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Normal form of $O(2)$ Hopf bifurcation in a model of a nonlinear optical system with diffraction and delay

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Abstract. In this paper we construct an $O(2)$-equivariant Hopf bifurcation normal form for a model of a nonlinear optical system with delay and diffraction in the feedback loop whose dynamics is governed by a system of coupled quasilinear diffusion equation and linear Schrödinger equation. The coefficients of the normal form are expressed explicitly in terms of the parameters of the model. This makes it possible to constructively analyze the phase portrait of the normal form and, based on the analysis, study the stability properties of the bifurcating rotating and standing waves.

Keywords: normal form, equivariant Hopf bifurcation, $O(2)$ symmetry, functional differential equation, delay, nonlinear optical system.

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1 Introduction

Nonlinear optical systems with nonlocal feedback often possess certain symmetries that – if carefully studied – can help one understand the typical pattern formation scenarios. For instance, Hopf bifurcation in the presence of $SO(2)$ symmetry gives rise to rotating waves: one-dimensional waves on a circle [9] or two-dimensional waves on a disc [11].

In its simplest form, Hopf bifurcation appears when two simple complex-conjugate eigenvalues of the linearized operator cross the imaginary axis with nonzero speed as a certain parameter is varied [12]. However, when the system is $O(2)$-symmetric, Hopf bifurcation becomes degenerate as the eigenvalues are double, each with a two-dimensional eigenspace. For nondelayed equations this situation was studied with the use of normal forms [4,7] and branching equations [8]. Before applying these ideas to a partial differential equation, one usually conducts a center manifold reduction and then proceeds to construct a normal form on the center manifold. Even for nondelayed equations the procedure is rather tedious (see [13] for a reaction-diffusion equation).

Teresa Faria extended this methodology to quasilinear functional differential equations (FDE) in Banach spaces [6]: she proposed a way to construct a normal form on a center

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manifold bypassing the explicit construction of the manifold and illustrated the approach on model problems. The method was successfully applied to a delayed diffusion FDE with $SO(2)$ symmetry to study the stability properties of one-dimensional rotating waves [9]. We note that, in the cited paper, the model lacks reflectional symmetry due to a transformation of the spatial argument in the feedback loop.

In [1], a model of a nonlinear optical system with diffraction and delay was studied that, unlike [9], includes just local spatial interactions and hence enjoys $O(2)$ symmetry. An $O(2)$-equivariant Hopf bifurcation permits not only rotating waves (both clockwise and counter-clockwise) but also standing waves. The present paper is devoted to the construction of an $O(2)$-equivariant Hopf bifurcation for a nonlinear optical system with diffraction and delay that makes it possible to analytically study the stability of the bifurcating rotating and standing waves [2].

2 Notation

By $H^2(\mathbb{C})$ we denote the standard Sobolev space of complex-valued functions on the interval $(0, 2\pi)$ that are Lebesgue square-integrable with their second derivative. By $H^2_{2\pi}(\mathbb{C})$ we denote the closed subspace of $H^2(\mathbb{C})$ of 2π-periodic functions. It itself becomes a Hilbert space once endowed with the suitable inner product and norm [10].

Given a unitary space X, we write $\langle \cdot, \cdot \rangle_X$ and $\| \cdot \|_X$ to denote its inner product and the corresponding norm. An inner product $\langle \cdot, \cdot \rangle$ with no subscript stands for the standard $L^2(0, 2\pi)$ inner product

$$\langle u, v \rangle = \int_0^{2\pi} u(x)\overline{v(x)}dx.$$

Given a Banach space X with the norm $\| \cdot \|_X$, by $C^k([a,b]; X)$ we denote the Banach space of k times continuously differentiable X-valued functions with the norm

$$\|u\|_{C^k([a,b];X)} = \sum_{j=0}^{k} \sup_{t \in [a,b]} \|u^{(j)}(t)\|_X.$$

Finally, given a function space $X(\mathbb{C})$ of complex-valued functions, we denote its real-valued counterpart by X. For example, $H^2(\mathbb{C})$ and H^2.

3 Main equation and auxiliary statements

We consider a one-dimensional model of a nonlinear optical system with a delayed feedback loop and diffraction therein (see [3] for the physical aspects of the problem)

$$\begin{aligned} &u_t + u = Du_{xx} + K|Be^{iu(t-T)}|^2, \quad x \in (0, 2\pi), \quad t > 0, \\ &u|_{x=0} = u|_{x=2\pi}, \quad u_x|_{x=0} = u_x|_{x=2\pi}. \end{aligned} \tag{3.1}$$

To describe the effects of diffraction in the paraxial approximation we employ a linear operator

$$B : H^2_{2\pi}(\mathbb{C}) \to H^2_{2\pi}(\mathbb{C}), \quad A_0(x) \mapsto A(x, z; A_0)|_{z=z_0},$$

that treats its input as the initial condition of a periodic initial-boundary value problem for the linear Schrödinger equation

$$\begin{aligned} &A_z + iA_{xx} = 0, \quad x \in (0, 2\pi), \quad z > 0, \\ &A|_{x=0} = A|_{x=2\pi}, \quad A_x|_{x=0} = A_x|_{x=2\pi}, \\ &A|_{z=0} = A_0(x) \end{aligned} \tag{3.2}$$

and propagates it along a distance $z = z_0$.

The sought real-valued function $u(x,t)$ represents the phase modulation of the light wave in the nonlinear Kerr slice. The parameters involved in the problem statement are: $D > 0$ is the effective diffusion coefficient (actually, $D = \tilde{D}/r^2$, where r is the circle radius); $K > 0$ is the nonlinearity coefficient (it is positive as it corresponds to Kerr-induced self-focusing of the light field); $T > 0$ is the temporal delay in the feedback loop; $z_0 > 0$ is the distance traversed by the light wave in the feedback loop (here, $z_0 = \tilde{z}_0/r^2$).

Lemma 3.1 ([1, 2]). *The operator B has a complete orthogonal system of eigenfunctions* $\exp(inx)$, $n \in \mathbb{Z}$, *in $H^2_{2\pi}(\mathbb{C})$. The corresponding eigenvalues are $\lambda_n(B) = \exp(in^2 z_0)$.*

Boundary value problem (3.1) admits spatially homogeneous equilibria $u(x,t) \equiv K$. Fixing a value $\hat{K}$ for the nonlinearity parameter and considering its perturbations $K(\mu) = \hat{K} + \mu$, we get a branch of constant solutions $u(x,t) \equiv K(\mu)$.

We set $u(x,t) = K(\mu) + v(x,t)$ to bring (3.1) to its local form in the vicinity of $K(\mu)$

$$\begin{aligned} &v_t + v = Dv_{xx} + K(\mu)\left(|Be^{iv(t-T)}|^2 - 1\right), \\ &v|_{x=0} = v|_{x=2\pi}, \quad v_x|_{x=0} = v_x|_{x=2\pi}. \end{aligned} \tag{3.3}$$

Taking out the linear part, we rewrite (3.3) as

$$\begin{aligned} &v_t + v = Dv_{xx} + L(\mu)v(t-T) + F(v(t-T),\mu), \quad v(t) \in H^2_{2\pi}, \\ &L(\mu)w \equiv -2K(\mu)\operatorname{Im} Bw, \quad F(w,\mu) = K(\mu)\left\{|B(e^{iw}-1)|^2 + 2\operatorname{Re} B(e^{iw} - 1 - iw)\right\}. \end{aligned}$$

Clearly, $L(\mu)$ can be expanded as follows:

$$L(\mu) = L_0 + \mu L_1, \quad L_0 = -2\hat{K}\operatorname{Im} B, \quad L_1 = -2\operatorname{Im} B.$$

Lemma 3.2 ([1, 2]). *The operator $F(w,\mu) : H^2_{2\pi} \times \mathbb{R} \to H^2_{2\pi}$ is analytic in the neighborhood of the origin. The operator F and its Fréchet derivatives $F_{w^n\mu^m}$ vanish at the origin when $n < 2$ or $m > 1$.*

Below are the (nonzero) quadratic and cubic Fréchet derivatives of F at the origin:

$$\begin{aligned} &F_{ww}(0,0)w^2 = 2\hat{K}\left\{|Bw|^2 - \operatorname{Re} Bw^2\right\}, \\ &F_{www}(0,0)w^3 = 2\hat{K}\left\{3\operatorname{Im}\left[Bw\overline{Bw^2}\right] + \operatorname{Im} Bw^3\right\}, \quad F_{ww\mu}(0,0)w^2\mu = 2\mu\left\{|Bw|^2 - \operatorname{Re} Bw^2\right\}. \end{aligned}$$

4 From FDE to ODE in Banach space

To rewrite boundary value problem (3.3) in the common FDE terms [6], we use a function space $\mathcal{C} = C([-T,0];X)$, $X = H^2_{2\pi}$, and a function $v_t \in \mathcal{C}$ that acts according to $v_t(\tau) = v(t+\tau)$

where $v \in X$; we also extend $L(\mu)$ onto $\mathcal{C}$ by $\tilde{L}(\mu)\varphi = L(\mu)\varphi(-T)$ so that $\tilde{L}(\mu)$ is linear and bounded in $\mathcal{C}$. We are ready to write (3.3) in its abstract form

$$\frac{d}{dt}v(t) = Av(t) + \tilde{L}_0 v_t + \tilde{F}(v_t, \mu), \quad v \in D(A). \tag{4.1}$$

Here $Aw = D\frac{d^2}{dx^2}w - w$, $D(A) = \{w \in X : Aw \in X\}$,

$$\tilde{F}(v_t, \mu) = F(v_t(-T), \mu) + \mu \tilde{L}_1 v_t = \sum_{n=2}^{\infty} \frac{1}{n!}\tilde{F}_n(v_t, \mu),$$

where $\tilde{F}_n$ are the n-th order terms in the expansion of $\tilde{F}$.

Consider the linearization of (4.1) at $v = 0$ and $\mu = 0$:

$$\frac{d}{dt}v(t) = Av(t) + \tilde{L}_0 v_t. \tag{4.2}$$

The corresponding characteristic equation is

$$Ay + \exp(-\lambda T)L_0 y - \lambda y = 0, \quad \lambda \in \mathbb{C}, \quad y \in D(A). \tag{4.3}$$

We restrict our attention to $y \in \{1, \sin(nx), \cos(nx)\} \subset D(A)$ as this is an orthogonal basis of eigenfunctions of both A and L_0 in X. Characteristic equation (4.3) is thus reduced to a countable family of equations

$$\Delta_n(\lambda) \equiv -1 - Dn^2 - 2\hat{K}\sin(n^2 z_0)e^{-\lambda T} - \lambda = 0, \quad \lambda \in \mathbb{C}, \quad n \in \mathbb{Z}_+. \tag{4.4}$$

For a Hopf bifurcation to occur we demand the following from the solutions $\lambda \in \mathbb{C}$ of (4.4):

1. For all solutions λ their real parts $\operatorname{Re}\lambda \le 0$.
2. They are $\operatorname{Re}\lambda = 0$ if and only if $\lambda = \pm i\nu_*$, $n = n_*$. (Hopf)

Remark 4.1. The first part of (Hopf) is unnecessary for the bifurcation itself but it makes the center manifold asymptotically stable. We do not mention the transversality condition explicitly for it is met automatically since

$$\left.\frac{d}{d\mu}\right|_{\mu=0} \operatorname{Re}\lambda = \frac{1}{\hat{K}} \frac{T(1 + Dn_*^2)^2 + T\nu_*^2 + 1 + Dn_*^2}{(T + TDn_*^2 + 1)^2 + T^2\nu_*^2} > 0.$$

Consider the generator $A_0 : \mathcal{C} \to \mathcal{C}$ of the flow of equation (4.2):

$$A_0\varphi = \dot{\varphi}, \quad D(A_0) = \left\{\varphi \in \mathcal{C}^1 : \varphi(0) \in D(A),\, \dot{\varphi}(0) = A\varphi(0) + \tilde{L}_0\varphi\right\}.$$

According to [6], the roots λ of characteristic equation (4.3) are the eigenvalues of A_0. As long as equation (3.3) is $O(2)$-equivariant, a four-dimensional eigenspace $P \subset \mathcal{C}$ is associated with $\lambda = \pm i\nu_*$ and is spanned by

$$\Phi = \left(\varphi_1 = \exp(in_*x + i\nu_*\tau),\ \varphi_2 = \exp(in_*x - i\nu_*\tau),\ \varphi_3 = \overline{\varphi_2},\ \varphi_4 = \overline{\varphi_1}\right) \subset \mathcal{C}(\mathbb{C}).$$

Note that

$$\frac{d}{d\tau}\Phi = \Phi\mathcal{J}, \quad \mathcal{J} = \operatorname{diag}(i\nu_*, -i\nu_*, i\nu_*, -i\nu_*).$$

Remark 4.2. On introducing a real vector space

$$\mathbb{E}^4 = \{(z_1, z_2, z_3, z_4)^T \in \mathbb{C}^4 : z_4 = \overline{z_1},\ z_2 = \overline{z_3}\},$$

we can represent P as $\{\Phi z : z \in \mathbb{E}^4\}$. This will allow us to facilitate computations as we will be working in $X(\mathbb{C})$ while technically staying in the context of real-valued functions X.

To decompose $\mathcal{C}$ into a direct sum of A_0-invariant subspaces, we introduce a space $\mathcal{C}^* \equiv \mathcal{C}([0,T];\, X)$ and a bilinear form $\ll \cdot, \cdot \gg\colon\ \mathcal{C}^* \times \mathcal{C} \to \mathbb{R}$

$$\ll \psi, \varphi \gg = \langle \varphi(0), \psi(0) \rangle_X + \int_{-T}^{0} \langle \varphi(\tau), L_0 \psi(\tau + T) \rangle_X \, d\tau.$$

It readily extends to a form $\ll \cdot, \cdot \gg\colon\ \mathcal{C}(\mathbb{C})^* \times \mathcal{C}(\mathbb{C}) \to \mathbb{C}$ that is antilinear in the first argument and linear in the second one:

$$\ll \psi, \varphi \gg = \langle \varphi(0), \psi(0) \rangle_{X(\mathbb{C})} + \int_{-T}^{0} \langle \varphi(\tau), L_0 \operatorname{Re} \psi(\tau + T) + i L_0 \operatorname{Im} \psi(\tau + T) \rangle_{X(\mathbb{C})} \, d\tau.$$

A formal adjoint with respect to $\ll \cdot, \cdot \gg$ operator A_0^* is defined as

$$A_0^* \psi = -\dot{\psi}, \quad D(A_0^*) = \left\{ \psi \in \mathcal{C}^{1^*} : \psi(0) \in D(A),\ -\dot{\psi}(0) = A\psi(0) + L_0 \psi(T) \right\}$$

and has the same imaginary eigenvalues. In the corresponding eigenspace we choose a basis Ψ that is biorthogonal to Φ. To this end we introduce

$$\tilde{\Phi} = \left(\tilde{\varphi}_1 = \exp(i n_* x + i \nu_* \tau),\ \tilde{\varphi}_2 = \exp(i n_* x - i \nu_* \tau),\ \tilde{\varphi}_3 = \overline{\tilde{\varphi}_2},\ \tilde{\varphi}_4 = \overline{\tilde{\varphi}_1}\right)^T \subset \mathcal{C}(\mathbb{C})^*$$

and evaluate the following:

$$\ll \tilde{\varphi}_j, \varphi_k \gg = \left\langle \varphi_k(0), \tilde{\varphi}_j(0) \right\rangle_{X(\mathbb{C})} \left[1 - 2\hat{K} \sin(n_*^2 z_0) e^{(-1)^j i \nu_* T} \int_{-T}^{0} e^{[(-1)^{k+1} - (-1)^{j+1}] i \nu_* \tau} d\tau \right].$$

We note that

- $\left\langle \varphi_k(0), \tilde{\varphi}_j(0) \right\rangle_{X(\mathbb{C})} = 0$ for (j,k) and (k,j) in $\{(1,3), (1,4), (2,3), (2,4)\}$
- $\left\langle \varphi_k(0), \tilde{\varphi}_j(0) \right\rangle_{X(\mathbb{C})} = 2\pi(1 + n_*^4)$ for (j,k) and (k,j) in $\{(1,2), (3,4)\}$ and $j = k$
- for $j - k$ odd,

  $$e^{(-1)^j i \nu_* T} \int_{-T}^{0} e^{[(-1)^{k+1} - (-1)^{j+1}] i \nu_* \tau} d\tau = \sin(\nu_* T)/\nu_*$$

  and, according to (Hopf),

  $$1 - 2\hat{K} \sin(n_*^2 z_0) \sin(\nu_* T)/\nu_* = 0$$

- for $j - k$ even,

  $$e^{(-1)^j i \nu_* T} \int_{-T}^{0} e^{[(-1)^{k+1} - (-1)^{j+1}] i \nu_* \tau} d\tau = T e^{(-1)^j i \nu_* T}$$

  and, according to (Hopf),

  $$1 - 2\hat{K} \sin(n_*^2 z_0) T e^{(-1)^j i \nu_* T} = 1 + T(1 + D n_*^2 - (-1)^j i \nu_*).$$

Thus

$$\ll \tilde{\Phi}, \Phi \gg = \operatorname{diag}(\kappa^{-1}, \overline{\kappa}^{-1}, \kappa^{-1}, \overline{\kappa}^{-1}), \quad \kappa^{-1} = 2\pi(1 + n_*^4)[1 + T(1 + Dn_*^2 + i\nu_*)],$$

and

$$\Psi = (\overline{\kappa}\tilde{\varphi}_1, \kappa\tilde{\varphi}_2, \overline{\kappa}\tilde{\varphi}_3, \kappa\tilde{\varphi}_4)^T$$

is biorthogonal to Φ, i.e. $\ll \Psi, \Phi \gg = I$. As a result, $Q = \left\{\varphi \in \mathcal{C} \ : \ \ll \Psi, \varphi \gg = (0,0,0,0)^T\right\}$ is invariant under the action of A_0 and $\mathcal{C} = P \oplus Q$.

To relax the constraints $D(A_0)$ we present an enlarged phase space $\mathcal{BC}$ [6] that is composed of functions of the form $\psi = \varphi + X_0\alpha$, $\varphi \in \mathcal{C}$, $\alpha \in X$, with a norm $\|\psi\|_{\mathcal{BC}} = \|\varphi\|_{\mathcal{C}} + \|\alpha\|_X$, where $X_0(\tau) = 0$, $-T \leq \tau < 0$, $X_0(0) = I$. In other words, $\mathcal{BC}$ comprises functions $[-T, 0] \to X$ that are uniformly continuous on $[-T, 0)$. The extension $\tilde{A}_0 : \mathcal{BC} \to \mathcal{BC}$ of the operator A_0 onto $\mathcal{BC}$ is defined as follows:

$$\tilde{A}_0\psi = \dot{\psi} + X_0[A\psi(0) + \tilde{L}_0\psi - \dot{\psi}(0)], \quad D(\tilde{A}_0) = \left\{\psi \in \mathcal{C}^1 : \psi(0) \in D(A)\right\} \equiv \mathcal{C}_0^1.$$

Finally, we can formulate equation (4.1) as an ordinary differential equation in $\mathcal{BC}$:

$$\frac{d}{dt}v = \tilde{A}_0 v + X_0[\tilde{F}(v, \mu)], \quad v(t) = v_t \in \mathcal{C}_0^1. \tag{4.5}$$

It is shown in [6] that $\pi(\varphi + X_0\alpha) = \Phi(\ll \Psi, \varphi \gg + \langle \alpha, \Psi(0)\rangle_X)$ is a continuous projection onto P, which commutes with $\tilde{A}_0$ on $\mathcal{C}_0^1$; hence $\mathcal{BC}$ is decomposed into a topological direct sum $\mathcal{BC} = P \oplus N(\pi)$. Going back to (4.5), we express $v(t) \in \mathcal{C}_0^1$ as a sum $v(t) = \Phi z(t) + y(t)$, where

$$z(t) = \ll \Psi, v(t) \gg \in \mathbb{E}^4, \quad y(t) = (I - \pi)v(t) \in N(\pi) \cap \mathcal{C}_0^1 = Q \cap \mathcal{C}_0^1 \equiv Q_0^1.$$

This leads to an equivalent system of differential equations in $\mathbb{E}^4 \times N(\pi)$, which we write down in a way that is suitable for the computation of the normal form:

$$\begin{aligned} \frac{d}{dt}z &= \mathcal{J}z + \sum_{j \geq 2} \frac{1}{j!} f_j^1(z, y, \mu), \\ \frac{d}{dt}y &= A_1 y + \sum_{j \geq 2} \frac{1}{j!} f_j^2(z, y, \mu), \end{aligned} \qquad z \in \mathbb{E}^4, \quad y \in Q_0^1 \subset N(\pi), \tag{4.6}$$

where $A_1 : N(\pi) \to N(\pi)$, $D(A_1) = Q_0^1$, is the restriction of $\tilde{A}_0$ and

$$f_j^1(z, y, \mu) = \langle \tilde{F}_j(\Phi z + y, \mu), \Psi(0)\rangle_X, \quad f_j^2(z, y, \mu) = (I - \pi)X_0\tilde{F}_j(\Phi z + y, \mu). \tag{4.7}$$

5 Normal form construction in the presence of $O(2)$ symmetry

To construct a normal form, one has to simplify the power series expansion of the vector field term by term: on the j-th step, the j-th order non-resonant terms are canceled out via a change of variables. For a fixed $j \in \mathbb{N}$ and a Banach space Y consider a space $V_j^p(Y)$ of homogeneous polynomials of degree j in p variables with coefficients from Y:

$$V_j^p(Y) = \left\{ \sum_{|q| = j} c_q w^q : q \in \mathbb{Z}_+^p, \ c_q \in Y \right\}.$$

We seek changes of the form $(z,y) = (\tilde{z},\tilde{y}) + \frac{1}{j!}(U^1_j(\tilde{z},\mu), U^2_j(\tilde{z},\mu))$, where $z,\tilde{z} \in \mathbb{E}^4$, $y,\tilde{y} \in Q^1_0$, $U^1_j \in V^5_j(\mathbb{E}^4)$, and $U^2_j \in V^5_j(Q^1_0)$.

Suppose we have already conducted the procedure for $1 \le l \le k-1$. Denote by $\tilde{f}_j = (\tilde{f}^1_j, \tilde{f}^2_j)$ the j-th order(in (z,y,μ)) terms we have obtained after the $(k-1)$-th step; denote by $g_j = (g^1_j, g^2_j)$ the j-th order terms after the k-th step. Then equations (4.6) take the form

$$\frac{d}{dt}\tilde{z} = \mathcal{J}\tilde{z} + \sum_{j\ge 2}\frac{1}{j!}g^1_j(\tilde{z},\tilde{y},\mu),$$
$$\frac{d}{dt}\tilde{y} = A_1\tilde{y} + \sum_{j\ge 2}\frac{1}{j!}g^2_j(\tilde{z},\tilde{y},\mu).$$

Here $g_j(\tilde{z},\tilde{y},\mu) = \tilde{f}_j(\tilde{z},\tilde{y},\mu)$, $2 \le j \le k-1$, and

$$g^1_k(\tilde{z},\tilde{y},\mu) = \tilde{f}^1_k(\tilde{z},\tilde{y},\mu) - (M^1_k U^1_k)(\tilde{z},\mu), \quad g^2_k(\tilde{z},\tilde{y},\mu) = \tilde{f}^2_k(\tilde{z},\tilde{y},\mu) - (M^2_k U^2_k)(\tilde{z},\mu),$$

where the operators M^1_k and M^2_k are defined as

$$(M^1_k h_1)(z,\mu) = \nabla_z h_1(z,\mu)\mathcal{J}z - \mathcal{J}\,[h_1(z,\mu)], \quad M^1_k : V^5_k(\mathbb{E}^4) \to V^5_k(\mathbb{E}^4),$$
$$(M^2_k h_2)(z,\mu) = \nabla_z h_2(z,\mu)\mathcal{J}z - A_1[h_2(z,\mu)], \quad M^2_k : V^5_k(Q^1_0) \subset V^5_k(N(\pi)) \to V^5_k(N(\pi)).$$

The terms we can cancel out are precisely the ones that lie in the images of M^1_k and M^2_k.

In [14] a center manifold that satisfies $\tilde{y} = 0$ is proved to exist. The flow on this center manifold is given by an ordinary differential equation in $\mathbb{E}^4$

$$\frac{d}{dt}\tilde{z} = \mathcal{J}\tilde{z} + \sum_{j\ge 2}\frac{1}{j!}g^1_j(\tilde{z},0,\mu).$$

To proceed we need to prescribe complementary subspaces to the images $R(M^1_k)$.

Lemma 5.1.

1. *Let $M^1_k(\mathbb{C}^4)$ be the extension of M^1_k onto the complex space $V^5_k(\mathbb{C}^4)$. Then it acts on monomials according to*
$$M^1_k(\mathbb{C}^4)[z^q\mu^l e_j] = iv_*(q_1 - q_2 + q_3 - q_4 + (-1)^j)z^q\mu^l e_j,$$
where $l + q_1 + q_2 + q_3 + q_4 = k$, $l \in \mathbb{Z}_+$, $q \in \mathbb{Z}^4_+$, and $\{e_j \ : \ j = 1,2,3,4\}$ is the standard basis in $\mathbb{C}^4$.

2. *The operator $M^1_k : V^5_k(\mathbb{E}^4) \to V^5_k(\mathbb{E}^4)$ is well-defined.*

3. *The kernel $N(M^1_2)$ has the following form*

$$N(M^1_2) = \mathrm{span}_{\mathbb{R}}\{z_1\mu e_1 + z_4\mu e_4,\ iz_1\mu e_1 - iz_4\mu e_4,\ z_3\mu e_1 + z_2\mu e_4,\ iz_3\mu e_1 - iz_2\mu e_4,$$
$$z_2\mu e_2 + z_3\mu e_3,\ iz_2\mu e_2 - iz_3\mu e_3,\ z_4\mu e_2 + z_1\mu e_3,\ iz_4\mu e_2 - iz_1\mu e_3\}.$$

4. *The kernel* $N(M_3^1)$ *has the following form*

$$\begin{aligned}N(M_3^1) = \operatorname{span}_{\mathbb{R}} \{ & z_1^2 z_2 e_1 + z_3 z_4^2 e_4,\ i z_1^2 z_2 e_1 - i z_3 z_4^2 e_4,\ z_1^2 z_4 e_1 + z_1 z_4^2 e_4,\ i z_1^2 z_4 e_1 - i z_1 z_4^2 e_4, \\ & z_2 z_3^2 e_1 + z_2^2 z_3 e_4,\ i z_2 z_3^2 e_1 - i z_2^2 z_3 e_4,\ z_3 z_4^2 e_1 + z_1 z_2^2 e_4,\ i z_3 z_4^2 e_1 - i z_1 z_2^2 e_4, \\ & z_1 \mu^2 e_1 + z_4 \mu^2 e_4,\ i z_1 \mu^2 e_1 - i z_4 \mu^2 e_4,\ z_3 \mu^2 e_1 + z_2 \mu^2 e_4,\ i z_3 \mu^2 e_1 - i z_2 \mu^2 e_4, \\ & z_3 z_4^2 e_2 + z_1^2 z_2 e_3,\ i z_3 z_4^2 e_2 - i z_1^2 z_2 e_3,\ z_1 z_4^2 e_2 + z_1^2 z_4 e_3,\ i z_1 z_4^2 e_2 - i z_1^2 z_4 e_3, \\ & z_2^2 z_3 e_2 + z_2 z_3^2 e_3,\ i z_2^2 z_3 e_2 - i z_2 z_3^2 e_3,\ z_1 z_2^2 e_2 + z_3^2 z_4 e_3,\ i z_1 z_2^2 e_2 - i z_3^2 z_4 e_3, \\ & z_2 \mu^2 e_2 + z_3 \mu^2 e_3,\ i z_2 \mu^2 e_2 - i z_3 \mu^2 e_3,\ z_4 \mu^2 e_2 + z_1 \mu^2 e_3,\ i z_4 \mu^2 e_2 - i z_1 \mu^2 e_3, \\ & z_1 z_2 z_3 e_1 + z_2 z_3 z_4 e_4,\ i z_1 z_2 z_3 e_1 - i z_2 z_3 z_4 e_4,\ z_1 z_3 z_4 e_1 + z_1 z_2 z_4 e_4, \\ & i z_1 z_3 z_4 e_1 - i z_1 z_2 z_4 e_4,\ z_1 z_2 z_4 e_2 + z_1 z_3 z_4 e_3,\ i z_1 z_2 z_4 e_2 - i z_1 z_3 z_4 e_3, \\ & z_2 z_3 z_4 e_2 + z_1 z_2 z_3 e_3,\ i z_2 z_3 z_4 e_2 - i z_1 z_2 z_3 e_3 \} \,.\end{aligned}$$

5. *Every* $V_k^5(\mathbb{E}^4)$ *can be decomposed as a direct sum* $V_k^5(\mathbb{E}^4) = R(M_k^1) \oplus N(M_k^1)$.

Proof. The first 4 statements are straightforward to verify. The last assertion follows from the fact that the adjoint – with respect to a suitable inner product in $V_k^5(\mathbb{E}^4)$ – operator $(M_k^1)^*$ has the same form as M_k^1 but is associated with the matrix $\mathcal{J}^*$ [5]. Since $\mathcal{J}^* = -\mathcal{J}$ then $(M_k^1)^* = -M_k^1$ and $N(M_k^1) = R(M_k^1)^\perp$. □

6 Computation of the normal form coefficients

We will construct the normal form up to the cubic terms. According to Lemma 5.1, we need to compute the following expressions:

$$g_2^1(z,0,\mu) = \mathcal{P}_{N(M_2^1)} \tilde{f}_2^1(z,0,\mu), \quad g_3^1(z,0,\mu) = \mathcal{P}_{N(M_3^1)} \tilde{f}_3^1(z,0,\mu),$$

where $\mathcal{P}_V$ is the projection onto V and $\tilde{f}_2 = f_2$. For the sake of brevity we will abuse some notation:

$$a e_1 + b e_3 + \text{c.c.} \equiv a e_1 + \bar{b} e_2 + b e_3 + \bar{a} e_4 \in \mathbb{E}^4, \quad a, b \in \mathbb{C}$$

Using (4.7) we can evaluate $\tilde{f}_2^1(z,0,\mu) = f_2^1(z,0,\mu)$:

$$\begin{aligned} f_2^1(z,0,\mu) = -4\mu \sin(n_*^2 z_0) 2\pi (1+n_*^4) \, [& \kappa (z_1 \exp(-i\nu_* T) + z_2 \exp(i\nu_* T)) e_1 + \\ & + \kappa (z_3 \exp(-i\nu_* T) + z_4 \exp(i\nu_* T)) e_3 + \text{c.c.}] \,. \end{aligned} \tag{6.1}$$

Thus

$$\frac{1}{2!} g_2^1(z,0,\mu) = A_1 z_1 \mu e_1 + A_1 z_3 \mu e_3 + \text{c.c.}, \quad A_1 = -2\sin(n_*^2 z_0) 2\pi (1+n_*^4) \kappa \exp(-i\nu_* T).$$

We will assume that $\operatorname{Re} A_1 \neq 0$. This, actually, follows directly from (Hopf) if we impose a constraint $n_*^2 z_0 < \pi$ that is well-aligned with the applicability of the paraxial approximation of light propagation.

After we have dealt with the quadratic terms, $\tilde{f}_3^1$ becomes

$$\begin{aligned} \tilde{f}_3^1(z,0,\mu) = f_3^1(z,0,\mu) + \frac{3}{2} \nabla_z f_2^1(z,0,\mu) U_2^1(z,\mu) \\ + \frac{3}{2} \nabla_y f_2^1(z,0,\mu) U_2^2(z,\mu) - \frac{3}{2} \nabla_z U_2^1(z,\mu) g_2^1(z,0,\mu). \end{aligned}$$

Hence it remains to project $f_3^1(z,0,\mu)$, $U_2^1(z,\mu)$, $U_2^2(z,\mu)$, and $g_2^1(z,0,\mu)$ onto the kernel $N(M_3^1)$.

Since $\operatorname{Re} A_1 \neq 0$, we only need to compute the terms that are at most linear in μ as higher order terms do not affect the qualitative behavior of the trajectories. So we set $\mu = 0$ to calculate the cubic terms.

We note immediately that $g_2^1(z,0,0) = (0,0,0,0)^T$. Recalling (4.7), we obtain

$$\mathcal{P}_{N(M_3^1)} f_3^1(z,0,0) = B_2(z_1^2 z_4 + 2z_1z_2z_3)e_1 + B_2(z_2z_3^2 + 2z_1z_3z_4)e_3 + \text{c.c.}$$

where $B_2 = 6\hat{K}\kappa 2\pi(1+n_*^4)(3\sin(n_*^2 z_0) - \sin(3n_*^2 z_0))\exp(-iv_* T)$.

From formula (6.1) we can derive that $U_2^1(z,0) = (M_2^1)^{-1}\mathcal{P}_{R(M_2^1)} f_2^1(z,0,0) = (0,0,0,0)^T$. To find the polynomial $U_2^2(z,0)$ we must solve

$$(M_2^2 U_2^2)(z,0) = f_2^2(z,0). \tag{6.2}$$

Set $h(z) \equiv U_2^2(z,0)$. We use (4.7) and the definition of M_2^2 to decipher equation (6.2):

$$\begin{aligned} &[\nabla_z h(z)]\,(\tau)\mathcal{J}z - \frac{d}{d\tau}\,[h(z)]\,(\tau) = -\Phi(\tau)\tilde{f}_2^1(z,0,0), \quad -T \le \tau < 0, \\ &[\nabla_z h(z)]\,(0)\mathcal{J}z - A\,[h(z)(0)] - \tilde{L}_0\,[h(z)] = \tilde{F}_2(\Phi z,0) - \Phi(0)\tilde{f}_2^1(z,0,0). \end{aligned} \tag{6.3}$$

Since Φ is continuous and $h \in V_2^5(Q_0^1)$, we can pass to a limit in the first equation of (6.3) as $\tau \to -0$ and subtract the result from the second equation. Note that $\tilde{f}_2^1(z,0,0)$ vanishes; then (6.3) transforms into

$$\begin{aligned} &\frac{d}{d\tau}\,[h(z)]\,(\tau) = [\nabla_z h(z)]\,(\tau)\mathcal{J}z, \quad -T \le \tau < 0, \\ &\frac{d}{d\tau}\,[h(z)]\,(0) - A\,[h(z)(0)] - \tilde{L}_0\,[h(z)] = \tilde{F}_2(\Phi z,0). \end{aligned} \tag{6.4}$$

The right hand side of the second equation of (6.4) evaluates as

$$\begin{aligned} \tilde{F}_2(\Phi z,0) = 2\hat{K}\left[1-\cos(4n_*^2 z_0)\right]\big[&(z_1\phi_1)^2 + (z_2\phi_2)^2 + (z_3\phi_3)^2 + (z_4\phi_4)^2 \\ &+ 2z_1z_2\exp(2in_*x) + 2z_3z_4\exp(-2in_*x)\big]. \end{aligned}$$

We now solve equations (6.4). To this end we express $h \in V_2^5(Q_0^1)$ as a linear combination of monomials

$$\begin{aligned} h(z) = h_{2000}z_1^2 + h_{0200}z_2^2 + h_{0020}z_3^2 + h_{0002}z_4^2 + 2h_{1100}z_1z_2 + 2h_{1010}z_1z_3 \\ + 2h_{1001}z_1z_4 + 2h_{0110}z_2z_3 + 2h_{0101}z_2z_4 + 2h_{0011}z_3z_4, \quad h_i \in Q_0^1(\mathbb{C}). \end{aligned}$$

Then $(\nabla_z h)(z)\mathcal{J}z = 2iv_*\left[h_{2000}z_1^2 - h_{0200}z_2^2 + h_{0020}z_3^2 - h_{0002}z_4^2 + 2h_{1010}z_1z_3 - 2h_{0101}z_2z_4\right]$, and we deduce that $h_{2000} = \overline{h_{0002}}$, $h_{0200} = \overline{h_{0020}}$, $h_{1010} = \overline{h_{0101}}$, $h_{1100} = \overline{h_{0011}}$, and $h_{1001}, h_{0110} \in Q_0^1$. On grouping the monomials, we obtain the following list of differential problems:

$$\left.\begin{aligned} &\frac{d}{d\tau}h_k(\tau) = \gamma_k h_k(\tau), \quad -T \le \tau < 0, \\ &\frac{d}{d\tau}h_k(0) - A[h_k(0)] - \tilde{L}_0^{\mathbb{C}}[h_k] = G_k, \end{aligned}\right. \quad k \in \{2000, 0200, 1010, 1100, 1001, 0110\},$$

where

$$\begin{aligned} &G_{2000} = 2\hat{K}\left[1-\cos(4n_*^2 z_0)\right]\varphi_1^2(-T), \quad G_{0020} = 2\hat{K}\left[1-\cos(4n_*^2 z_0)\right]\varphi_3^2(-T), \\ &G_{1100} = 2\hat{K}\left[1-\cos(4n_*^2 z_0)\right]\exp(2in_*x), \quad G_{1010} = G_{1001} = G_{0110} = 0, \\ &\gamma_{2000} = \gamma_{0020} = \gamma_{1010} = 2iv_*, \quad \gamma_{1100} = \gamma_{1001} = \gamma_{0110} = 0. \end{aligned}$$

Each problem has a unique solution:

$$
\begin{aligned}
&h_{2000} = C_{2000}\varphi_1^2, \quad C_{2000} = -2\hat{K}[1-\cos(4n_*^2 z_0)](\Delta_{2n_*}(2i\nu_*))^{-1}\exp(-2i\nu_* T),\\
&h_{0020} = C_{0020}\varphi_3^2, \quad C_{0020} = C_{2000},\\
&h_{1100} = C_{1100}\exp(2in_* x), \quad C_{1100} = -2\hat{K}[1-\cos(4n_*^2 z_0)](\Delta_{2n_*}(0))^{-1},\\
&h_{1010} = h_{1001} = h_{0110} = 0.
\end{aligned}
$$

Having found $U_2^2(z,0) = h(z)$, we can calculate the remaining term of $g_3^1(z,0,0)$:

$$
\begin{aligned}
&\mathcal{P}_{N(M_3^1)}\nabla_y f_2^1(z,0,0)[h(z)] = (C_2 z_1^2 z_4 + 2D_2 z_1 z_2 z_3)e_1 + (C_2 z_2 z_3^2 + 2D_2 z_1 z_3 z_4)e_3 + \text{c.c.},\\
&C_2 = 4\hat{K}[\cos(3n_*^2 z_0) - \cos(n_*^2 z_0)]\kappa C_{2000} 2\pi(1+n_*^4)\exp(-i\nu_* T),\\
&D_2 = 4\hat{K}[\cos(3n_*^2 z_0) - \cos(n_*^2 z_0)]\kappa C_{1100} 2\pi(1+n_*^4)\exp(-i\nu_* T).
\end{aligned}
$$

Accumulating all the cubic terms, we find

$$
\frac{1}{3!}g_3^1(z,0,0) = (A_2^{(1)} z_1^2 z_4 + A_2^{(2)} z_1 z_2 z_3)e_1 + (A_2^{(1)} z_2 z_3^2 + A_2^{(2)} z_1 z_3 z_4)e_3 + \text{c.c.},
$$

where $A_2^{(1)} = (B_2 + C_2)/6$ and $A_2^{(2)} = (B_2 + D_2)/3$.

This concludes our computation as we have obtained all the quadratic and cubic terms (that are at most linear in μ) of the sought normal form

$$
\frac{d}{dt}z = \mathcal{J}z + \frac{1}{2!}g_2^1(z,0,\mu) + \frac{1}{3!}g_3^1(z,0,0) + \underline{O}(|z|\mu^2 + |(z,\mu)|^4). \tag{6.5}
$$

Passing to polar coordinates $z_1 = \rho_1\exp(i\omega_1)$ and $z_3 = \rho_3\exp(i\omega_3)$ in (6.5), we get our final statement.

Theorem 6.1. *Let* (Hopf) *and* $n_*^2 z_0 < \pi$ *hold. Then the flow of* (3.3) *on a center manifold is governed by the following normal form*

$$
\begin{aligned}
&\frac{d}{dt}\rho_1 = \rho_1(K_1\mu + K_2^{(1)}\rho_1^2 + K_2^{(2)}\rho_3^2) + \underline{O}(\rho_1\mu^2 + |(\rho_1,\rho_3,\mu)|^4),\\
&\frac{d}{dt}\omega_1 = \nu_* + \underline{O}(|(\rho_1,\rho_3,\mu)|),\\
&\frac{d}{dt}\rho_3 = \rho_3(K_1\mu + K_2^{(1)}\rho_3^2 + K_2^{(2)}\rho_1^2) + \underline{O}(\rho_3\mu^2 + |(\rho_1,\rho_3,\mu)|^4),\\
&\frac{d}{dt}\omega_3 = \nu_* + \underline{O}(|(\rho_1,\rho_3,\mu)|),
\end{aligned}
$$

where $K_1 = \operatorname{Re} A_1 \neq 0$, $K_2^{(1)} = \operatorname{Re} A_2^{(1)}$, *and* $K_2^{(2)} = \operatorname{Re} A_2^{(2)}$.

7 Conclusion

In this paper we constructed an $O(2)$-equivariant Hopf bifurcation normal form for a model of a nonlinear optical system with delay and diffraction in the feedback loop. The coefficients were expressed explicitly in terms of the parameters of the model. This makes it possible to constructively analyze the phase portrait of the normal form and, based on the analysis, study the stability properties of the bifurcating rotating and standing waves (see [2]).

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