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# Infinitely many nontrivial solutions for a class of biharmonic equations via variant fountain theorems

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**Abstract** In this paper, we investigate the existence of infinitely many solutions for a class of biharmonic equations where the nonlinearity involves a combination of superlinear and asymptotically linear terms. The solutions are obtained from a variant version of Fountain Theorem.

*Keywords:* Biharmonic equation; Nonlinearity with superlinear and asymptotically linear terms; Variant fountain theorems.

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## 1. Introduction

In this paper, we are concerned with the multiplicity of solutions for the following biharmonic problem:

$$\begin{cases} \Delta^2 u + a\Delta u = f(x,u) + g(x,u), \quad \text{in} \quad \Omega, \\ u = \Delta u = 0, \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta^2$ is the biharmonic operator, $a$ is a real parameter, $\Omega \subset R^N$ is a bounded domain with smooth boundary $\partial\Omega$, $N \geq 3$. We assume that $f(x,u)$ and $g(x,u)$ satisfy some of the following conditions:
$(g_1)$: $g \in C(\overline{\Omega} \times R, R)$ is odd in $u$.
$(g_2)$: $g(x,u) = bu + g_1(x,u)$, where $b$ is a real parameter.
$(g_3)$: There exist $q \in (1,2)$, $c_1 > 0$ such that

$$|g_1(x,u)| \leq c_1|u|^{q-1}, \quad \text{for } x \in \Omega \text{ and } u \in R.$$

$(f_1)$: $f \in C(\overline{\Omega} \times R, R)$ is odd in $u$.
$(f_2)$: There exists $C > 0$ such that $|f(x,u)| \leq C(1 + |u|^{p-1})$ for $x \in \Omega$ and $u \in R$, where $2 < p < 2^*$, $2^* = \frac{2N}{N-2}$ .
$(f_3)$ : $\lim_{|u|\to\infty} \frac{F(x,u)}{u^2} = \infty$ uniformly for $x \in \Omega$, where $F(x,u) = \int_0^u f(x,s)\mathrm{d}s$.
$(f_4)$: $f(x,u)u \geq 0$ for $u > 0$.
$(f_5)$: There exist $0 < \mu < 2^*$, $c_2 > 0$ and $L > 0$ such that $H(x,u) \geq c_2|u|^\mu$ for $|u| \geq L$ and $x \in \Omega$, where $H(x,u) = \frac{1}{2}f(x,u)u - F(x,u)$ .

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Biharmonic equations have been studied by many authors. In [4], Lazer and Mckenna considered the biharmonic problem:

$$\begin{cases} \Delta^2 u + a\Delta u = d[(u+1)^+ - 1], & \text{in} \quad \Omega, \\ u = \Delta u = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

where $u^+ = \max\{u, 0\}$ and $d \in R$. They pointed out that this type of nonlinearity furnishes a model to study traveling waves in suspension bridges. In [5], the authors got $2k-1$ solutions when $N = 1$ and $d > \lambda_k(\lambda_k - c)$ ($\lambda_k$ is the sequence of the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$) via the global bifurcation method. In [10], a negative solution of (1.2) was considered when $d \geq \lambda_1(\lambda_1 - c)$ by a degree argument. If the nonlinearity $d[(u+1)^+ - 1]$ is replaced by a general function $f(x, u)$, one has the following problem:

$$\begin{cases} \Delta^2 u + a\Delta u = f(x, u), & \text{in} \quad \Omega, \\ u = \Delta u = 0, & \text{on} \quad \partial\Omega. \end{cases} \tag{1.3}$$

In [7], the authors proved the existence of two or three solutions of problem (1.3) for a more general nonlinearity $f$ by using a variational method. In [12], positive solutions of problem (1.3) were got when $f$ satisfies the local superlinearity and sublinearity. For other related results, see [6], [13], [15] and the references therein. We emphasize that all the papers mentioned above are concerned with the case $a < \lambda_1$ only. So far as we know, little has been done for the case $a \geq \lambda_1$. In particular, the authors in [8] considered the case $a \geq \lambda_1$ and got the existence of multiple solutions of problem (1.3).

Our aim in the present paper is to investigate the existence of infinitely many large energy solutions of problem (1.1) in the case $a < \lambda_1$ and $a \geq \lambda_1$. Usually, in order to obtain the existence of infinitely many solutions for superlinear problems, the nonlinearity is assumed to satisfy the following $(AR)$ condition due to Ambrosetti-Rabinowitz [1]:

$(AR)$: There is $\alpha > 2$ such that for $u \neq 0$ and $x \in \Omega$,

$$0 < \alpha F(x, u) \leq u f(x, u),$$

where $F(x, u) = \int_0^u f(x, s)\mathrm{d}s$. This condition implies that $F(x, u) \geq c_3|u|^\alpha - c_4$, where $c_3$, $c_4$ are two positive constants. It is well known that the $(AR)$ condition guarantees the boundedness of the $(PS)_c$ sequence for the corresponding functional. Then we can apply the Symmetric Pass Theorem in [9] or the Fountain Theorem in [11] to get the desired result. In this paper, the nonlinearity involves a combination of superlinear and asymptotically linear terms. Moreover, the superlinear term doesn't satisfy the $(AR)$ condition. Thus, it is difficult to derive the boundedness of the $(PS)_c$ sequence for the corresponding functional. However, motivated by the variant Fountain Theorem established in [14], we can overcome the difficulty.

Before stating our main results we give some notations. Throughout this paper, we denote by $C$ a universal positive constant unless otherwise specified and we set $L^s(\Omega)$ the usual Lebesgue space equipped with the norm $\|u\|_s := (\int_\Omega |u|^s \mathrm{d}x)^{\frac{1}{s}}$, $1 \leq s < \infty$. Let $\lambda_k$ $(k = 1, 2, \cdots)$ denote the eigenvalues and $\varphi_k$ $(k = 1, 2, \cdots)$ the corresponding normalized eigenfunctions of the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega. \end{cases}$$

Here, we repeat each eigenvalue according to its (finite) multiplicity. Then, $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \cdots$ and $\lambda_k \to \infty$ as $k \to \infty$.

The main results of this paper are summarized in the following theorems. To the best of our knowledge, the conclusions are new.

**Theorem 1.1.** Assume that $f$ satisfies $(f_1) - (f_5)$, $g$ satisfies $(g_1) - (g_3)$. Then, given $a < \lambda_1$ and $b < \lambda_1(\lambda_1 - a)$, problem (1.1) has infinitely many solutions $\{u_n\}$ satisfying

$$\frac{1}{2}\left(\int_\Omega |\Delta u_n|^2 \mathrm{d}x - a\int_\Omega |u_n|^2 \mathrm{d}x\right) - \int_\Omega F(x, u_n)\mathrm{d}x - \int_\Omega G(x, u_n)\mathrm{d}x \to \infty \quad as \ \ n \to \infty, \tag{1.4}$$

for $\mu > \max\{\frac{N}{2}(p-2), q\}$, where $G(x,u) = \int_0^u g(x,s)\mathrm{d}s$.

**Theorem 1.2.** Assume that $f$, $g$ satisfy conditions of Theorem 1.1. Then, given $a < \lambda_1$ and $b \geq \lambda_1(\lambda_1 - a)$, problem (1.1) has infinitely many solutions $\{u_n\}$ satisfying (1.4) for $\mu > q$ and $\mu \geq (p-1)\frac{2N}{N+2}$.

**Theorem 1.3.** Assume that $f$, $g$ satisfy conditions of Theorem 1.1. Let $a \geq \lambda_1$ and $\frac{1}{2}\lambda_j < a \leq \frac{1}{2}\lambda_{j+1}$ for some $j \in N$. Then, if $\lambda_i(\frac{1}{2}\lambda_i - a) \leq b < \lambda_{i+1}(\frac{1}{2}\lambda_{i+1} - a)$ for some $i \in N$, $i \geq j+1$, problem (1.1) has infinitely many solutions $\{u_n\}$ satisfying (1.4) for $\mu > q$ and $\mu \geq (p-1)\frac{2N}{N+2}$.

# 2. Variational setting and Variant fountain theorem

Let $\Omega \subset R^N$ be a bounded smooth domain, $H = H^2(\Omega)\bigcap H_0^1(\Omega)$ be the Hilbert space equipped with the inner product

$$(u,v)_H = \int_\Omega \Delta u \Delta v \mathrm{d}x,$$

which induces the norm

$$\|u\|_H = (\int_\Omega |\Delta u|^2 \mathrm{d}x)^{\frac{1}{2}}.$$

For $u \in H$, denote

$$I(u) = \frac{1}{2}\left(\int_\Omega |\Delta u|^2\mathrm{d}x - a\int_\Omega |\nabla u|^2 \mathrm{d}x\right) - \int_\Omega F(x,u)\mathrm{d}x - \int_\Omega G(x,u)\mathrm{d}x.$$

From $(f_1)-(f_2)$ and $(g_1)-(g_3)$, we have $I \in C^1(H)$. Moreover, a critical point of $I$ in $H$ is a weak solution of (1.1).

We need the following variant fountain theorem introduced in [14] to handle the problem.

Let $E$ be a Banach space with the norm $\|.\|$ and $E = \overline{\bigoplus_{j\in N} X_j}$ with $\dim X_j < \infty$ for any $j \in N$. Set $Y_k = \bigoplus_{j=1}^k X_j$, $Z_k = \overline{\bigoplus_{j=k}^\infty X_j}$ and

$$B_k = \{u \in Y_k : \|u\| \leq \rho_k\}, \qquad N_k = \{u \in Z_k : \|u\| = r_k\} \quad \text{for} \quad \rho_k > r_k > 0.$$

Consider the following $C^1$ functional $\Phi_\lambda : E \to R$ defined by:

$$\Phi_\lambda(u) := A(u) - \lambda B(u), \qquad \lambda \in [1,2].$$

We assume that

$(F_1)$: $\Phi_\lambda$ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$. Furthermore, $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for all $(\lambda, u) \in [1,2] \times E$.

$(F_2)$: $B(u) \geq 0$ for all $u \in E$; $A(u) \to \infty$ or $B(u) \to \infty$ as $\|u\| \to \infty$.

Let, for $k \geq 2$,

$$\begin{aligned}
\Gamma_k &:= \{\gamma \in C(B_k, E) : \gamma \text{ } is \text{ } odd, \gamma|_{\partial B_k} = id\},\\
c_k(\lambda) &:= \inf_{\gamma\in\Gamma_k} \max_{u\in B_k} \Phi_\lambda(\gamma(u)),\\
b_k(\lambda) &:= \inf_{u\in Z_k,\ \|u\|=r_k} \Phi_\lambda(u),\\
a_k(\lambda) &:= \max_{u\in Y_k,\ \|u\|=\rho_k} \Phi_\lambda(u).
\end{aligned}$$

**Theorem 2.1.** Assume $(F_1)$ and $(F_2)$. If $b_k(\lambda) > a_k(\lambda)$ for all $\lambda \in [1,2]$, then $c_k(\lambda) \geq b_k(\lambda)$ for all $\lambda \in [1,2]$. Moreover, for a.e. $\lambda \in [1,2]$, there exists a sequence $\{u_n^k(\lambda)\}_{n=1}^\infty$ such that

$$\sup_n \|u_n^k(\lambda)\| < \infty, \qquad \Phi'_\lambda(u_n^k(\lambda)) \to 0 \quad \text{and} \quad \Phi_\lambda(u_n^k(\lambda)) \to c_k(\lambda) \quad \text{as} \ n \to \infty.$$

## 3. The case $a < \lambda_1$ and $b \in R$

For $a < \lambda_1$, define a norm $u \in H$ as follows:

$$\|u\| = \left( \int_\Omega |\Delta u|^2 \mathrm{d}x - a \int_\Omega |\nabla u|^2 \mathrm{d}x \right)^{\frac{1}{2}}.$$

We note that the norm $\|.\|$ is a equivalent norm on $H$. In this section we use the norm $\|.\|$. It is well known that $\wedge_k = \lambda_k(\lambda_k - a)$, $k = 1, 2, \cdots$, are eigenvalues of the eigenvalue problem

$$\begin{cases} \Delta^2 u + a\Delta u = \wedge u, & \text{in} \quad \Omega, \\ u = \Delta u = 0, & \text{on} \quad \partial\Omega, \end{cases}$$

$\varphi_k$, $k = 1, 2, \cdots$ are the corresponding eigenfunctions. Furthermore, the set of $\{\varphi_k\}$ is an orthogonal basis on the Hilbert space $H$. Let $X_j := span\{\varphi_j\}$, $j \in N$ and set $Y_k = \bigoplus_{j=1}^k X_j$, $Z_k = \overline{\bigoplus_{j=k}^\infty X_j}$.

Observe that the following inequality holds:

$$\|u\|^2 \geq \wedge_k \int_\Omega u^2 \mathrm{d}x, \qquad \forall u \in Z_k. \tag{3.1}$$

We start with some technical lemmas.

Consider $I_\mu : H \to R$ defined by

$$I_\mu(u) := \frac{1}{2}\|u\|^2 - \mu \int_\Omega F(x, u)\mathrm{d}x - \int_\Omega G(x, u)\mathrm{d}x, \qquad \mu \in [1, 2].$$

**Lemma 3.1.** Assume that $f$ satisfies $(f_1) - (f_4)$, $g$ satisfies $(g_1) - (g_3)$. Then, given $a < \lambda_1$ and $b < \wedge_1$, there exists $k_0 \in N$, such that for $k \geq k_0$, there exist $\overline{c}_k \geq \overline{b}_k > 0$, $\overline{b}_k \to \infty$ as $k \to \infty$. Moreover, fix $k \geq k_0$, there exist $\mu_n \to 1$ as $n \to \infty$ and $\{u_n\}_{n=1}^\infty \subset H$ such that

$$I'_{\mu_n}(u_n) = 0, \qquad I_{\mu_n}(u_n) \in [\overline{b}_k, \overline{c}_k].$$

**Proof.** We note that for $u \in H$,

$$\|u\|^2 \geq \wedge_1 \int_\Omega u^2 \mathrm{d}x. \tag{3.2}$$

Then it is easy to prove $(F_1) - (F_2)$ hold. By $(f_2)$, there holds

$$|F(x, u)| \leq C(|u| + |u|^p). \tag{3.3}$$

$(g_3)$ implies that

$$|G_1(x, u)| \leq C|u|^q, \tag{3.4}$$

where $G_1(x, u) = \int_0^u g_1(x, s)\mathrm{d}s$. For $2 < p < 2^*$, let

$$\beta_k := \sup_{u \in Z_k, \|u\|=1} \|u\|_p. \tag{3.5}$$

Then $\beta_k \to 0$ as $k \to \infty$ following the method of Lemma 3.8 in [11]. Hence, combining (3.3) – (3.5) , we obtain that for $u \in Z_k$,

$$\begin{aligned} I_\mu(u) &= \frac{1}{2}\|u\|^2 - \mu \int_\Omega F(x, u)\mathrm{d}x - \int_\Omega G(x, u)\mathrm{d}x \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}b \int_\Omega u^2 \mathrm{d}x - C\|u\|^q - C\|u\| - C\beta_k^p \|u\|^p. \end{aligned} \tag{3.6}$$

For simplicity, we only need to consider the case $0 \leq b < \wedge_1$. (3.2) and (3.6) imply that for $u \in Z_k$,

$$I_\mu(u) \geq \frac{1}{2}\left(1 - \frac{b}{\wedge_1}\right)\|u\|^2 - C(\|u\|^q + \|u\|) - C\beta_k^p \|u\|^p.$$

Choosing $\bar{r}_k := \left(\frac{2Cp\wedge_1\beta_k^p}{\wedge_1-b}\right)^{\frac{1}{2-p}}$, we obtain that for $u\in Z_k$, $\|u\|=\bar{r}_k$,

$$\begin{aligned}
I_\mu(u) &\geq \left[\frac{1}{4}\left(1-\frac{b}{\wedge_1}\right)\|u\|^2 - C\beta_k^p\|u\|^p\right] + \left[\frac{1}{4}\left(1-\frac{b}{\wedge_1}\right)\|u\|^2 - C(\|u\|^q+\|u\|)\right] \\
&\geq \frac{p-2}{4p}\left(1-\frac{b}{\wedge_1}\right)\left(\frac{2Cp\wedge_1\beta_k^p}{\wedge_1-b}\right)^{\frac{2}{2-p}} + \left[\frac{1}{4}\left(1-\frac{b}{\wedge_1}\right)\bar{r}_k^2 - C(\bar{r}_k^q+\bar{r}_k)\right] \\
&:= \bar{b}_k + \left[\frac{1}{4}\left(1-\frac{b}{\wedge_1}\right)\bar{r}_k^2 - C(\bar{r}_k^q+\bar{r}_k)\right].
\end{aligned}$$

From $\beta_k\to 0$ as $k\to\infty$, we have $\bar{r}_k\to\infty$ as $k\to\infty$. Thus, there exists $k_0\in N$, such that for $k\geq k_0$, $\left[\frac{1}{4}\left(1-\frac{b}{\wedge_1}\right)\bar{r}_k^2 - C(\bar{r}_k^q+\bar{r}_k)\right]\geq 0$. Therefore, for $k\geq k_0$,

$$\bar{b}_k(\mu) := \inf_{u\in Z_k,\|u\|=\bar{r}_k} I_\mu(u)\geq \bar{b}_k.$$

Moreover, $\bar{b}_k\to\infty$ as $k\to\infty$ from $\beta_k\to 0$ as $k\to\infty$. On the other hand, by $(f_1)$ and $(f_3)$, we obtain that for any $M>0$, there exists $C(M)>0$ such that

$$F(x,u)\geq M|u|^2 - C(M). \tag{3.7}$$

Combining $(g_2)$, $(f_4)$, (3.4) and (3.7), there holds

$$\begin{aligned}
I_\mu(u) &\leq \frac{1}{2}\|u\|^2 + \frac{1}{2}|b|\int_\Omega u^2\mathrm{d}x + C\int_\Omega |u|^q\mathrm{d}x - \int_\Omega F(x,u)\mathrm{d}x \\
&\leq C\|u\|^2 + C\|u\|^q - M\int_\Omega |u|^2\mathrm{d}x + C(M)meas(\Omega).
\end{aligned}$$

Choosing $M>0$ large enough, we have that for $u\in Y_k$,

$$I_\mu(u)\leq -C\|u\|^2 + C\|u\|^q + C,$$

using the equivalence of all norms on the finite dimensional space $Y_k$ . Now we choose $\bar{\rho}_k>0$ large enough, such that $\bar{\rho}_k>\bar{r}_k$ and

$$\bar{a}_k(\mu) := \max_{u\in Y_k,\|u\|=\bar{\rho}_k} I_\mu(u)\leq 0.$$

Thus, the conditions of Theorem 2.1 are satisfied for $k\geq k_0$. For $k\geq k_0$, from Theorem 2.1, we obtain that for all $\mu\in[1,2]$, $\bar{c}_k(\mu)\geq\bar{b}_k(\mu)\geq\bar{b}_k$, where $\bar{c}_k(\mu) := \inf_{\gamma\in\Gamma_k}\max_{u\in B_k} I_\mu(\gamma(u))$, $B_k := \{u\in Y_k : \|u\|\leq\bar{\rho}_k\}$ and $\Gamma_k := \{\gamma\in C(B_k,H) : \gamma \textit{ is odd}, \gamma|_{\partial B_k} = id\}$. Moreover, $\bar{c}_k(\mu)\leq\sup_{u\in B_k} I(u) := \bar{c}_k$. Fix $k\geq k_0$, we have that for *a.e.* $\mu\in[1,2]$, there exists a sequence $\{u_n^k(\mu)\}_{n=1}^\infty$ such that

$$\sup_n\|u_n^k(\mu)\|<\infty,\quad I'_\mu(u_n^k(\mu))\to 0\quad \textit{and}\;\; I_\mu(u_n^k(\mu))\to\bar{c}_k(\mu)\geq\bar{b}_k\quad \textit{as}\;\; n\to\infty.$$

Recalling that $\bar{c}_k(\mu)\leq\bar{c}_k$, by standard argument, we conclude that $\{u_k^n(\mu)\}_{n=1}^\infty$ has a convergent subsequence. Suppose $u_n^k(\mu)\to u^k(\mu)$ as $n\to\infty$. We get $I'_\mu(u^k(\mu))=0$, $I_\mu(u^k(\mu))\in[\bar{b}_k,\bar{c}_k]$, for almost every $\mu\in[1,2]$. So, when $\mu_n\to 1$ with $\mu_n\in[1,2]$, we find a sequence $u^k(\mu_n)$ (denote by $u_n$ for simplicity) satisfying $I'_{\mu_n}(u_n)=0$, $I_{\mu_n}(u_n)\in[\bar{b}_k,\bar{c}_k]$.

Consider $I_\mu^*: H\to R$ defined by

$$I_\mu^*(u) := \frac{1}{2}\|u\|^2 - \mu\int_\Omega[F(x,u)+\frac{1}{2}bu^2]\mathrm{d}x - \int_\Omega G_1(x,u)\mathrm{d}x,\qquad \mu\in[1,2],$$

where $G_1(x,u)=\int_0^u g_1(x,s)\mathrm{d}s$.

**Lemma 3.2.** Assume that $f$, $g$ satisfy conditions of Lemma 3.1. Then, given $a<\lambda_1$ and $b\geq\wedge_1$, there exists $k_0'\in N$, such that for $k\geq k_0'$, there exist $c_k^*\geq b_k^*>0$, $b_k^*\to\infty$ as $k\to\infty$. Moreover, fix $k\geq k_0'$, there exist $\mu_n\to 1$ as $n\to\infty$ and $\{u_n\}_{n=1}^\infty\subset H$ such that

$$I_{\mu_n}^{*'}(u_n)=0,\qquad I_{\mu_n}^*(u_n)\in[b_k^*,c_k^*].$$

**Proof.** It is easy to prove $(F_1)-(F_2)$ hold. $(3.3)-(3.5)$ imply that for $u \in Z_k$,

$$\begin{aligned} I_\mu^*(u) &= \frac{1}{2}\|u\|^2 - \mu\int_\Omega [F(x,u)+\frac{1}{2}bu^2]\mathrm{d}x - \int_\Omega G_1(x,u)\mathrm{d}x \\ &\geq \frac{1}{2}\|u\|^2 - b\int_\Omega u^2\mathrm{d}x - C\|u\|^q - C\|u\| - C\beta_k^p\|u\|^p. \end{aligned} \tag{3.8}$$

Since $b \geq \wedge_1$, there exists $j \in N$, such that $\frac{\wedge_j}{2} \leq b < \frac{\wedge_{j+1}}{2}$. Combining (3.1) and (3.8), we obtain that for $u \in Z_k$, $k \geq j+1$,

$$I_\mu^*(u) \geq \frac{1}{2}\left(1-\frac{2b}{\wedge_{j+1}}\right)\|u\|^2 - C(\|u\|^q+\|u\|) - C\beta_k^p\|u\|^p.$$

For $k \geq j+1$, choosing $r_k^* := \left(\frac{2Cp\wedge_{j+1}\beta_k^p}{\wedge_{j+1}-2b}\right)^{\frac{1}{2-p}}$, we obtain that for $u \in Z_k$, $\|u\| = r_k^*$,

$$\begin{aligned} I_\mu^*(u) &\geq \left[\frac{1}{4}\left(1-\frac{2b}{\wedge_{j+1}}\right)\|u\|^2 - C\beta_k^p\|u\|^p\right] + \left[\frac{1}{4}\left(1-\frac{2b}{\wedge_{j+1}}\right)\|u\|^2 - C(\|u\|^q+\|u\|)\right] \\ &\geq \frac{p-2}{4p}\left(1-\frac{2b}{\wedge_{j+1}}\right)\left(\frac{2Cp\wedge_{j+1}\beta_k^p}{\wedge_{j+1}-2b}\right)^{\frac{2}{2-p}} + \left[\frac{1}{4}\left(1-\frac{2b}{\wedge_{j+1}}\right)r_k^{*2} - C(r_k^{*q}+r_k^*)\right] \\ &:= b_k^* + \left[\frac{1}{4}\left(1-\frac{2b}{\wedge_{j+1}}\right)r_k^{*2} - C(r_k^{*q}+r_k^*)\right]. \end{aligned}$$

It is easy to see that $r_k^* \to \infty$ as $k \to \infty$. Thus, there exists $k_0' \geq j+1$, $k_0' \in N$, such that for $k \geq k_0'$,

$$b_k^*(\mu) := \inf_{u\in Z_k, \|u\|=r_k^*} I_\mu^*(u) \geq b_k^*.$$

Moreover, $b_k^* \to \infty$ as $k \to \infty$ . On the other hand, combining $(f_4)$, (3.4) and (3.7), we obtain that for any $M > 0$, there exists $C(M) > 0$, such that

$$\begin{aligned} I_\mu^*(u) &\leq \frac{1}{2}\|u\|^2 - \int_\Omega F(x,u)\mathrm{d}x - \int_\Omega G_1(x,u)\mathrm{d}x \\ &\leq C\|u\|^2 + C\|u\|^q - M\|u\|_2^2 + C(M)meas(\Omega). \end{aligned}$$

Choosing $M > 0$ large enough, we have that for $u \in Y_k$,

$$I_\mu^*(u) \leq -C\|u\|^2 + C\|u\|^q + C.$$

Thus, we can choose $\rho_k^* > 0$ large enough, such that $\rho_k^* > r_k^*$ and

$$a_k^*(\mu) := \max_{u\in Y_k, \|u\|=\rho_k^*} I_\mu^*(u) \leq 0.$$

The rest of the proof is just the same as Lemma 3.1, we omit it.

**Proof of Theorem 1.1.** $(f_1)-(f_4)$ and $(g_1)-(g_3)$ imply that Lemma 3.1 holds. Fix $k \geq k_0$,we claim that the sequence $\{u_n\}$ of Lemma 3.1 is bounded under assumptions of Theorem 1.1. Indeed, $(f_1)$ and $(f_5)$ imply that

$$\frac{1}{2}f(x,u)u - F(x,u) \geq C|u|^\mu - C. \tag{3.9}$$

Together with $(g_2)-(g_3)$, there holds

$$\begin{aligned} &I_{\mu_n}(u_n) - \frac{1}{2}(I'_{\mu_n}(u_n), u_n) \\ =&\mu_n\int_\Omega [\frac{1}{2}f(x,u_n)u_n - F(x,u_n)]\mathrm{d}x + \int_\Omega [\frac{1}{2}g(x,u_n)u_n - G(x,u_n)]\mathrm{d}x \\ \geq& C\int_\Omega |u_n|^\mu\mathrm{d}x - C\int_\Omega |u_n|^q\mathrm{d}x - C. \end{aligned}$$

Since

$$I_{\mu_n}(u_n) - \frac{1}{2}(I'_{\mu_n}(u_n), u_n) = I_{\mu_n}(u_n) \in [\bar{b}_k, \bar{c}_k],$$

we get

$$\|u_n\|_\mu \le C, \tag{3.10}$$

in view of $\mu > q$. (3.10) implies that

$$\|u_n\|_q \le C. \tag{3.11}$$

Thus, from $(f_2)$, $(g_2)-(g_3)$, (3.2), (3.11) and $(I'_{\mu_n}(u_n), u_n) = 0$,

$$\begin{aligned}\|u_n\|^2 &= \mu_n \int_\Omega f(x,u_n)u_n \mathrm{d}x + \int_\Omega g(x,u_n)u_n \mathrm{d}x \\ &\le b\int_\Omega u_n^2 \mathrm{d}x + C\int_\Omega (|u_n| + |u_n|^p)\mathrm{d}x + C \\ &\le \frac{b}{\wedge_1}\|u_n\|^2 + C\int_\Omega (|u_n| + |u_n|^p)\mathrm{d}x + C,\end{aligned}$$

which implies that

$$\|u_n\|^2 \le C\int_\Omega (|u_n| + |u_n|^p)\mathrm{d}x + C. \tag{3.12}$$

Note that $\frac{N}{2}(p-2) < (p-1)\frac{2N}{N+2} < p$, we will consider two cases.
Case 1. $\mu > q$ and $\mu \ge (p-1)\frac{2N}{N+2}$.
From (3.10) and (3.12),

$$\begin{aligned}\|u_n\|^2 &\le C\|u_n\| + C\int_\Omega |u_n|^{p-1}|u_n|\mathrm{d}x + C \\ &\le C\|u_n\| + C\|u_n\|_{2^*}\|u_n\|^{p-1}_{(p-1)\frac{2N}{N+2}} + C \\ &\le C\|u_n\| + C\|u_n\|\|u_n\|_\mu^{p-1} + C \\ &\le C\|u_n\| + C, \end{aligned} \tag{3.13}$$

which implies that $\|u_n\| \le C$.
Case 2. $\mu > q$ and $\frac{N}{2}(p-2) < \mu < p$.
We need the following well known inequality (3.14).
If $0 < \mu < p < 2^*$ and $t \in (0,1)$ are such that $\frac{1}{p} = \frac{1-t}{\mu} + \frac{t}{2^*}$ then

$$\|u\|_p \le \|u\|_\mu^{1-t}\|u\|_{2^*}^t, \qquad \forall u \in L^\mu \cap L^{2^*}. \tag{3.14}$$

Combining (3.10), (3.12) and (3.14), there holds

$$\begin{aligned}\|u_n\|^2 &\le C\|u_n\| + C\|u_n\|_\mu^{(1-t)p}\|u_n\|_{2^*}^{tp} + C \\ &\le C\|u_n\| + C\|u_n\|_{2^*}^{tp} + C \\ &\le C\|u_n\| + C\|u_n\|^{tp} + C.\end{aligned} \tag{3.15}$$

Observing that the condition $\mu > \frac{N}{2}(p-2)$ is equivalent to $tp < 2$, we conclude from (3.15) that $\|u_n\| \le C$. The claim is proved. Combining with Lemma 3.1 and by standard argument, we obtain that $\{u_n\}$ has a convergent subsequence (denote by $u_n$ for simplicity). Since $u_n$ is relevant to the choice of $k$, we suppose that $u_n \to u^k$ in $H$, as $n \to \infty$. We note that

$$I(u_n) = I_{\mu_n}(u_n) + (\mu_n - 1)\int_\Omega F(x,u_n)\mathrm{d}x. \tag{3.16}$$

Since $\sup_n \|u_n\| < \infty$, we conclude that $\int_\Omega F(x,u_n)\mathrm{d}x$ stays bounded as $n \to \infty$. Recalling that $I_{\mu_n}(u_n) \in [\overline{b}_k, \overline{c}_k]$, we get

$$I(u^k) = \lim_{n\to\infty} I(u_n) \in [\overline{b}_k, \overline{c}_k]. \tag{3.17}$$

On the other hand, we have

$$(I'(u_n), v) = (I'_{\mu_n}(u_n), v) + (\mu_n - 1)\int_\Omega f(x,u_n)v\mathrm{d}x \quad \text{for all} \;\; v \in H. \tag{3.18}$$

Combining with $I'_{\mu_n}(u_n) = 0$ and $\sup_n \|u_n\| < \infty$,we obtain that

$$\lim_{n\to\infty}(I'(u_n), v) = 0 \quad \text{for all} \;\; v \in H.$$

Since $I \in C^1(H)$, we have $I'(u_n) \to I'(u^k)$ in $H^*$. Thus, $(I'(u^k), v) = 0, \ \forall v \in H$. Combining with (3.17) and $\overline{b}_k \to \infty$ as $k \to \infty$, we know that the conclusion of Theorem 1.1 holds.

**Proof of Theorem 1.2.** $(f_1) - (f_4)$ and $(g_1) - (g_3)$ imply that Lemma 3.2 holds. Fix $k \geq k_0'$,we claim that the sequence $\{u_n\}$ of Lemma 3.2 is bounded under assumptions of Theorem 1.2. Seeking a contradiction we suppose that $\|u_n\| \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|}$. Up to a sequence, we get

$$\begin{aligned}
&v_n \rightharpoonup v \ \ \textit{weakly} \ \ \textit{in} \ \ H, \\
&v_n \to v \ \ \textit{strongly} \ \ \textit{in} \ \ L^t(\Omega), \qquad 1 \leq t < 2^*, \\
&v_n(x) \to v(x) \qquad \textit{a.e.} \ \ x \in \Omega.
\end{aligned}$$

We consider two cases.
Case 1. $v \neq 0$ in $H$.
By $I^*_{\mu_n}(u_n) \in [b^*_k, c^*_k]$, there holds

$$\frac{1}{2}\|u_n\|^2 - \mu_n \int_\Omega [F(x, u_n) + \frac{1}{2} b u_n^2] \mathrm{d}x - \int_\Omega G_1(x, u_n) \mathrm{d}x \geq b^*_k.$$

Divided by $\|u_n\|^2$ in both sides of the above equality, we get

$$\begin{aligned}
\frac{1}{2} + o(1) \geq &\int_\Omega \frac{F(x, u_n)}{\|u_n\|^2} \mathrm{d}x + \frac{b}{2}\int_\Omega v_n^2 \mathrm{d}x \\
= &\int_\Omega \frac{F(x, u_n)}{\|u_n\|^2} \mathrm{d}x + \frac{b}{2}\int_\Omega v^2 \mathrm{d}x,
\end{aligned} \tag{3.19}$$

in view of $(f_4)$ and (3.4). On the other hand, set $\Omega_1 := \{x \in \Omega, v(x) \neq 0\}$. Since $meas(\Omega_1) > 0$ and for $x \in \Omega_1$,

$$\lim_{n \to \infty} \frac{F(x, u_n)}{\|u_n\|^2} = +\infty,$$

using Fatou's lemma, we obtain that

$$\lim_{n \to \infty} \int_\Omega \frac{F(x, u_n)}{\|u_n\|^2} \mathrm{d}x \geq \lim_{n \to \infty} \int_{\Omega_1} \frac{F(x, u_n)}{\|u_n\|^2} \mathrm{d}x = +\infty,$$

which contradicts (3.19).
Case 2. $v = 0$ in $H$.
Since $b \geq \wedge_1$, there exists $i \in N$, such that $\frac{1}{2}\wedge_i \leq b < \frac{1}{2}\wedge_{i+1}$. We note that $H = Y_i \bigoplus Y_i^\perp$, where $Y_i = \bigoplus_{j=1}^i X_j$. Decompose $u_n$ as $u_n = u_{n1} + u_{n2}$, where $u_{n1} \in Y_i$ and $u_{n2} \in Y_i^\perp$. From (3.1) and $I^{*'}_{\mu_n}(u_n) = 0$, we have

$$\begin{aligned}
0 = &(I^{*'}_{\mu_n}(u_n), u_{n2}) \\
= &\|u_{n2}\|^2 - \mu_n \int_\Omega b u_{n2}^2 \mathrm{d}x - \mu_n \int_\Omega f(x, u_n) u_{n2} \mathrm{d}x - \int_\Omega g_1(x, u_n) u_{n2} \mathrm{d}x \\
\geq &\|u_{n2}\|^2 - 2b \int_\Omega u_{n2}^2 \mathrm{d}x - \mu_n \int_\Omega f(x, u_n) u_{n2} \mathrm{d}x - \int_\Omega g_1(x, u_n) u_{n2} \mathrm{d}x \\
\geq &\left(1 - \frac{2b}{\lambda_{i+1}}\right) \|u_{n2}\|^2 - \mu_n \int_\Omega f(x, u_n) u_{n2} \mathrm{d}x - \int_\Omega g_1(x, u_n) u_{n2} \mathrm{d}x.
\end{aligned} \tag{3.20}$$

Combining $(f_2)$, $(g_3)$ and (3.20), there holds

$$\begin{aligned}
\|u_{n2}\|^2 \leq &C \int_\Omega (|u_{n2}| + |u_n|^{p-1}|u_{n2}|) \mathrm{d}x + C \int_\Omega |u_n|^{q-1}|u_{n2}| \mathrm{d}x \\
\leq &C\|u_{n2}\| + C\int_\Omega |u_n|^{p-1}|u_{n2}| \mathrm{d}x + C\|u_{n2}\|_2 \|u_n\|^{q-1}_{2(q-1)} \\
\leq &C\|u_{n2}\| + C\int_\Omega |u_n|^{p-1}|u_{n2}| \mathrm{d}x + C\|u_{n2}\| \|u_n\|^{q-1} \\
\leq &C\|u_n\| + C\|u_n\|^q + C\int_\Omega |u_n|^{p-1}|u_{n2}| \mathrm{d}x.
\end{aligned} \tag{3.21}$$

Divided by $\|u_n\|^2$ in both sides of (3.21) and noting that $\|u_n\| \to \infty$,

$$\frac{\|u_{n2}\|^2}{\|u_n\|^2} \leq o(1) + \frac{C\int_\Omega |u_n|^{p-1}|u_{n2}|\mathrm{d}x}{\|u_n\|^2}. \tag{3.22}$$

Arguing as in the proof of Theorem 1.1, we conclude that for $\mu > q$, $\|u_n\|_\mu \leq C$. Thus, for $\mu > q$ and $\mu \geq (p-1)\frac{2N}{N+2}$, (3.22) implies that

$$\begin{aligned}\frac{\|u_{n2}\|^2}{\|u_n\|^2} \leq & o(1) + \frac{C\|u_{n2}\|_{2^*}\|u_n\|^{p-1}_{(p-1)\frac{2N}{N+2}}}{\|u_n\|^2} \\ \leq & o(1) + \frac{C\|u_{n2}\|\|u_n\|^{p-1}_\mu}{\|u_n\|^2} \\ \leq & o(1) + \frac{C\|u_n\|}{\|u_n\|^2} \\ = & o(1). \end{aligned} \tag{3.23}$$

Set $v_{n1} = \frac{u_{n1}}{\|u_n\|}$ and $v_{n2} = \frac{u_{n2}}{\|u_n\|}$. Then, $v_n = v_{n1} + v_{n2}$. (3.23) implies that $v_{n2} \to 0$ strongly in $L^2(\Omega)$. Note that $v_n \to 0$ strongly in $L^2(\Omega)$, we obtain that $v_{n1} \to 0$ strongly in $L^2(\Omega)$. Thus, $v_{n1} \to 0$ strongly in $H$, using the equivalence of all norms on the finite dimensional space. Observe that $v_{n2} \to 0$ strongly in $H$, we have $v_n \to 0$ strongly in $H$, contradicts with $\|v_n\| = 1$. Thus, the claim is proved. The rest of the proof is just the same as Theorem 1.1, we omit the details.

# 4. The case $a \geq \lambda_1$ and $b \geq 0$

In this section we use the norm $\|.\|_H$. It is well known that $\mu_j = \lambda_j^2$, $j = 1, 2, \cdots$, are eigenvalues of the eigenvalue problem

$$\begin{cases} \Delta^2 u = \mu u, & \text{in} \quad \Omega, \\ u = \Delta u = 0, & \text{on} \quad \partial\Omega, \end{cases}$$

$\varphi_j$, $j = 1, 2, \cdots$ are the corresponding eigenfunctions. Furthermore, the set of $\{\varphi_j\}$ is an orthogonal basis on the Hilbert space $H$. Let $X_j := span\{\varphi_j\}$, $j \in N$ and set $U_k = \bigoplus_{j=1}^k X_j$, $V_k = \overline{\bigoplus_{j=k}^\infty X_j}$.

Using the Lax-Milgram Theorem, we deduce that for any $g \in L^r(\Omega)$, $\frac{2N}{N+2} \leq r < \infty$, there exists unique $u \in H$, such that

$$\int_\Omega \Delta u \Delta \varphi \mathrm{d}x = \int_\Omega g\varphi \mathrm{d}x, \quad \forall \varphi \in H. \tag{4.1}$$

From [2], we have

$$\|u\|_{W^{4,r}} \leq C\|g\|_r \tag{4.2}$$

and

$$\Delta^2 u = g \quad in \quad \Omega, \quad \gamma_0(u) = \gamma_0(\Delta u) = 0. \tag{4.3}$$

Here, $\gamma_0(u)$ and $\gamma_0(\Delta u)$ are the traces on the boundary $\partial\Omega$. That is, $\gamma_0$ is a linear continuous operator such that $\gamma_0(v) = v|_{\partial\Omega}$ for all $v \in C(\bar{\Omega})$. Let

$$E := \{u; u \in W^{4,r}(\Omega), \gamma_0(u) = \gamma_0(\Delta u) = 0\}$$

be the linear space equipped with the $W^{4,r}$ norm. It is easy to see that $E$ is a Banach space. Then we can conclude that for any $g \in L^r(\Omega)$, there exists unique $u \in E$ satisfying $\Delta^2 u = g$ and $\|u\|_{W^{4,r}} \leq C\|g\|_r$, where $\Delta^2$ is a linear operator from $E$ to $L^r(\Omega)$. Thus, the inverse operator $(\Delta^2)^{-1}$ is a linear bounded operator from $L^r(\Omega)$ to $E$. The restriction $\frac{2N}{N+2} \leq r < \infty$ ensures that the imbedding $E \hookrightarrow H$ is compact. Hence, the operator $(\Delta^2)^{-1}$ is compact from $L^r(\Omega)$ to $H$.

We observe that the operator $\Delta$ is a linear bounded operator from $H$ to $L^2(\Omega)$. Then the operator $(\Delta^2)^{-1}\Delta$ is compact from $H$ to $H$. On the other hand, we recall that $N_f$ is the Nemytskii operator defined by $(N_f u)(x) = f(x, u(x))$ for $x \in \Omega$. From $(f_1) - (f_2)$ and Proposition 5 in [3], we know that $N_f$ is continuous from $L^p(\Omega)$ to $L^{\frac{p}{p-1}}(\Omega)$ and maps bounded sets into bounded sets. Combining with the compact imbedding $H \hookrightarrow L^p(\Omega)$ and $\frac{p}{p-1} > \frac{2N}{N+2}$, we have $(\Delta^2)^{-1}N_f$ is a compact operator from $H$ to

$H$. Similarly, $(\Delta^2)^{-1}N_{g_1}$ is compact from $H$ to $H$, where $N_{g_1}$ is defined by $(N_{g_1}u)(x) = g_1(x, u(x))$ for $x \in \Omega$.

Now, for $\mu \in [1,2]$, we can define

$$A_\mu(u) := (\Delta^2)^{-1}\left[-a\mu\Delta u + \mu f(x,u) + b\mu u + g_1(x,u)\right], \qquad \mu \in [1,2].$$

Moreover, $A_\mu$ is compact from $H$ to $H$.

Consider $J_\mu : H \to R$ defined by

$$J_\mu(u) := \frac{1}{2}\|u\|_H^2 - \mu\int_\Omega [F(x,u) + \frac{1}{2}a|\nabla u|^2 + \frac{1}{2}bu^2]\mathrm{d}x - \int_\Omega G_1(x,u)\mathrm{d}x, \qquad \mu \in [1,2].$$

**Lemma 4.1.** For $\mu \in [1,2]$, there holds

$$\left\langle J'_\mu(u), u - A_\mu(u)\right\rangle_{H^*,H} = \|u - A_\mu(u)\|_H^2. \tag{4.4}$$

**Proof.** Let $u \in H$ and set $v = A_\mu(u)$. It is easy to see that

$$\int_\Omega \Delta u\Delta v\mathrm{d}x = \int_\Omega u\left[-a\mu\Delta u + \mu f(x,u) + b\mu u + g_1(x,u)\right]\mathrm{d}x$$

and

$$\int_\Omega \Delta v\Delta v\mathrm{d}x = \int_\Omega v\left[-a\mu\Delta u + \mu f(x,u) + b\mu u + g_1(x,u)\right]\mathrm{d}x.$$

Then

$$\begin{aligned}
&\left\langle J'_\mu(u), u - A_\mu(u)\right\rangle_{H^*,H} \\
&= \int_\Omega \Delta u(\Delta u - \Delta v)\mathrm{d}x - a\mu\int_\Omega \nabla u\nabla(u-v)\mathrm{d}x - \mu\int_\Omega f(x,u)(u-v)\mathrm{d}x - b\mu\int_\Omega u(u-v)\mathrm{d}x \\
&\quad - \int_\Omega g_1(x,u)(u-v)\mathrm{d}x \\
&= \int_\Omega (\Delta u - \Delta v)^2\mathrm{d}x.
\end{aligned}$$

**Lemma 4.2.** Assume that $f$, $g$ satisfy conditions of Lemma 3.1. Let $a \geq \lambda_1$ and $\frac{1}{2}\lambda_j < a \leq \frac{1}{2}\lambda_{j+1}$ for some $j \in N$. Then , if $\lambda_i(\frac{1}{2}\lambda_i - a) \leq b < \lambda_{i+1}(\frac{1}{2}\lambda_{i+1} - a)$ for some $i \in N$, $i \geq j+1$, there exists $k''_0 \in N$, such that for $k \geq k''_0$, there exist $\tilde{c}_k \geq \tilde{b}_k > 0$, $\tilde{b}_k \to \infty$ as $k \to \infty$. Moreover, fix $k \geq k''_0$, there exist $\mu_n \to 1$ as $n \to \infty$ and $\{u_n\}_{n=1}^\infty \subset H$, such that

$$J'_{\mu_n}(u_n) = 0, \qquad J_{\mu_n}(u_n) \in [\tilde{b}_k, \tilde{c}_k].$$

**Proof.** It is easy to prove $(F_1) - (F_2)$ hold. For $2 < p < 2^*$, let

$$\alpha_k := \sup_{u \in V_k, \|u\|_H = 1} \|u\|_p. \tag{4.5}$$

Then $\alpha_k \to 0$ as $k \to \infty$ following the method of Lemma 3.8 in [11]. Combining (3.3) – (3.4) and (4.5), we obtain that for $u \in V_k$,

$$\begin{aligned}
J_\mu(u) &= \frac{1}{2}\|u\|_H^2 - \mu\int_\Omega [F(x,u) + \frac{1}{2}a|\nabla u|^2 + \frac{1}{2}bu^2]\mathrm{d}x - \int_\Omega G_1(x,u)\mathrm{d}x \\
&\geq \frac{1}{2}\|u\|_H^2 - \int_\Omega [a|\nabla u|^2 + bu^2]\mathrm{d}x - C\|u\|_H^q - C\|u\|_H - C\alpha_k^p\|u\|_H^p.
\end{aligned} \tag{4.6}$$

We note that $\left\{\frac{\varphi_j}{\|\varphi_j\|_H}\right\}_{j=1}^\infty$ is an orthonormal basis of $H$. Then, for $u \in V_k$, we can write

$$u = \sum_{j=k}^\infty c_j \frac{\varphi_j}{\|\varphi_j\|_H}$$

for $c_j = \left(u, \frac{\varphi_j}{\|\varphi_j\|_H}\right)_H$, the series converging in $H$. In additional,

$$\|u\|_H^2 = \sum_{j=k}^{\infty} c_j^2. \tag{4.7}$$

Denote $u_m = \sum_{j=k}^{m} c_j \frac{\varphi_j}{\|\varphi_j\|_H}$, where $m \in N$, $m \geq k$ . Thus,

$$\lim_{m\to\infty} \|u_m - u\|_H = 0.$$

Recall that $H_0^1(\Omega)$ is the Hilbert space equipped with the inner product

$$(u, v)_{H_0^1} = \int_\Omega \nabla u \nabla v \mathrm{d}x,$$

which induces the norm

$$\|u\|_{H_0^1} = (\int_\Omega |\nabla u|^2 \mathrm{d}x)^{\frac{1}{2}}.$$

We note that for $u \in H$,

$$\int_\Omega |\nabla u|^2 \mathrm{d}x \leq C \int_\Omega |\Delta u|^2 \mathrm{d}x.$$

Thus,

$$\lim_{m\to\infty} \|u_m - u\|_{H_0^1} = 0.$$

Now, we rewrite

$$u = \sum_{j=k}^{\infty} c_j \frac{\|\varphi_j\|_{H_0^1}}{\|\varphi_j\|_H} \frac{\varphi_j}{\|\varphi_j\|_{H_0^1}},$$

the series converging in $H_0^1(\Omega)$. Note that $\left\{ \frac{\varphi_j}{\|\varphi_j\|_{H_0^1}} \right\}_{j=1}^{\infty}$ is an orthonormal basis of $H_0^1(\Omega)$, we have

$$\|u\|_{H_0^1}^2 = \sum_{j=k}^{\infty} c_j^2 \frac{\|\varphi_j\|_{H_0^1}^2}{\|\varphi_j\|_H^2}. \tag{4.8}$$

Combining (4.7) $-$ (4.8), we obtain that for $u \in V_k$,

$$\|u\|_H^2 \geq \lambda_k \|u\|_{H_0^1}^2. \tag{4.9}$$

Similarly, we can deduce that for $u \in V_k$,

$$\|u\|_{H_0^1}^2 \geq \lambda_k \|u\|_2^2. \tag{4.10}$$

(4.6) and (4.9) $-$ (4.10) imply that for $u \in V_k$, $k \geq i+1$,

$$J_\mu(u) \geq \left[\frac{1}{2} - \frac{1}{\lambda_{i+1}}\left(a + \frac{b}{\lambda_{i+1}}\right)\right] \|u\|_H^2 - C\|u\|_H^q - C\|u\|_H - C\alpha_k^p \|u\|_H^p. \tag{4.11}$$

For $k \geq i+1$, choosing $\tilde{r}_k := \left(\frac{2Cp\lambda_{i+1}\alpha_k^p}{\lambda_{i+1} - 2\left(a + \frac{b}{\lambda_{i+1}}\right)}\right)^{\frac{1}{2-p}}$, we have that for $u \in V_k$, $\|u\| = \tilde{r}_k$,

$$\begin{aligned}
J_\mu(u) \geq & \left\{ \frac{1}{2}\left[\frac{1}{2} - \frac{1}{\lambda_{i+1}}\left(a + \frac{b}{\lambda_{i+1}}\right)\right] \|u\|_H^2 - C\alpha_k^p \|u\|_H^p \right\} \\
& + \left\{ \frac{1}{2}\left[\frac{1}{2} - \frac{1}{\lambda_{i+1}}\left(a + \frac{b}{\lambda_{i+1}}\right)\right] \|u\|_H^2 - C\|u\|_H^q - C\|u\|_H \right\} \\
\geq & \left(\frac{1}{2} - \frac{1}{p}\right)\left[\frac{1}{2} - \frac{1}{\lambda_{i+1}}\left(a + \frac{b}{\lambda_{i+1}}\right)\right] \left(\frac{2Cp\lambda_{i+1}\alpha_k^p}{\lambda_{i+1} - 2\left(a + \frac{b}{\lambda_{i+1}}\right)}\right)^{\frac{2}{2-p}} \\
& + \frac{1}{2}\left[\frac{1}{2} - \frac{1}{\lambda_{i+1}}\left(a + \frac{b}{\lambda_{i+1}}\right)\right] \tilde{r}_k^2 - C\tilde{r}_k^q - C\tilde{r}_k \\
:= & \tilde{b}_k + \frac{1}{2}\left[\frac{1}{2} - \frac{1}{\lambda_{i+1}}\left(a + \frac{b}{\lambda_{i+1}}\right)\right] \tilde{r}_k^2 - C\tilde{r}_k^q - C\tilde{r}_k.
\end{aligned} \tag{4.12}$$

We note that $\tilde{r}_k \to \infty$ as $k \to \infty$. Thus, there exists $k_0'' \geq j+1$, $k_0'' \in N$, such that for $k \geq k_0''$,

$$\tilde{b}_k(\mu) := \inf_{u \in V_k, \|u\|_H = \tilde{r}_k} J_\mu(u) \geq \tilde{b}_k.$$

Moreover, $\tilde{b}_k \to \infty$ as $k \to \infty$. On the other hand, combining $(f_4)$, (3.4) and (3.7), we obtain that for any $M > 0$, there exists $C(M) > 0$, such that

$$\begin{aligned} J_\mu(u) &\leq \frac{1}{2}\|u\|_H^2 - \int_\Omega F(x,u)\mathrm{d}x - \int_\Omega G_1(x,u)\mathrm{d}x \\ &\leq \frac{1}{2}\|u\|_H^2 + C\|u\|_H^q - M\|u\|_2^2 + C(M)meas(\Omega). \end{aligned}$$

Choosing $M > 0$ large enough, we have that for $u \in U_k$,

$$J_\mu(u) \leq -C\|u\|_H^2 + C\|u\|_H^q + C.$$

Thus, we can choose $\tilde{\rho}_k > 0$ large enough, such that $\tilde{\rho}_k > \tilde{r}_k$ and

$$\tilde{a}_k(\mu) := \max_{u \in U_k, \|u\| = \tilde{\rho}_k} J_\mu(u) \leq 0.$$

Thus, the conditions of Theorem 2.1 are satisfied for $k \geq k_0''$. For $k \geq k_0''$, from Theorem 2.1, we obtain that for all $\mu \in [1,2]$, $\tilde{c}_k(\mu) \geq \tilde{b}_k(\mu) \geq \tilde{b}_k$, where $\tilde{c}_k(\mu) := \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} J_\mu(\gamma(u))$, $B_k := \{u \in U_k : \|u\| \leq \tilde{\rho}_k\}$ and $\Gamma_k := \{\gamma \in C(B_k, H) : \gamma \ is \ odd, \gamma|_{\partial B_k} = id\}$. Moreover, $\tilde{c}_k(\mu) \leq \sup_{u \in B_k} J(u) := \tilde{c}_k$. Fix $k \geq k_0''$, we have that for *a.e.* $\mu \in [1,2]$, there exists a sequence $\{u_n^k(\mu)\}_{n=1}^\infty$ such that

$$\sup_n \|u_n^k(\mu)\|_H < \infty, \quad J_\mu'(u_n^k(\mu)) \to 0 \quad and \ \ J_\mu(u_n^k(\mu)) \to \tilde{c}_k(\mu) \geq \tilde{b}_k \quad as \ \ n \to \infty.$$

Thus, from Lemma 4.1, we conclude that

$$u_n^k(\mu) - A_\mu(u_n^k(\mu)) \to 0 \quad strongly \ \ in \ \ H.$$

Recalling that $A_\mu$ is compact from $H$ to $H$, combining with $\sup_n \|u_n^k(\mu)\|_H < \infty$, we deduce that $\{u_k^n(\mu)\}_{n=1}^\infty$ has a convergent subsequence. Suppose $u_n^k(\mu) \to u^k(\mu)$ as $n \to \infty$. We get $J_\mu'(u^k(\mu)) = 0$, $J_\mu(u^k(\mu)) \in [\tilde{b}_k, \tilde{c}_k]$, for almost every $\mu \in [1,2]$. So, when $\mu_n \to 1$ with $\mu_n \in [1,2]$, we find a sequence $u^k(\mu_n)$ (denote by $u_n$ for simplicity) satisfying $J_{\mu_n}'(u_n) = 0$, $J_{\mu_n}(u_n) \in [\tilde{b}_k, \tilde{c}_k]$.

**Proof of Theorem 1.3.** $(f_1)-(f_4)$ and $(g_1)-(g_3)$ imply that Lemma 4.2 holds. Fix $k \geq k_0''$,we claim that the sequence $\{u_n\}$ of Lemma 4.2 is bounded under assumptions of Theorem 1.3. Seeking a contradiction we suppose that $\|u_n\|_H \to \infty$. Let $w_n = \frac{u_n}{\|u_n\|_H}$. Up to a sequence, we get

$$\begin{aligned} &w_n \rightharpoonup w \ \ weakly \ \ in \ \ H, \\ &w_n \to w \ \ strongly \ \ in \ \ L^t(\Omega), \quad 1 \leq t < 2^*, \\ &w_n(x) \to w(x) \quad a.e. \ \ x \in \Omega. \end{aligned}$$

We consider two cases.

Case 1. $w \neq 0$ in $H$.

By $J_{\mu_n}(u_n) \in [\tilde{b}_k, \tilde{c}_k]$, there holds

$$\frac{1}{2}\|u_n\|_H^2 \geq \tilde{b}_k + \int_\Omega F(x,u_n)\mathrm{d}x + \int_\Omega G_1(x,u_n)\mathrm{d}x.$$

Divided by $\|u_n\|_H^2$ in both sides of the above equality and in view of (3.4), we get

$$\frac{1}{2} \geq o(1) + \int_\Omega \frac{F(x,u_n)}{\|u_n\|_H^2}\mathrm{d}x. \tag{4.13}$$

Set $\Omega_2 := \{x \in \Omega, w(x) \neq 0\}$. Since $meas(\Omega_2) > 0$ and for $x \in \Omega_2$,

$$\lim_{n\to\infty} \frac{F(x,u_n)}{\|u_n\|_H^2} = +\infty,$$

using Fatou's lemma, we obtain that

$$\lim_{n\to\infty}\int_\Omega \frac{F(x,u_n)}{\|u_n\|_H^2}\mathrm{d}x \geq \lim_{n\to\infty}\int_{\Omega_2} \frac{F(x,u_n)}{\|u_n\|_H^2}\mathrm{d}x = +\infty,$$

which contradicts (4.13).

Case 2. $w = 0$ in $H$.

We note that $H = Y_i \bigoplus Y_i^\perp$, where $Y_i = \bigoplus_{j=1}^i X_j$. Decompose $u_n$ as $u_n = u_{n1} + u_{n2}$, where $u_{n1} \in Y_i$ and $u_{n2} \in Y_i^\perp$. In view of (4.9) $-$ (4.10), we have

$$\begin{aligned}
0 =&(J'_{\mu_n}(u_n), u_{n2})\\
=&\|u_{n2}\|_H^2 - \mu_n \int_\Omega [a|\nabla u_{n2}|^2 + bu_{n2}^2]\mathrm{d}x - \mu_n \int_\Omega f(x,u_n)u_{n2}\mathrm{d}x - \int_\Omega g_1(x,u_n)u_{n2}\mathrm{d}x\\
\geq&\|u_{n2}\|_H^2 - 2\int_\Omega [a|\nabla u_{n2}|^2 + bu_{n2}^2]\mathrm{d}x - \mu_n \int_\Omega f(x,u_n)u_{n2}\mathrm{d}x - \int_\Omega g_1(x,u_n)u_{n2}\mathrm{d}x\\
\geq&\left[1 - \frac{1}{\lambda_{i+1}}\left(a + \frac{b}{\lambda_{i+1}}\right)\right]\|u_{n2}\|_H^2 - \mu_n \int_\Omega f(x,u_n)u_{n2}\mathrm{d}x - \int_\Omega g_1(x,u_n)u_{n2}\mathrm{d}x.
\end{aligned} \tag{4.14}$$

Combining $(f_2)$, $(g_3)$ and (4.14), there holds

$$\|u_{n2}\|_H^2 \leq C\|u_n\|_H + C\|u_n\|_H^q + C\int_\Omega |u_n|^{p-1}|u_{n2}|\mathrm{d}x. \tag{4.15}$$

Divided by $\|u_n\|_H^2$ in both sides of (4.15),

$$\frac{\|u_{n2}\|_H^2}{\|u_n\|_H^2} \leq o(1) + \frac{C\int_\Omega |u_n|^{p-1}|u_{n2}|\mathrm{d}x}{\|u_n\|_H^2}. \tag{4.16}$$

Arguing as in the proof of Theorem 1.2, we conclude that $w_n \to 0$ strongly in $H$, contradicts with $\|w_n\|_H = 1$. Thus, the claim is proved. Now we will prove the sequence $\{u_n\}$ has a convergent subsequence. Observe that

$$(J'(u_n), v) = (J'_{\mu_n}(u_n), v) + (\mu_n - 1)\int_\Omega [f(x,u_n)v + a\nabla u_n \nabla v + bu_n v]\mathrm{d}x \quad \textit{for all } v \in H.$$

Combining with $J'_{\mu_n}(u_n) = 0$ and $\sup_n \|u_n\|_H < \infty$,we have

$$\lim_{n\to\infty}(J'(u_n), v) = 0 \quad \textit{for all } v \in H.$$

Thus, from Lemma 4.1, we conclude that

$$u_n - A_1(u_n) \to 0 \quad \textit{strongly in } H.$$

Recalling that $A_1$ is compact from $H$ to $H$, together with $\sup_n \|u_n\|_H < \infty$, we know that $\{u_n\}$ has a convergent subsequence. Since $u_n$ is relevant to the choice of $k$, we suppose that $u_n \to u^k$ in $H$, as $n \to \infty$. We note that

$$J(u_n) = J_{\mu_n}(u_n) + (\mu_n - 1)\int_\Omega [F(x,u_n) + \frac{1}{2}a|\nabla u_n|^2 + \frac{1}{2}bu_n^2]\mathrm{d}x.$$

Combining with $J_{\mu_n}(u_n) \in [\tilde{b}_k, \tilde{c}_k]$, we get

$$J(u^k) = \lim_{n\to\infty} J(u_n) \in [\tilde{b}_k, \tilde{c}_k].$$

Besides,

$$J'(u^k) = \lim_{n\to\infty} J'(u_n) = 0.$$

Recalling that $\tilde{b}_k \to \infty$ as $k \to \infty$, we get the conclusion of Theorem 1.3.


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**References**


[1] A. Ambrosetti, P. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.* 14 (1973) 349-381.

[2] F. Bernis, J. Garcia-Azorero, I. Peral, Existence and multiplicity of nontrivial solutions in semilinear critical problems of fourth order, *Adv. Differential Equations* 1 (2)(1996) 219-240.

[3] G. Dinca, P. Jebelean, J. Mawhin, Variational and topological methods for Dirichlet problems with p-Laplacian, *Portugal. Math.* 58 (2001) 339-378.

[4] A. C. Lazer, P. J. Mckenna, Large amplitude periodic oscillations in suspension bridges: Some new connections with nonlinear analysis, *SIAM Rev.* 32 (1990) 537-578.

[5] A. C. Lazer, P. J. Mckenna, Global bifurcation and a theorem of Tarantello, *J. Math. Anal. Appl.* 181 (1994) 648-655.

[6] X. Liu, Y. Huang, On sign-changing solution for a fourth-order asymptotically linear elliptic problem, *Nonlinear Anal.* 72 (2010) 2271-2276.

[7] A. M. Micheletti, A. Pistoia, Multiplicity results for a fourth-order semilinear elliptic problem, *Nonlinear Anal.* 31 (1998) 895-908.

[8] A. M. Micheletti, A. Pistoia, Nontrivial solutions for some fourth order semilinear elliptic problems, *Nonlinear Anal.* 34 (1998) 509-523.

[9] P. H. Rabinowitz, Minimax Methods in Critical Point Theory with Applications to Differential Equations, *in: CBMS Reg. Conf. Series Math.* vol. 65, Amer. Math. Soc., Providence, RI, 1986.

[10] G. Tarantello, A note on a semilinear elliptic problem, *Differential Integral Equations* 5 (1992) 561-566.

[11] M. Willem, Minimax Theorems. Birkhäuser, Boston, 1996.

[12] G. Xu, J. Zhang, Existence results for some fourth-order nonlinear elliptic problems of local superlinearity and sublinearity, *J. Math. Anal. Appl.* 281 (2003) 633-640.

[13] Y. Yang, J. Zhang, Existence of solutions for some fourth-order nonlinear elliptic problems, *J. Math. Anal. Appl.* 351 (2009) 128-137.

[14] W. Zou, Variant Fountain Theorems and their applications, *Manuscripta Math.* 104 (2001) 343-358.

[15] J. Zhou, X. Wu, Sign-changing solutions for some fourth-order nonlinear elliptic problems, *J. Math. Anal. Appl.* 342 (2008) 542-558.


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