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Optimal decay estimates for solutions to damped second order ODE's

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Abstract. In this paper we derive optimal decay estimates for solutions to second order ordinary differential equations with weak damping. The main assumptions are Kurdyka–Łojasiewicz gradient inequality and its inverse.

Keywords: Kurdyka–Łojasiewicz inequality, rate of convergence to equilibrium, second order equation with damping.

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1 Introduction

In this paper we study long-time behavior for solutions of damped second order ordinary differential equations

$$\ddot{u} + g(\dot{u}) + \nabla E(u) = 0, \tag{SOP}$$

where $E \in C^2(\Omega)$, Ω being an open connected subset of $\mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1-function satisfying $\langle g(v), v\rangle \geq 0$ on $\mathbb{R}^n$. This last condition means that the term $g(\dot{u})$ in (SOP) has a damping effect. It is easy to see that energy

$$\mathcal{E}(u, \dot{u}) = \frac{1}{2}\|\dot{u}\|^2 + E(u)$$

is nonincreasing along solutions. In fact, if u is a classical solution to (SOP), then

$$\frac{d}{dt}\mathcal{E}(u(t), \dot{u}(t)) = -\langle g(v), v\rangle \leq 0.$$

If $u : [0, +\infty) \to \Omega$ is a global solution and φ belongs to the ω-limit set of u, then $\mathcal{E}(u(t), \dot{u}(t)) \to \mathcal{E}(\varphi, 0) = E(\varphi)$ as $t \to +\infty$. In this paper, we derive the exact rate of convergence of $\mathcal{E}(u(t), \dot{u}(t))$ to $E(\varphi)$.

Our main assumption is the Kurdyka–Łojasiwicz gradient inequality (see [10])

$$\Theta(|E(u) - E(\varphi)|) \leq \|\nabla E(u)\|. \tag{KLI}$$

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For linear g, the optimal decay estimate was derived in [2]. For nonlinear g (typically satisfying $g'(0) = 0$) some decay estimates were shown in [3,7,8]. Here we derive better decay estimates under additional assumptions on E and we show that these estimates are optimal. We will assume that E satisfies an inverse to (KLI) and some estimates on the second gradient and that g has certain behavior near zero. The present result generalizes the one from [5, Theorem 20] where we worked with the Łojasiewicz gradient inequality, i.e. (KLI) with $\Theta(s) = s^{1-\theta}$ for a constant $\theta \in (0, \frac{1}{2}]$ (see [11]). It also generalizes the result by Haraux (see [9]) and Abdelli, Anguiano, Haraux (see [1]). The present result applies e.g. to functions E and g having the growth near origin as

$$s^a \ln^{r_1}(1/s) \ln^{r_2}(\ln(1/s)) \dots \ln^{r_k}(\ln \dots \ln(1/s)) \tag{1.1}$$

for some constants $a, r_1, \dots, r_k$. It also applies to functions E with a non-strict local minimum in φ.

The paper is organized as follows. In Section 2 we present our notations, basic definitions and the main result. Section 3 contains the proof of the main result.

2 Notations and the main result

By $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ we denote the usual norm and scalar product on $\mathbb{R}^d$. For nonnegative functions $f, g : G \subset \mathbb{R}^d \to \mathbb{R}$ we write $g(x) = O(f(x))$ on G if there exists $C > 0$ such that $g(x) \leq Cf(x)$ for all $x \in G$. We say that $g(x) = O(f(x))$ for $x \to a$ if $g(x) = O(f(x))$ on a neighborhood of a. If $f(x) = O(g(x))$ and $g(x) = O(f(x))$, we write $f \sim g$.

We say that a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $f(0) = 0$ and $f(s) > 0$ for $s > 0$

- is *admissible* if f is nondecreasing and there exists $c > 0$ such that $sf'_\pm(s) \leq cf(s)$ for all $s > 0$,
- has *property (K)* if for every $K > 0$ there exists $C(K) > 0$ such that $f(Ks) \leq C(K)f(s)$ holds for all $s > 0$,
- is *C-sublinear* if there exists $C > 0$ such that $f(t + s) \leq C(f(t) + f(s))$ holds for all t, $s > 0$.

It is easy to see that admissible functions are C-sublinear and have property (K) (for proof see Appendix of [4]). Further, for nondecreasing functions property (K) is equivalent to C-sublinearity. Moreover, every concave function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is admissible and satisfies $sf'_\pm(s) \leq f(s)$.

Let us introduce the inverse Kurdyka–Łojasiewicz inequality

$$\Theta_1(|E(u) - E(\varphi)|) \geq \|\nabla E(u)\| \tag{IKLI}$$

and an inequality for the second gradient

$$\|\nabla^2 E(u)\| \leq \Gamma(\|\nabla E(u)\|). \tag{2.1}$$

When we say that inequality (KLI) (resp. (IKLI), (2.1)) holds on a set U it means that the inequality holds for all $u \in U$ with a given fixed φ and Θ (resp. Θ_1, Γ).

By a solution to (SOP) we always mean a classical solution defined on $[0, +\infty)$. By $R(u) = \{u(t) : t \geq 0\}$ we denote the *range of* u. We say that a solution is precompact if $R(u)$ is precompact in Ω (the domain of E). The ω-limit set of u is

$$\omega(u) = \{\varphi \in \Omega : \exists t_n \nearrow +\infty, \ u(t_n) \to \varphi\}.$$

By c, C, $\tilde{c}$, $\tilde{C}$ we denote generic constants, their values can change from line to line or from expression to expression.

The main result of the present paper is the following.

Theorem 2.1. *Let u be a precompact solution to (SOP) and $\varphi \in \omega(u)$. Let $E(\cdot) \geq E(\varphi)$ on $R(u)$ and let E satisfy (KLI), (IKLI) and (2.1) on $R(u)$ with admissible functions Θ, Θ_1 and Γ, such that $\Theta(s) \sim \Theta_1(s)$ and $\Gamma(\Theta(s)) \sim \Theta(s)\Theta'(s)$ for $s \to 0+$. Let g satisfies*

$$\langle g(v), v\rangle \geq ch(\|v\|)\|v\|^2, \qquad \|g(v)\| \leq Ch(\|v\|)\|v\| \tag{2.2}$$

with an admissible function h satisfying

$$\Theta(s) \geq c\sqrt{s}\, h(\sqrt{s}) \tag{2.3}$$

for some $c > 0$ and all $s \geq 0$. Let us denote

$$\chi(s) = sh(\sqrt{s}), \qquad \Phi_\chi = \int \frac{1}{\chi(s)} ds \tag{2.4}$$

and assume that $\psi(s) = s^2 h(s)$ is convex. Then

$$c(-\Phi_\chi)^{-1}(Ct) \leq \mathcal{E}(u(t), \dot{u}(t)) - \mathcal{E}(\varphi, 0) \leq C(-\Phi_\chi)^{-1}(ct)$$

for some c, $C > 0$ and all t large enough.

Let us first mention that if $E(u) = \|u\|^p$, $p \geq 2$, then (KLI), (IKLI) hold with $\Theta(s) \sim \Theta_1(s) = Cs^{1-\theta}$, $\theta = \frac{1}{p}$ and (2.1) holds with $\Gamma(s) = Cs^{\frac{1-2\theta}{1-\theta}}$. If $h(s) = s^\alpha$, $\alpha \in (0,1)$, then condition (2.3) becomes $\alpha \geq 1 - 2\theta$ and $(-\Phi_\chi)^{-1}(ct) = Ct^{-\frac{2}{\alpha}}$. In this case, we obtain the same result as [5, Theorem 20] and also [9].

Remark 2.2.

1. If $(\Phi_\chi)^{-1}$ has property (K), then the statement of Theorem 2.1 can be written as $\mathcal{E}(u(t), \dot{u}(t)) - E(\varphi) \sim (-\Phi_\chi)^{-1}(t)$.

2. We can see that the energy decay depends on h only. In particular, it is independent of Θ.

3. It is enough to assume that all the assumptions except $\langle g(v), v\rangle > 0$ for all $v \neq 0$ hold on a small neigborhood of zero, resp. a small neighborhood of $\omega(u)$.

4. It follows from (KLI) and [2, Proposition 2.8] that $\Theta(s) = O(\sqrt{s})$. Hence, by (2.3) function h must be bounded on a neighborhood of zero and $\Phi_\chi(t) \to -\infty$ as $t \to 0+$. So, it is not important which primitive function Φ_χ we take and we have $(-\Phi_\chi)^{-1}(t) \to 0$ as $t \to +\infty$.

5. Theorem 2.1 does not imply that $u(t) \to \varphi$ as $t \to +\infty$. In fact, in [6, Theorem 4] we have shown that $u(t) \to \varphi$ if h is large enough, in particular if $\int_0^\varepsilon \frac{1}{\Theta(s)h(\Theta(s))} < +\infty$. If this condition is not satisfied, it may happen that $\omega(u)$ contains more than one point.

6. If φ is an asymptotically stable equilibrium for the gradient system $\dot{u} + \nabla E(u) = 0$ (e.g. if E has a strict local minimum in φ and is convex on a neighborhood of φ) and (KLI), (IKLI) hold on a neighborhood of φ, then by [5, Corollary 5] we have $\|x - \varphi\| \sim \Phi_\Theta(E(x) - E(\varphi))$ on a neighborhood of φ where $\Phi_\Theta(t) = \int_0^t \frac{1}{\Theta}$. In this case, for any solution starting in a neighborhood of φ we have
$$c(-\Phi_\chi)^{-1}(Ct) \leq \|v(t)\|^2 + \Phi_\Theta^{-1}(\|u(t) - \varphi\|) \leq C(-\Phi_\chi)^{-1}(ct)$$
and, especially,
$$\|u(t) - \varphi\| \leq \Phi_\Theta(C(-\Phi_\chi)^{-1}(ct)),$$
so $u(t) \to \varphi$. We do not have the estimate for $\|u(t) - \varphi\|$ from below since, at least in one-dimensional case, the solution oscillates and $u(t_n) = \varphi$ for a sequence $t_n \nearrow +\infty$ (see [9]).

Example 2.3. Let us consider $E(u) = F(\|u\|)$ with a real function F having a strict local minimum $F(0) = 0$ and satisfying on a right neighborhood of zero $CF(s) \geq sF'(s) \geq (1 + \varepsilon)F(s)$ and $sF''(s) \sim F'(s)$. Moreover, we assume that $(F')^{-1}$ has property (K). (It is easy to show that any analytic function $F(s) = \sum_{k=2m}^\infty a_k s^k$, $a_{2m} > 0$ and any function of the form (1.1) with $a > 2$, $r_i \in \mathbb{R}$ or $a = 2$, $r_1 = \cdots = r_{j-1} = 0$, $r_j < 0$, $r_{j+1}, \ldots, r_k \in \mathbb{R}$ satisfy these assumptions.) Then (KLI), (IKLI) holds with $\Theta(s) = C\frac{s}{F^{-1}(s)}$, since
$$\Theta(E(u)) = \Theta(F(\|u\|)) = C\frac{F(\|u\|)}{\|u\|} \sim F'(\|u\|) = \|\nabla E(u)\|.$$
Further, (2.1) holds with $\Gamma(s) = C\frac{s}{(F')^{-1}(s)}$ since
$$\|\nabla^2 E(u)\| \leq CF''(\|u\|) \sim \frac{F'(\|u\|)}{\|u\|} \sim \Gamma(F'(\|u\|)) = \Gamma(\|\nabla E(u)\|),$$
where the first inequality is due to the fact that the diagonal resp. nondiagonal terms of $\nabla^2 E(u)$ are
$$F''(\|u\|)\frac{u_i^2}{\|u\|^2} \quad \text{resp.} \quad \frac{u_i u_j}{\|u\|^2}\left(F''(\|u\|) - \frac{F'(\|u\|)}{\|u\|}\right),$$
so they are estimated by $CF''(\|u\|)$. Further, we have
$$\Theta'(F(s)) = \frac{\frac{d}{ds}\Theta(F(s))}{F'(s)} = \frac{\frac{d}{ds}\frac{F(s)}{s}}{F'(s)} = \frac{F'(s)s - F(s)}{s^2 F'(s)} = \frac{1}{s}\left(1 - \frac{F(s)}{sF'(s)}\right) \sim \frac{1}{s},$$
so
$$\Theta(F(s))\Theta'(F(s)) \sim \frac{1}{s}\Theta(F(s)) \sim \frac{1}{s^2}F(s)$$
and
$$\Gamma(\Theta(F(s))) \sim \frac{\Theta(F(s))}{(F')^{-1}(\Theta(F(s)))} \sim \frac{F(s)}{s(F')^{-1}(\frac{F(s)}{s})} \sim \frac{F(s)}{s(F')^{-1}(F'(s))} = \frac{F(s)}{s^2},$$
hence $\Gamma(\Theta(s)) \sim \Theta(s)\Theta'(s)$. Then, for any g satisfying (2.2) with a function h small enough (such that (2.3) holds) Theorem 2.1 can be applied and we obtain the exact energy decay which depends on h only and not on F. In particular, if $h(s) = s^\alpha$ we have $\mathcal{E}(u(t), v(t)) \sim t^{-\frac{2}{\alpha}}$ and if h is of the form (1.1), we have by [4, Lemmas 6.5, 6.6]
$$\mathcal{E}(u(t), v(t)) \sim t^{-\frac{2}{a}} \ln^{-\frac{r_1}{a}}(\ln 1/t) \ldots \ln^{-\frac{r_k}{a}}(\ln \ldots \ln 1/t).$$
Let us mention that if h is equal to (1.1) and such that $cs \leq h(s) \leq c$ near zero (i.e. $a \in [0, 1]$ and if $a \in \{0, 1\}$ we have a sign condition on the first nonzero number r_i), then $\psi(s) = s^2 h(s)$ is convex near zero.

3 Proof of Theorem 2.1

Let us write $v(t)$ instead of $\dot{u}(t)$ and $\mathcal{E}(t)$ instead of $\mathcal{E}(u(t), v(t))$. We also often write u, v instead of $u(t)$, $v(t)$.

First of all, since u is precompact $\{E(u(t)) : t \geq 0\}$ is bounded. Therefore, $\{\mathcal{E}(t) : t \geq 0\}$ is bounded, hence v is bounded and by (SOP) also $\ddot{u} = \dot{v}$ is bounded. Since

$$\int_0^t \langle g(v), v\rangle = \mathcal{E}(0) - \mathcal{E}(t) \leq K,$$

we have $\langle g(v), v\rangle \in L^1((0, +\infty))$. Then boundedness of $\dot{v}$ yields convergence of $\langle g(v(t)), v(t)\rangle$ to 0. Hence $v(t) \to 0$ as $t \to +\infty$ and it follows that $\mathcal{E}(t) \to \mathcal{E}(\varphi, 0)$. So, we can assume without loss of generality that $E(\varphi) = 0$, $\mathcal{E}(\varphi, 0) = 0$.

In the rest of the proof we will work with

$$H(t) = \mathcal{E}(t) + \varepsilon B(E(u(t)))\langle \nabla E(u(t)), v\rangle,$$

where

$$B(s) = \begin{cases} \frac{1}{\Theta(s)^2} s h(\sqrt{s}) & s > 0 \\ 0 & s = 0 \end{cases}$$

and $\varepsilon > 0$ is small enough. Let us mention that B can be unbounded in a neighborhood of zero, but due to (2.3) we have $\Theta(s)B(s) \leq C\sqrt{s}$, hence H is continuous even in the points where $E(u(t)) = 0$ and in these points we have $H(t) = \mathcal{E}(t)$. Let us denote $M := \{t \geq 0 : E(u(t)) > 0\}$ and $M^c = \{t \geq 0 : \ E(u(t)) = 0\}$.

We show that $H(t) \sim \mathcal{E}(t)$. On M^c it is trivial. On M we apply (IKLI), Cauchy–Schwarz and Young inequalities and $\Theta(s)B(s) \leq C\sqrt{s}$ and we obtain

$$\begin{aligned} |\varepsilon B(E(u))\langle \nabla E(u(t)), v\rangle| &\leq \varepsilon C B(E(u))\Theta(E(u))\|v\| \\ &\leq \varepsilon C B(E(u))^2 \Theta(E(u))^2 + \varepsilon C\|v\|^2 \\ &\leq \varepsilon C \mathcal{E}(t), \end{aligned}$$

hence

$$(1 - \varepsilon C)\mathcal{E}(t) \leq H(t) \leq (1 + \varepsilon C)\mathcal{E}(t)$$

and taking $\varepsilon > 0$ small enough we obtain $H(t) \sim \mathcal{E}(t)$.

The next step is to show that

$$0 \leq -H'(t) \sim h(\|v\|)\|v\|^2 + E(u)h\left(\sqrt{E(u)}\right). \tag{3.1}$$

Let us first estimate $B'(s)$. For any $s > 0$ we have

$$B'(s) = \frac{B(s)}{s}\left(1 + \frac{h'(\sqrt{s})\sqrt{s}}{h(\sqrt{s})} - 2\frac{s\Theta'(s)}{\Theta(s)}\right) \in \left[\frac{B(s)}{s}(1 - 2C), \frac{B(s)}{s}(1 + C)\right],$$

where the equality follows by definition of B and the rest from admissibility of h and Θ (the two fractions in round bracket are nonnegative and bounded above by a constant). Hence, $|sB'(s)| \leq CB(s)$.

Let $t \in M$. Let us compute $H'(t)$ and use the fact that u solves (SOP) to get

$$\begin{aligned} H'(t) = & -\langle g(v), v\rangle - \varepsilon B(E(u))\|\nabla E(u)\|^2 \\ & + \varepsilon B'(E(u))\langle \nabla E(u), v\rangle^2 \\ & + \varepsilon B(E(u))\langle \nabla^2 E(u) v, v\rangle \\ & + \varepsilon B(E(u))\langle \nabla E(u), -g(v)\rangle . \end{aligned} \tag{3.2}$$

Due to (2.2) we have $\langle g(v), v\rangle \sim h(\|v\|)\|v\|^2$ and by definition of B, (KLI) and (IKLI) we immediately have $B(E(u))\|\nabla E(u)\|^2 \sim E(u)h(\sqrt{E(u)})$. So,

$$\langle g(v), v\rangle + \varepsilon B(E(u))\|\nabla E(u)\|^2 \sim h(\|v\|)\|v\|^2 + \varepsilon C E(u) h\left(\sqrt{E(u)}\right).$$

We show that the second, third and fourth lines of (3.2) are smaller than this term, then (3.1) is proved.

The second line of (3.2) is less than

$$\varepsilon C \frac{B(E(u))}{E(u)} \Theta(E(u))^2 \|v\|^2 \le \varepsilon C h\left(\sqrt{E(u)}\right)\|v\|^2.$$

Since Γ has property (K) and satisfies $\Gamma(\Theta(s)) \sim \Theta(s)\Theta'(s) \le C s^{-1}\Theta(s)^2$ and due to (IKLI) and definition of B, the third line in (3.2) is less than

$$\varepsilon C B(E(u))\Gamma(\|\nabla E(u)\|)\|v\|^2 \le \varepsilon C h\left(\sqrt{E(u)}\right)\|v\|^2.$$

If $E(u) \le 4C\|v\|^2$, then (h satisfies property (K)) we have $h\big(\sqrt{E(u)}\big)\|v\|^2 \le \tilde{C} h(\|v\|)\|v\|^2$ and if $E(u) \ge 4C\|v\|^2$, then $h\big(\sqrt{E(u)}\big)\|v\|^2 \le \frac{1}{4C} h\big(\sqrt{E(u)}\big)E(u)$. So, in either case we have that lines two and three in (3.2) are less than

$$\varepsilon C h(\|v\|)\|v\|^2 + \frac{1}{4}\varepsilon h\left(\sqrt{E(u)}\right)E(u),$$

so they are less than the first line in (3.2) since we can make εC small by taking ε small enough. The last line in (3.2) is (by definition of B and (2.3)) less than

$$\begin{aligned} \varepsilon C B(E(u))\|\nabla E\| h(\|v\|)\|v\| &\le \varepsilon C \frac{1}{\Theta(E(u))} E(u) h\left(\sqrt{E(u)}\right) h(\|v\|)\|v\| \\ &\le \varepsilon C \sqrt{E(u)} h(\|v\|)\|v\|. \end{aligned}$$

Applying the Young inequality $ab \le \psi(a) + \tilde{\psi}(b)$ with $\psi(s) = s^2 h(s)$ and the convex conjugate $\tilde{\psi}$ we get

$$\begin{aligned} \varepsilon C \sqrt{E(u)} h(\|v\|)\|v\| &\le \frac{1}{4}\varepsilon \psi\left(\sqrt{E(u)}\right) + \varepsilon C \tilde{\psi}(\|v\| h(\|v\|)) \\ &\le \frac{1}{4}\varepsilon E(u) h\left(\sqrt{E(u)}\right) + \varepsilon C h(\|v\|)\|v\|^2 \end{aligned}$$

since $\tilde{\psi}(sh(s)) \le C s^2 h(s)$ due to Lemma 3.1 below. Now, (3.1) is proven on M. If $E(u(t)) \to 0$ for $t \to t_0$, we can see that $H'(t) \to -\langle g(v(t_0)), v(t_0)\rangle = \mathcal{E}'(t_0)$ (due to the estimates above, all terms on the right-hand side of (3.2) except the first one tend to zero). By continuity of H, we have $H' = \mathcal{E}'$ on M^c, in particular (3.1) holds on M^c.

We show that $\chi(H(t)) \sim -H'(t)$. In fact,

$$\begin{aligned}
\chi(H(t)) &\le \chi(C(\|v\|^2 + E(u)))) \\
&\le C(\chi(\|v\|^2) + \chi(E(u))) \\
&= C\left(h(\|v\|)\|v\|^2 + E(u)h\left(\sqrt{E(u)}\right)\right) \\
&\le -CH'(t),
\end{aligned}$$

where we applied monotonicity in the first line, C-sublinearity and property (K) in the second line (χ has these properties by Lemma 3.2 below), definition of χ in the third line and (3.1) in the last inequality. On the other hand, by Lemma 3.2 also the inverse inequalities in C-sublinearity and property (K) are valid, so we have

$$\begin{aligned}
\chi(H(t)) &\ge \chi(c(\|v\|^2 + E(u)))) \\
&\ge c(\chi(\|v\|^2) + \chi(E(u))) \\
&= c\left(h(\|v\|)\|v\|^2 + E(u)h\left(\sqrt{E(u)}\right)\right) \\
&\ge -cH'(t),
\end{aligned}$$

so $\chi(H(t)) \sim -H'(t)$ is proved.

Let $T = \sup\{t \ge 0 : \ H(t) > 0\}$. For any $t \in (0, T)$ we have proved

$$-\frac{d}{dt}\Phi_\chi(H(t)) = -\frac{H'(t)}{\chi(H(t))} \in [c, C].$$

Integrating this relation from t_0 to t we obtain

$$c(t - t_0) - \Phi_\chi(H(t_0)) \le -\Phi_\chi(H(t)) \le C(t - t_0) - \Phi_\chi(H(t_0)). \tag{3.3}$$

If $T < +\infty$, then we can see that $-\Phi_\chi(H(t))$ is bounded on $(0, T)$, hence $0 < \lim_{t\to T-} H(t) = H(T)$, contradiction. Therefore, $T = +\infty$, (3.3) holds for all $t > 0$ and for t large enough we have

$$\tilde{c}t \le c(t - t_0) - \Phi_\chi(H(t_0)) \le -\Phi_\chi(H(t)) \le C(t - t_0) - \Phi_\chi(H(t_0)) \le \tilde{C}t.$$

Hence

$$c(-\Phi_\chi)^{-1}(\tilde{C}t) \le H(t) \sim \mathcal{E}(u(t), v(t)) \le C(-\Phi_\chi)^{-1}(\tilde{c}t),$$

which completes the proof of Theorem 2.1.

Lemma 3.1. *Let $\psi(s) = s^2h(s)$ and $\tilde{\psi}(r) = \sup\{rs - \psi(s) : s \ge 0\}$ be the convex conjugate to ψ. Then there exists $C > 0$ such that $\tilde{\psi}(sh(s)) \le Cs^2h(s)$ for all $s \ge 0$.*

Proof. Since ψ is convex, the one-sided derivatives $\psi'_\pm(s) = s^2h'_\pm(s) + 2sh(s)$ are nondecreasing functions and the interval $[\psi'_-(s), \psi'_+(s)]$ is nonempty. Take $s_0 > 0$ arbitrarily and take $r \in [\psi'_-(s_0), \psi'_+(s_0)]$. Then the function $s \mapsto rs - \psi(s)$ attains its maximum in s_0, hence $\tilde{\psi}(r) = rs_0 - s_0^2h(s_0)$. Since $r \ge \psi'_-(s_0) = s_0^2h'_-(s_0) + 2s_0h(s_0) \ge s_0h(s_0)$ and $\tilde{\psi}$ is increasing, we have $\tilde{\psi}(s_0h(s_0)) \le \tilde{\psi}(r) = rs_0 - s_0^2h(s_0) \le \psi'_+(s_0)s_0 - s_0^2h(s_0) = s_0^3h'_+(s_0) + 2s_0^2h(s_0) - s_0^2h(s_0) \le (c + 2 - 1)s_0^2h(s_0)$. ☐

Lemma 3.2. *Function $\chi(s) = sh(\sqrt{s})$ is C-sublinear and it has property (K). Moreover, $\chi(s+t) \geq \frac{1}{2}(\chi(s)+\chi(t))$ for all $s, t > 0$ and for every $c > 0$ there exists $\tilde{c} > 0$ such that $\chi(cs) \geq \tilde{c}\chi(s)$.*

Proof. Since h has property (K), we have for a fixed $K > 0$

$$\chi(Ks) = Ksh(\sqrt{K}\sqrt{s}) \leq KsC(\sqrt{K})h(\sqrt{s}) = KC(\sqrt{K})\chi(s).$$

So, χ has property (K) and since it is increasing, it is also C-sublinear. Since χ is increasing, we also have $\chi(s+t) \geq \chi(s)$, $\chi(s+t) \geq \chi(t)$ and therefore $\chi(s+t) \geq \frac{1}{2}(\chi(s)+\chi(t))$. From property (K) we have for any fixed $c > 0$

$$\chi(s) = \chi\left(\frac{1}{c}cs\right) \leq C\left(\frac{1}{c}\right)\chi(cs) = \frac{1}{\tilde{c}}\chi(cs)$$

and the last property is proven. □

References

[1] M. Abdelli, M. Anguiano, A. Haraux, Existence, uniqueness and global behavior of the solutions to some nonlinear vector equations in a finite dimensional Hilbert space, *Nonlinear Anal.* **161**(2017), 157–181. https://doi.org/10.1016/j.na.2017.06.001; MR3672999; Zbl 1381.34033

[2] P. Bégout, J. Bolte, M. A. Jendoubi, On damped second-order gradient systems, *J. Differential Equations* **259**(2015), No. 7, 3115–3143. https://doi.org/10.1016/j.jde.2015.04.016; MR3360667; Zbl 1347.34082

[3] T. Bárta, Rate of convergence to equilibrium and Łojasiewicz-type estimates, *J. Dynam. Differential Equations* **29**(2017), No. 4, 1553–1568. https://doi.org/10.1007/s10884-016-9549-z; MR3736148; Zbl 06829893

[4] T. Bárta, Decay estimates for solutions of abstract wave equations with general damping function, *Electron. J. Differential Equations* **2016**, No. 334, 1–17. MR3604779; Zbl 1368.35191

[5] T. Bárta, Sharp and optimal decay estimates for solutions of gradient-like systems, preprint, 2017.

[6] T. Bárta, R. Chill, E. Fašangová, Every ordinary differential equation with a strict Lyapunov function is a gradient system, *Monatsh. Math.* **166**(2012), 57–72. https://doi.org/10.1007/s00605-011-0322-4; MR2901252; Zbl 1253.37019

[7] I. Ben Hassen, A. Haraux, Convergence and decay estimates for a class of second order dissipative equations involving a non-negative potential energy, *J. Funct. Anal.* **260**(2011), No. 10, 2933–2963. https://doi.org/10.1016/j.jfa.2011.02.010; MR2774060; Zbl 1248.34092

[8] L. Chergui, Convergence of global and bounded solutions of a second order gradient like system with nonlinear dissipation and analytic nonlinearity. *J. Dynam. Differential Equations* **20**(2008), No. 3, 643–652. https://doi.org/10.1007/s10884-007-9099-5; MR2429439; Zbl 1167.34017

[9] A. Haraux, Sharp decay estimates of the solutions to a class of nonlinear second order ODE's, *Anal. Appl. (Singap.)* **9**(2011), No. 1, 49–69. https://doi.org/10.1142/S021953051100173X; MR2763360; Zbl 1227.34052

[10] K. Kurdyka, On gradients of functions definable in *o*-minimal structures, *Ann. Inst. Fourier (Grenoble)* **48**(1998), 769–783. https://doi.org/10.5802/aif.1638; MR1644089; Zbl 0934.32009

[11] S. Łojasiewicz, Une propriété topologique des sous-ensembles analytiques réels (in French), in: *Colloques internationaux du C.N.R.S.: Les équations aux dérivées partielles, Paris (1962), Editions du C.N.R.S.*, Paris, 1963, pp. 87–89. https://doi.org/10.1006/jdeq.1997.3393; MR0160856; Zbl 0915.34060