) + b_i f(y_{i+1})\right)^{\frac{1}{s}}\right|\\
&\quad\leq P|x_{n-k} - y_{n-k}|\\
&\qquad+ L_d\sum_{j\geq n}\left|\frac{1}{r_j}\right|^{\frac{1}{s}}\left(\sum_{i\geq j}|a_i(g(x_i) - g(y_i))| + \sum_{i\geq j}|b_i(f(x_{i+1}) - f(y_{i+1}))|\right) \qquad (3.9)\\
&\quad\leq P|x_{n-k} - y_{n-k}| + L_d L_g\sum_{j\geq n}\left|\frac{1}{r_j}\right|^{\frac{1}{s}}\sum_{i\geq j}|a_i||x_i - y_i|\\
&\qquad+ L_d L_f\sum_{j\geq n}\left|\frac{1}{r_j}\right|^{\frac{1}{s}}\sum_{i\geq j}|b_i||x_{i+1} - y_{i+1}| + L_d\left[\varepsilon_g\sum_{j\geq n}\left|\frac{1}{r_j}\right|^{\frac{1}{s}}\sum_{i\geq j}|a_i| + \varepsilon_f\sum_{j\geq n}\left|\frac{1}{r_j}\right|^{\frac{1}{s}}\sum_{i\geq j}|b_i|\right]\\
&\quad\leq P\,\mathrm{Diam}(X_{n-k}) + \varepsilon_n,
\end{aligned}$$

where

$$\varepsilon_n := 2dL_d \left[L_g \sum_{j\geq n} \left|\frac{1}{r_j}\right|^{\frac{1}{s}} \sum_{i\geq j} |a_i| + L_f \sum_{j\geq n} \left|\frac{1}{r_j}\right|^{\frac{1}{s}} \sum_{i\geq j} |b_i| \right] + L_d \left[\varepsilon_g \sum_{j\geq n} \left|\frac{1}{r_j}\right|^{\frac{1}{s}} \sum_{i\geq j} |a_i| + \varepsilon_f \sum_{j\geq n} \left|\frac{1}{r_j}\right|^{\frac{1}{s}} \sum_{i\geq j} |b_i| \right].$$

So, from (3.9) and the properties of the upper limit, as $\lim_n \varepsilon_n = 0$ we infer

$$\limsup_n \operatorname{Diam}(T(B)_n) \leq P \limsup_n \operatorname{Diam}(B_{n-k}) = P \limsup_n \operatorname{Diam}(B_n),$$

and therefore, $\mu(T(B)) \leq P\mu(B)$. Then, by Theorem 2.3, T has a fixed point.

Finally, once the existence of fixed points of the mapping T has been proved, we will show the relationship between the fixed points of T and the solutions of the equation (1.1). Let $\xi := (\xi_n)_{n\geq 0} \in C$ be a fixed point of T, that is, $\xi_n = T(\xi)_n$ for each $n \geq 0$. Then, according to definition of T

$$\xi_n + p_n \xi_{n-k} = -\sum_{j\geq n} \left(\frac{1}{r_j} \sum_{i\geq j} (a_i g(\xi_i) + b_i f(\xi_{i+1})) \right)^{1/s} \quad \text{for all } n \in \mathbb{N}_{n_4}, \tag{3.10}$$

which leads us to the following equation

$$\Delta(\xi_n + p_n \xi_{n-k})^s = -\frac{1}{r_n} \sum_{i\geq n} (a_i g(\xi_i) + b_i f(\xi_{i+1})) \quad \text{for all } n \in \mathbb{N}_{n_4}.$$

Using again the operator Δ for both sides of the above equation:

$$\Delta\big(r_n \big(\Delta(\xi_n + p_n \xi_{n-k})\big)^s\big) = -a_n g(\xi_n) - b_n f(\xi_{n+1}) \quad \text{for all } n \in \mathbb{N}_{n_4},$$

and so, the terms $\xi_{n_4}, \xi_{n_4+1}, \ldots$ of the sequence ξ fulfills the equation (1.1). If $n_4 = k$ the proof is ended, otherwise we need to find the $n_4 - k + 1$ previous terms of the solution of the equation (1.1) (recall that such equation is defined for each $n \geq k$, and $k \leq n_4$). We can use the following formula, which is obtained directly from (3.10):

$$\xi_{n-k+l} = \frac{1}{p_{n+l}} \left(-\xi_{n+l} + \sum_{j\geq n+l} \left(\frac{1}{r_j} \sum_{i\geq j} (a_i g(\xi_i) + b_i f(\xi_{i+1})) \right)^{\frac{1}{s}} \right) \quad \text{for all } n \in \mathbb{N}_{n_4},$$

for each $l = 0, 1, \ldots, k-1$. So, the equation (1.1) has a bounded solution and the proof is now complete. □

Remark 3.4. As is noted in [9, Remark 3.3], unlike most of the problems solved by fixed point techniques, the whole sequence solution of the equation (1.1) is not obtained through a fixed point method, but through backward iteration. Such procedure must be applied, as the iteration which defines (1.1) is an iteration with memory, that is, we have to know also earlier terms in order to start the iteration.

As we have point out above, in [13] the linear growth condition (3.3) is required on the function f. But, note that in the proof of the above theorem, we consider $f : [-d, d] \longrightarrow \mathbb{R}$ which satisfies condition (3.3) as f is uniformly continuous on the compact $[-d, d]$. Then, [13, Theorem 2] can be stated under more general conditions, namely as follows.

Corollary 3.5. *Let the nonlinear second order neutral difference equation*

$$\Delta\big(r_n\Delta(x_n + p_n x_{n-k})\big) + a_n g(x_n) = 0 \quad \textit{for all } n \in \mathbb{N}_k,$$

where $k \in \mathbb{N}$, $a, p : \mathbb{N}_0 \longrightarrow \mathbb{R}$, $r : \mathbb{N}_0 \longrightarrow \mathbb{R} \setminus \{0\}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ continuous are given. Assume the following conditions:

(D1) the sequence $p : \mathbb{N}_0 \longrightarrow \mathbb{R}$ satisfies:

$$-1 < \liminf_n p_n \leq \limsup_n p_n < 1;$$

(D2) the sequence $a : \mathbb{N}_0 \longrightarrow \mathbb{R}$ and $r : \mathbb{N}_0 \longrightarrow \mathbb{R} \setminus \{0\}$ satisfy:

$$\sum_{n\geq 0} \left|\frac{1}{r_n}\right| \sum_{i\geq n} |a_i| < +\infty.$$

Then, the above equation has some bounded solution $x : \mathbb{N}_k \longrightarrow \mathbb{R}$.

We conclude this section with an example.

Example 3.6. Let the equation

$$\Delta\left((-1)^n \Delta\left(x_n + \frac{1}{2}x_{n-3}\right)^{1/3}\right) + \frac{1}{2^n}\left(g(x) + f(x_{n+1})\right) = 0. \tag{3.11}$$

Then, for $g(x) := x^5$ and $f(x) := x^{5/3}$ the existence of bounded solutions of the above equation has been proved in [9, Example 3.2], as f is locally Lipschitzian. However, by the well known Rademacher's theorem (see, for instance, [8, §3.1.6, p. 216]), if f is locally Lipschitzian in an open $U \subset \mathbb{R}$, then it is differentiable at almost every $x \in U$. So, if $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous but nowhere differentiable, f can not be locally Lipschitzian. A classical example of a continuous but nowhere differentiable function is the so called Weierstrass function, defined as

$$f(x) := \sum_{n\geq 0} a^n \cos(b^n \pi x) \qquad \text{for all } x \in \mathbb{R},$$

where $0 < a < 1$, b is a positive odd integer and $ab > 1 + 3\pi/2$. Therefore, for this f we can not apply [9, Theorem 3.1] to equation (3.11), while Theorem 3.3 states the existence of bounded solutions for such equation.

4 On the stability of the solutions

Following [6] (see also [9]), we give the following definition of asymptotically stable solution:

Definition 4.1. Let $C \subset \ell_\infty$ non-empty and bounded. A solution $x : \mathbb{N}_k \longrightarrow \mathbb{R}$ of (1.1) is said to be asymptotically stable in C if the following conditions are satisfied:

(a) $x \in C$;

(b) if $y \in C$ is a solution of (1.1), then for every $\varepsilon > 0$ there is $N := N(\varepsilon) \in \mathbb{N}_k$ such that $|x_n - y_n| \leq \varepsilon$ for each $n \geq N$.

The asymptotically stable property (in other sense than the above defined) of the solutions has been analyzed in [9, 13] where is required that the involved functions be Lipschitzian in the whole set $\mathbb{R}$.

Next, we give the following result.

Proposition 4.2. *For each $d > 0$ the equation (1.1) has at least one bounded solution $x : \mathbb{N}_k \longrightarrow \mathbb{R}$ asymptotically stable in $C_d := \{x \in \ell_\infty : |x_n| \leq d', 0 \geq n < n_4, |x_n| \leq d, \forall n \geq n_4\}$, with $d' > 0$ an arbitrary number, for a suitable $n_4 \in \mathbb{N}$.*

Proof. Fixed $d > 0$, let $T : C_d \longrightarrow C_d$ and $n_4 \geq k$ be as in the proof of Theorem 3.3. We know that T has a fixed point in C_d. Denote by S_d the set of fixed points of T.

Then, by Theorem 3.3, there is $x \in S_d$ such that $T(x)_n = x_n$ for each $n \geq 0$. Let us note that if $y \in C_d$ is a solution of the equation (1.1) then, in view of (3.9), we have

$$|x_n - y_n| = |T(x)_n - T(y)_n| \leq P|x_{n-k} - y_{n-k}| + \varepsilon_n,$$

for each $n \geq n_4$, where ε_n has been defined in the proof of Theorem 3.3. So, as $\lambda := \limsup_n |x_n - y_n| = \limsup_n |x_{n-k} - y_{n-k}|$ we infer from the above inequality that

$$\lambda \leq P\lambda + \limsup_n \varepsilon_n,$$

and as $\limsup_n \varepsilon_n = 0$ and $0 < P < 1$, must be $\lambda = 0$. Therefore, given any $\varepsilon > 0$ there is an integer $N := N(\varepsilon) \geq n_4$ such that

$$|x_n - y_n| \leq \varepsilon \quad \text{for all } n \geq N,$$

and the result follows. □

We close the paper with an example.

Example 4.3. Let the equation

$$\Delta\left((-1)^n \Delta\left(x_n + \frac{1}{2}x_{n-3}\right)^{1/3}\right) + \frac{1}{2^n} f(x_{n+1}) = 0,$$

posed in [9, Example 4.2]) with $f(x) := -x + \sin(\pi x/2)$ for each $x \in \mathbb{R}$, which is Lipschitzian in the whole set $\mathbb{R}$.

Fixed any $d > 0$, let C_d be as in Proposition 4.2. Then, by virtue of [9, Theorem 4.1], the above equation has at least one asymptotically stable solution in C_d. However, if f is not Lipschitzian (for instance, the Weierstrass function of Example 3.6) [9, Theorem 4.1] can not be applied.

On the other hand, for every continuous $f : \mathbb{R} \longrightarrow \mathbb{R}$, we can check easily that the conditions of Theorem 3.3 are satisfied and therefore the existence of solutions holds. Consequently, by Proposition 4.2, the above equation has at least one solution asymptotically stable in C_d.

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References

[1] R. P. Agarwal, *Difference equations and inequalities. Theory, methods, and applications*, Monographs and Textbooks in Pure and Applied Mathematics, Vol. 228, Marcel Dekker Inc., New York, 2000. MR1740241

[2] R. P. Agarwal, M. Bohner, S. R. Grace, D. O'Regan, *Discrete oscillation theory*, Hindawi Publishing Corporation, New York, 2005. MR2179948; https://doi.org/10.1155/9789775945198

[3] R. R. Akhmerov, M. I. Kamenskii, A. S. Potapov, A. E. Rodkina, B. N. Sadovskii, *Measure of noncompactness and condensing operators*, Birkhäuser Verlag, Basel, 1992. MR1153247; https://doi.org/10.1007/978-3-0348-5727-7.

[4] J. Banaś, On measures of noncompactness in Banach spaces, *Comment. Math. Univ. Carolin.* **21**(1980), No. 1, 131–143. MR0566245

[5] J. Banaś, K. Goebel, *Measures of noncompactness in Banach spaces*, Lecture Notes in Pure and Applied Mathematics, Vol. 60, Marcel Dekker Inc., New York, 1980. MR0591679

[6] J. Banaś, B. Rzepka, An application of measure of noncompactness in study of asymptotic stability, *Appl. Math. Lett.* **16**(2003), No. 1, 1–6. MR1938185; https://doi.org/10.1016/S0893-9659(02)00136-2

[7] G. Darbo, Punti uniti in trasformazioni a codominio non compatto (in Italian), *Rend. Sem. Mat. Univ. Padova* **24**(1955), 84–92. MR0070164

[8] H. Federer, *Geometric measure theory*, Springer-Verlag, New York, 1969. MR0257325; https://doi.org/10.1007/978-3-642-62010-2

[9] M. Galewski, R. Jankowski, M. Nockowska-Rosiak, E. Schmeidel, On the existence of bounded solutions for nonlinear second order neutral difference equations, *Electron. J. Qual. Theory Differ. Equ.* **2014**, No. 72, 1–12. MR3304198; https://doi.org/10.14232/ejqtde.2014.1.72

[10] W. G. Kelley, A. C. Peterson, *Difference equations. An introduction with applications*, Academic Press, San Diego, 2001. MR1765695

[11] V. Kharkov, Positive solutions of the Emden–Fowler difference equation, *J. Difference Equ. Appl.* **19**(2013), No. 2, 234–260. MR3021785; https://doi.org/10.1080/10236198.2011.634805

[12] R. A. Mickens, *Difference equations theory: applications and advanced topics*, Taylor & Francis Group, New York, 2015.

[13] E. Schmeidel, An application of measures of noncompactness in the investigation of boundedness of solutions of second-order neutral difference equations, *Adv. Difference Equ.* **2013**, 2013:91, 9 pp. MR3049968; https://doi.org/10.1186/1687-1847-2013-91

[14] E. Schmeidel, Z. Zbaszyniak, An application of Darbo's fixed point theorem in the investigation of periodicity of solutions of difference equations, *Comput. Math. Appl.* **64**(2012) No. 7, 2185–2191. MR2966854; https://doi.org/10.1016/j.camwa.2011.12.025

[15] S. Stević, Asymptotic behaviour of second-order difference equation, *ANZIAM J.* **46**(2004) No. 1, 157–170. MR2075520; https://doi.org/10.1017/S1446181100013742

[16] R. Vanderbei, *Uniform continuity is almost Lipschitz continuity*, Technical Report SOR-91–11, Statistics and Operations Research Series, Princeton University, 1991.