Electronic Journal of Qualitative Theory of Differential Equations
2017, No. **31**, 1–7; doi: 10.14232/ejqtde.2017.1.31 http://www.math.u-szeged.hu/ejqtde/

Three spectra inverse Sturm–Liouville problems with overlapping eigenvalues

Shouzhong Fu ✉ [1], **Zhong Wang** [1] and **Guangsheng Wei** [2]

[1] College of Mathematics and Statistics, Zhaoqing University, Zhaoqing 526061, PR China
[2] College of Mathematics and Information Science, Shaanxi Normal University, Xi'an 710062, PR China

Received 28 January 2017, appeared 9 May 2017
Communicated by Miklós Horváth

Abstract. In the paper we show that the Dirichlet spectra of three Sturm–Liouville differential operators defined on the intervals $[0,1]$, $[0,a]$ and $[a,1]$ for some $a \in (0,1)$ fixed, together with the knowledge of the normalizing constants corresponding to the overlapped eigenvalues, uniquely determine the potential q on $[0,1]$. In this situation we also provide the algorithm for recovering the potential by using the given spectral data.

Keywords: eigenvalue, normalizing constant, inverse eigenvalue problem.

2010 Mathematics Subject Classification: 34A55, 34B24.

1 Introduction

In this paper we are concerned with the inverse problem for recovering the potential q on interval $[0,1]$ of a Sturm–Liouville equation

$$lu := -u'' + q(x)u = \lambda u \tag{1.1}$$

using the three spectra $\sigma(L)$, $\sigma(L^-)$ and $\sigma(L^+)$ corresponding to three Sturm–Liouville operators L, L^- and L^+. These operators are generated in L^2 spaces by the differential expressions l defined on $[0,1]$, $[0,a]$ and $[a,1]$, respectively, and the Dirichlet boundary conditions

$$u(0) = 0 = u(1), \tag{1.2}$$
$$u(0) = 0 = u(a), \tag{1.3}$$
$$u(a) = 0 = u(1). \tag{1.4}$$

Here the potential $q \in L^2[0,1]$ is a real-valued function and $a \in (0,1)$ is fixed.

Recently there has been much interest in three spectra inverse problems (see [1–7, 9, 16] and the references therein). This problem was first investigated by Pivovarchik [15] under condition that $a = 1/2$ and $\sigma(L)$ and $\sigma(L^{\pm})$ are all Dirichlet spectra. Further investigation has

✉ Corresponding author. Email: shzhfu@sina.com

been carried out by Gesztesy and Simon [9] under the more general situations of $a \in (0,1)$ and no-Dirichlet spectra. In particular, Gesztesy and Simon [9] proved uniqueness of the reconstructed q whenever the three spectra do not overlap and suggested a counterexample to uniqueness otherwise. They also raised an interesting assertion: We believe the analysis of the situation where $\sigma(L^-)\cap\sigma(L^+)$ has k-points will yield k-parameter sets of potentials q consistent with the given sets of eigenvalues (see [9, p. 91]).

One of our purposes of this paper is to answer affirmatively the above assertion. More specifically, we shall show that the uniqueness result remains valid by using the three spectra provided that the normalizing constants α_n corresponding to λ_n are employed as the additional spectral data, when $\lambda_n \in \sigma(L^-)\cap\sigma(L^+)$. This implies that when $\sigma(L^-)\cap\sigma(L^+)$ has k-points, there then exists a k-parameter set of potentials q corresponding to the given three spectra. Our second purpose here is to reconstruct potentials in terms of the three spectra, together with the normalizing constants $\{\alpha_n\}_{n\in\Lambda}$ where $\Lambda\subset\mathbb{N}$ satisfying $\sigma(L^+)\cap\sigma(L^-)=\{\lambda_n\}_{n\in\Lambda}$. The method used here is similar to that used solving half-inverse problem [15] and three spectra inverse problem without the overlapping eigenvalues [14].

Section 2 includes the uniqueness theorem and its proof. Section 3 gives the algorithm for the reconstruction of q in terms of given spectral data.

2 Uniqueness

In this section we shall prove our uniqueness result concerning with three spectra inverse Sturm–Liouville problems with overlapping eigenvalues.

Denote by $u_-(x,\lambda)$ and $u_+(x,\lambda)$ the solutions of Eq. (1.1) with the initial conditions

$$u_-(0,\lambda)=u_-'(0,\lambda)-1=0, \tag{2.1}$$

$$u_+(1,\lambda)=u_+'(1,\lambda)-1=0. \tag{2.2}$$

It is known (see, e.g., [8] and [13, Lemma 3.4.2 and proof of Theorem 3.4.1]) that, for each $x\in[0,1]$, $u_\pm(x,\lambda)$ are entire functions of λ and the eigenvalues $\{\lambda_n\}_{n=1}^\infty$ of the operator L are precisely zeros of the function $\Delta(\lambda)$ defined by

$$\Delta(\lambda)=\begin{vmatrix} u_-(a,\lambda) & u_+(a,\lambda)\\ u_-'(a,\lambda) & u_+'(a,\lambda)\end{vmatrix}=u_-(1,\lambda), \tag{2.3}$$

where $\Delta(\lambda)$ is called the characteristic function of L, which also has the following representation:

$$\Delta(\lambda)=\prod_{k=1}^{\infty}\left(\frac{\lambda_k-\lambda}{k^2\pi^2}\right). \tag{2.4}$$

Similarly, letting $\sigma(L^+)=\{\mu_n^+\}_{n=1}^\infty$ and $\sigma(L^-)=\{\mu_n^-\}_{n=1}^\infty$, then we have

$$\Delta^+(\lambda):=u_+(a,\lambda)=(1-a)\prod_{k=1}^{\infty}\left(\frac{\mu_n^+-\lambda}{k^2\pi^2/(1-a)^2}\right), \tag{2.5}$$

$$\Delta^-(\lambda):=u_-(a,\lambda)=a\prod_{k=1}^{\infty}\left(\frac{\mu_n^--\lambda}{k^2\pi^2/a^2}\right), \tag{2.6}$$

where $\Delta^\pm(\lambda)$ denote the characteristic functions of the operators $L^\pm$. Moreover, for any $\lambda_n\in\sigma(L)$, the normalizing constant, α_n, associated with the λ_n is defined by

$$\alpha_n=\int_0^1 u_-^2(x,\lambda_n)\mathrm{d}x. \tag{2.7}$$

In virtue of the above preliminaries, we are now in a position to give the uniqueness result of this paper.

Theorem 2.1. *Let the Sturm–Liouville differential operators L and $L^{\pm}$ be defined by (1.1)–(1.4). Let*

$$\sigma(L^{+})\cap\sigma(L^{-})=\{\lambda_n\}_{n\in\Lambda} \tag{2.8}$$

where $\Lambda(\subset\mathbb{N})$ is the associated index set. Then the potential q is uniquely determined almost everywhere on $[0,1]$ by the three spectra $\sigma(L)$, $\sigma(L^{\pm})$ and the normalizing constants $\{\alpha_n\}_{n\in\Lambda}$.

Proof. Let us consider another operators $\tilde{L}$, $\tilde{L}^{-}$ and $\tilde{L}^{+}$ of the same form (1.1)–(1.4) but with different coefficient $\tilde{q}(x)$ on $[0,1]$. Denote by $\tilde{\Delta}(\lambda)$ and $\tilde{\Delta}^{\pm}(\lambda)$ the characteristic functions of operators $\tilde{L}$ and $\tilde{L}^{\pm}$. Then by the hypotheses of Theorem 2.1, these operators and L and $L^{\pm}$ have common spectra $\sigma(L)$ and $\sigma(L^{\pm})$ respectively; and normalizing constants α_n for $n\in\Lambda$ corresponding to the common eigenvalues $\lambda_n\in\sigma(L)\cap\sigma(L^{\pm})$.

From (2.3)–(2.6) we have $u_{\pm}(a,\lambda)=\tilde{u}_{\pm}(a,\lambda)$ and $u_{-}(1,\lambda)=\tilde{u}_{-}(1,\lambda)$. This together with (2.3) implies

$$\begin{vmatrix} u_{-}(a,\lambda) & u_{+}(a,\lambda) \\ u'_{-}(a,\lambda)-\tilde{u}'_{-}(a,\lambda) & u'_{+}(a,\lambda)-\tilde{u}'_{+}(a,\lambda)\end{vmatrix}=\Delta(\lambda)-\tilde{\Delta}(\lambda)\equiv 0. \tag{2.9}$$

We first prove that $u'_{\pm}(a,\mu_k^{\pm})=\tilde{u}'_{\pm}(a,\mu_k^{\pm})$ for all $k\in\mathbb{N}$. If $\mu_k^{-}\notin\sigma(L)$ then by (2.5) and (2.6) we have $u_{+}(a,\mu_k^{-})\neq 0$, $u_{-}(a,\mu_k^{-})=0$ and therefore $u'_{-}(a,\mu_k^{-})=\tilde{u}'_{-}(a,\mu_k^{-})$. Similar argument yields $u'_{+}(a,\mu_k^{+})=\tilde{u}'_{+}(a,\mu_k^{+})$ for all $\mu_k^{+}\notin\sigma(L)$.

On the other hand, if there exist k_1, k_2 and $k\in\mathbb{N}$ such that $\mu_{k_1}^{-}=\mu_{k_2}^{+}=\lambda_k\in\sigma(L)$, then $u_{-}(a,\lambda_k)=0=u_{+}(a,\lambda_k)$. This together with (2.9) yields

$$\begin{vmatrix} \dot{u}_{-}(a,\lambda_k) & \dot{u}_{+}(a,\lambda_k) \\ u'_{-}(a,\lambda_k)-\tilde{u}'_{-}(a,\lambda_k) & u'_{+}(a,\lambda_k)-\tilde{u}'_{+}(a,\lambda_k)\end{vmatrix}=0. \tag{2.10}$$

Here $\dot{u}_{\pm}=\partial u_{\pm}/\partial\lambda$. Since λ_k is a simple eigenvalue of the operators L and $L^{\pm}$, it follows that $\dot{u}_{\pm}(a,\lambda_k)\neq 0$. If

$$u'_{\pm}(a,\lambda_k)-\tilde{u}'_{\pm}(a,\lambda_k)\neq 0, \tag{2.11}$$

then we have the following equality

$$\frac{\dot{u}_{-}(a,\lambda_k)}{\dot{u}_{+}(a,\lambda_k)}=\frac{u'_{-}(a,\lambda_k)-\tilde{u}'_{-}(a,\lambda_k)}{u'_{+}(a,\lambda_k)-\tilde{u}'_{+}(a,\lambda_k)}. \tag{2.12}$$

Note that $\dot{\Delta}(\lambda_k)=-\alpha_k\kappa_k$ where $\kappa_k=u'_{+}(a,\lambda_k)/u'_{-}(a,\lambda_k)$ (see [8]). Since $\alpha_k=\tilde{\alpha}_k$ and $\Delta(\lambda)=\tilde{\Delta}(\lambda)$ for all $\lambda\in\mathbb{C}$, it follows that $\kappa_k=\tilde{\kappa}_k$. This combined with (2.11) gives

$$\frac{u'_{-}(a,\lambda_k)}{u'_{+}(a,\lambda_k)}=\frac{\tilde{u}'_{-}(a,\lambda_k)}{\tilde{u}'_{+}(a,\lambda_k)}=\frac{u'_{-}(a,\lambda_k)-\tilde{u}'_{-}(a,\lambda_k)}{u'_{+}(a,\lambda_k)-\tilde{u}'_{+}(a,\lambda_k)}=\frac{\dot{u}_{-}(a,\lambda_k)}{\dot{u}_{+}(a,\lambda_k)}. \tag{2.13}$$

Moreover, because each zero of the characteristic function $\Delta(\lambda)$ is simple, we have

$$\dot{\Delta}(\lambda_k)=\begin{vmatrix} \dot{u}_{-}(a,\lambda_k) & \dot{u}_{+}(a,\lambda_k) \\ u'_{-}(a,\lambda_k) & u'_{+}(a,\lambda_k)\end{vmatrix}\neq 0, \tag{2.14}$$

which contradicts (2.13). Therefore, $u'_{\pm}(a,\lambda_k)=\tilde{u}'_{\pm}(a,\lambda_k)$ for all $k\in\Lambda$.

By the discussion above, we see that $u'_\pm(a,\mu_k^\pm) - \tilde{u}'_\pm(a,\mu_k^\pm) = 0$ for all $k \in \mathbb{N}$. Let us consider the function $F(\lambda)$ defined as

$$F_-(\lambda) = \frac{u'_-(a,\lambda) - \tilde{u}'_-(a,\lambda)}{u_-(a,\lambda)}. \tag{2.15}$$

It is clear to check that $F(\lambda)$ is an entire function and it has the following representation

$$F_-(\lambda) = \frac{u'_-(a,\lambda)}{u_-(a,\lambda)} - \frac{\tilde{u}'_-(a,\lambda)}{\tilde{u}_-(a,\lambda)} =: \tilde{m}_-(a,\lambda) - m_-(a,\lambda). \tag{2.16}$$

Here $m_-(a,\lambda)$ is called the Weyl m-function associated with the Sturm–Liouville operator L^-. It is known that $m_-(a,\lambda) = i\sqrt{\lambda} + o(1)$ as $|\lambda| \to \infty$ in any sector $\varepsilon < \operatorname{Arg}(\lambda) < \pi - \varepsilon$ for $\varepsilon > 0$, where $\sqrt{\lambda}$ is the square root branch with $\operatorname{Im}(\sqrt{\lambda}) \geq 0$, which implies that $m_-(a,iy) - \tilde{m}_-(a,iy) = o(1)$ as $y \to +\infty$. It together with (2.16) yields $F_-(\lambda) = 0$ and therefore $u'_-(a,\lambda) = \tilde{u}'_-(a,\lambda)$ for all $\lambda \in \mathbb{C}$.

The above argument implies that $m_\pm(a,\lambda) \equiv \tilde{m}_\pm(a,\lambda)$. By the uniqueness result of Marchenko [13], we have either $q = \tilde{q}$ on $[0,a]$ by using $m_-(a,\lambda) \equiv \tilde{m}_-(a,\lambda)$ or $q = \tilde{q}$ on $[a,1]$ by using $m_+(a,\lambda) \equiv \tilde{m}_+(a,\lambda)$. The proof is complete. □

3 Reconstruction

By the uniqueness theorem above, we know that the solution of the inverse problem discussed by us, if exists, is unique. In this section we provide the way of recovering the potential q by using three Dirichlet spectra $\sigma(L)$, $\sigma(L^\pm)$ and (if necessary) the normalizing constants $\{\alpha_n\}_{n\in\Lambda}$ corresponding to the overlapped eigenvalues $\lambda_n \in \sigma(L)$, where $\Lambda = \{n : \lambda_n \in \sigma(L^-) \cap \sigma(L^+)\}$. The method used here is similar to that used solving half-inverse problem [15] and three spectra inverse problem [14] without the overlapping eigenvalues.

Without loss of generality, we always assume the eigenvalues $\{\lambda_k\}_{k=1}^\infty$, $\{\mu_k^\pm\}_{k=1}^\infty$ are all positive. Denote by $s^2 = \lambda$ and $\gamma^2 = \mu$. Under these notations, we shall use increasing sequences $\{s_k\}_{k\in\mathbb{Z}\setminus\{0\}}$ $(s_{-k} = -s_k,\ (s_k)^2 = \lambda_k$ for $k \in \mathbb{N})$ and $\{\gamma_k^\pm\}_{k\in\mathbb{Z}\setminus\{0\}}$ $(\gamma_{-k}^\pm = -\gamma_k^\pm,\ (\gamma_k^\pm)^2 = \mu_k^\pm$ for $k \in \mathbb{N})$, and $\{\alpha_n\}_{n\in\Lambda}$ to recovering q.

Based on condition that the potential q exists corresponding to our spectral data, according to [13, p. 32] we have

$$u_-(x,\lambda) = \frac{\sin(sx)}{s} + \int_0^x K_1(x,t)\frac{\sin(st)}{s}\mathrm{d}t, \tag{3.1}$$

$$u_+(x,\lambda) = \frac{\sin\big(s(1-x)\big)}{s} + \int_x^1 K_2(x,t)\frac{\sin\big(s(1-t)\big)}{s}\mathrm{d}t. \tag{3.2}$$

Using (3.1) and (3.2), we obtain

$$u'_-(a,\lambda) = \cos(sa) + K_1\frac{\sin(sa)}{s} + \frac{\psi_1(s)}{s}, \tag{3.3}$$

$$u'_+(a,\lambda) = \cos\big(s(1-a)\big) - K_2\frac{\sin\big(s(1-a)\big)}{s} + \frac{\psi_2(s)}{s}, \tag{3.4}$$

where $\psi_1(0) = 0$, $\psi_2(0) = 0$,

$$K_1 = \frac{1}{2}\int_0^a q(t)\mathrm{d}t, \qquad K_2 = \frac{1}{2}\int_a^1 q(t)\mathrm{d}t,$$

$$\psi_1(s) = \int_0^a K'_{1,x}(a,t)\sin(st)\mathrm{d}t, \qquad \psi_2(s) = \int_a^1 K'_{2,x}(a,t)\sin\big(s(1-t)\big)\mathrm{d}t. \tag{3.5}$$

Moreover, the $K_1(x,t)$ and $K_2(x,t)$ possess partial derivatives of the first order, and belonging to $L^2(0,a)$ and $L^2(a,1)$, respectively, as a function of each of its variables when the other variable is fixed. From [10, Theorem 8] we see that two functional sequences $\{\sin(\gamma_k^- t)\}_{k=1}^{\infty}$ and $\{\sin(\gamma_k^+(t-1))\}_{k=1}^{\infty}$ are Riesz bases of $L^2[0,a]$ and $L^2[a,1]$, respectively. This together with the fact that $K'_{1,x}(a,t) \in L^2(0,a)$ and $K'_{2,x}(a,t) \in L^2(a,1)$ implies

$$\{\psi_1(\gamma_k^-)\}_{k\in\mathbb{Z}\setminus\{0\}},\ \{\psi_2(\gamma_k^+)\}_{k\in\mathbb{Z}\setminus\{0\}} \in l_2 \tag{3.6}$$

(see [13, Lemma 1.4.3]). On the other hand, letting $\mathcal{L}^a$ denote the class of entire functions of exponential type $\le a$, which belong to $L^2(-\infty,\infty)$ for real λ, then we have $\psi_1 \in \mathcal{L}^a$ and $\psi_2 \in \mathcal{L}^{(1-a)}$.

Now we are in a position to construct q. We begin by introducing the following lemma.

Lemma 3.1. *We have*

$$u'_-(a,\mu_k^-) = \begin{cases} -\dfrac{\Delta(\mu_k^-)}{u_+(a,\mu_k^-)} & \text{if } \mu_k^- \notin \sigma(L), \\ -\dfrac{\alpha_n\dot{\Delta}(\mu_k^-)}{\dot{\Delta}(\mu_k^-)\dot{u}_-(a,\mu_k^-) + \alpha_n\dot{u}_+(a,\mu_k^-)} & \text{if } \mu_k^- = \lambda_n \in \sigma(L), \end{cases} \tag{3.7}$$

and

$$u'_+(a,\mu_k^+) = \begin{cases} \dfrac{\Delta(\mu_k^+)}{u_-(a,\mu_k^+)} & \text{if } \mu_k^+ \notin \sigma(L), \\ \dfrac{\big(\dot{\Delta}(\mu_k^+)\big)^2}{\dot{\Delta}(\mu_k^+)\dot{u}_-(a,\mu_k^+) + \alpha_n\dot{u}_+(a,\mu_k^+)} & \text{if } \mu_k^+ = \lambda_n \in \sigma(L). \end{cases} \tag{3.8}$$

Proof. By (2.3), if $\lambda = \mu_k^- \notin \sigma(L)$ then we have $u_+(a,\mu_k^-) \neq 0$ and therefore $u'_-(a,\mu_k^-) = -\triangle(\mu_k^-)/u_+(a,\mu_k^-)$; moreover, if $\mu_k^- = \lambda_n \in \sigma(L)$ then we have $\mu_k^- \in \sigma(L^+)$. This together with (2.3) implies

$$\dot{\Delta}(\mu_k^-) = \begin{vmatrix} \dot{u}_- & \dot{u}_+ \\ u'_- & u'_+ \end{vmatrix} (a,\mu_k^-). \tag{3.9}$$

Note that $\dot{\Delta}(\mu_k^-) = -\alpha_n\kappa_n$ and $\kappa_n = u'_+(a,\mu_k^-)/u'_-(a,\mu_k^-)$, which together with (3.9) yields (3.7). Similar argument implies that (3.8) remains true. The proof is complete. □

The above lemma helps us to solve the values of $\psi_1(\gamma_k^-)$ and $\psi_2(\gamma_k^+)$ for $k \in \mathbb{Z}\setminus\{0\}$ in terms of (3.3) and (3.4). Thus, for $k \in \mathbb{N}$ we have

$$\psi_1(\gamma_k^-) = \gamma_k^-\left(u'_-(a,\mu_k^-) - \cos(a\gamma_k^-) - K_1\frac{\sin(a\gamma_k^-)}{\gamma_k^-}\right), \tag{3.10}$$

$$\psi_1(\gamma_{-k}^-) = \gamma_{-k}^-\left(u'_-(a,\mu_k^-) - \cos(a\gamma_{-k}^-) - K_1\frac{\sin(a\gamma_{-k}^-)}{\gamma_{-k}^-}\right);$$

$$\psi_2(\gamma_k^+) = \gamma_k^+\left(u'_+(a,\mu_k^+) + \cos\big((1-a)\gamma_k^+\big) + K_2\frac{\sin\big((1-a)\gamma_k^+\big)}{\gamma_k^+}\right), \tag{3.11}$$

$$\psi_2(\gamma_{-k}^+) = \gamma_{-k}^+\left(u'_+(a,\mu_k^+) + \cos\Big((1-a)\gamma_{-k}^+\Big) + K_2\frac{\sin\Big((1-a)\gamma_{-k}^+\Big)}{\gamma_{-k}^+}\right),$$

where $\gamma_{-k}^- = -\gamma_k^-$, $\gamma_{-k}^+ = -\gamma_k^+$. Moreover, it is easy to check from (3.1) and (3.2) that both functions $su_-(a,\lambda)$ and $su_+(a,\lambda)$ are of sine-type (see, e.g., [11]), that is, there exist positive numbers m, M and p such that

$$me^{|\operatorname{Im} s|a} \leq |su_-(a,\lambda)| \leq Me^{|\operatorname{Im} s|a},$$
$$me^{|\operatorname{Im} s|(1-a)} \leq |su_+(a,\lambda)| \leq Me^{|\operatorname{Im} s|(1-a)}$$

for $|\operatorname{Im} s| > p$. We use Theorem A in [11] and find

$$\psi_1(s) = su_-(a,s^2) \sum_{k=-\infty,k\neq 0}^{\infty} \frac{\psi_1(\gamma_k^-)}{\frac{\mathrm{d}su_-(a,s^2)}{\mathrm{d}s}|_{s=\gamma_k^-}(s-\gamma_k^-)}, \tag{3.12}$$

$$\psi_2(s) = su_+(a,s^2) \sum_{k=-\infty,k\neq 0}^{\infty} \frac{\psi_2(\gamma_k^+)}{\frac{\mathrm{d}su_+(a,s^2)}{\mathrm{d}s}|_{s=\gamma_k^+}(s-\gamma_k^+)}, \tag{3.13}$$

where $\psi_1(0) = \psi_2(0) = 0$. The series on the right-hand sides of (3.12) and (3.13) converge uniformly on any compact subdomain of $\mathbb{C}$ and in $L_2(-\infty,\infty)$ for real λ to functions which belong to $\mathcal{L}^a$ and $\mathcal{L}^{1-a}$. Then we find the characteristic functions $u'_-(a,\lambda)$ and $u'_+(a,\lambda)$ of the Dirichlet-Neumann Sturm–Liouville problems on the intervals $[0,a]$ and $[a,1]$, respectively. Denote by $\{\nu_k^-\}_{-\infty,k\neq 0}^{\infty}$ and $\{\nu_k^+\}_{-\infty,k\neq 0}^{\infty}$ the sets of zeros of the functions $u'_-(a,\lambda)$ and $u'_+(a,\lambda)$.

We are able to construct the part of the potential $q \in L_2(0,a)$ by the two spectra $\{\gamma_k^-\}_{-\infty,k\neq 0}^{\infty}$ and $\{\nu_k^-\}_{-\infty,k\neq 0}^{\infty}$ and the part of the potential $q \in L_2(a,1)$ by the two spectra $\{\gamma_k^+\}_{-\infty,k\neq 0}^{\infty}$ and $\{\nu_k^+\}_{-\infty,k\neq 0}^{\infty}$ via the procedure of [12, 13].

The method of reconstruction of potentials is described by the following algorithm.

Algorithm 3.2. *Suppose that the three spectra $\sigma(L) = \{\lambda_n\}_{n=1}^{\infty}$, $\sigma(L^-) = \{\mu_n^-\}_{n=1}^{\infty}$ and $\sigma(L^+) = \{\mu_n^+\}_{n=1}^{\infty}$ corresponding to Sturm–Liouville operators L, L^- and L^+ defined by (1.1)–(1.4) and the normalizing constants $\{\alpha_n\}_{n\in\Lambda}$ where $\Lambda = \{n : \lambda_n \in \sigma(L^-) \cap \sigma(L^+)\}$ are given. Then the algorithm of constructing q from the given data follows.*

(1) Construct $\Delta(\lambda)$ and $u_\pm(a,\lambda)$ by (2.3)–(2.6) in Section 2.

(2) Find $\psi_1(s)$ and $\psi_2(s)$ by (3.12)–(3.13) and K_1 and K_2 by the asymptotics of $\{\mu_n^\pm\}_{n=1}^{\infty}$:

$$\mu_n^+ = \left(\frac{n\pi}{a}\right)^2 + \frac{K_1}{a} + o(1), \qquad \mu_n^+ = \left(\frac{n\pi}{1-a}\right)^2 + \frac{K_2}{1-a} + o(1);$$

and then find $u'_-(a,\lambda)$ and $u'_+(a,\lambda)$ by (3.3) and (3.4).

(3) Solve the zeros, $\{\nu_k^-\}_{-\infty,k\neq 0}^{\infty}$ and $\{\nu_k^+\}_{-\infty,k\neq 0}^{\infty}$, of functions $u'_-(a,\lambda)$ and $u'_+(a,\lambda)$.

(4) Using the algorithm of Gelfand and Levitan (see [12,13]) to construct the (unique) potential q on $[0,a]$ and $[a,1]$ by $\{\gamma_k^\pm\}_{k\in\mathbb{Z}\setminus\{0\}}$ and $\{\nu_k^\pm\}_{k\in\mathbb{Z}\setminus\{0\}}$.

Acknowledgements

The authors would like to thank the referees for their insightful suggestions and comments which improved and strengthened the presentation of this manuscript.

The research was supported in part by the NNSF of China (No. 11571212, 11171295).

References

[1] O. Boyko, O. Martinyuk, V. Pivovarchik, Higher order Nevanlinna functions and the inverse three spectra problem, *Opuscula Math.* **36**(2016), 301–314. MR3472167; url

[2] O. Boyko, O. Martinyuk, V. Pivovarchik, On a generalization of the three spectra inverse problem, *Methods Funct. Anal. Topology* **22**(2016), 48–61. MR3522863

[3] O. Boyko, V. Pivovarchik, C.F. Yang, On solvability of three spectra problem, *Math. Nachr.* **289**(2016), 1727–1738. MR3563902; url

[4] M. C. Drignei, Uniqueness of solutions to inverse Sturm–Liouville problems with $L^2(0,a)$ potential using three spectra, *Adv. Appl. Math.* **42**(2009), 471–482. MR2511011; url

[5] M. C. Drignei, Constructibility of an solution to an inverse Sturm–Liouville problem using three Dirichlet spectra, *Inverse Problems* **26**(2010), 025003, 29 pp. MR2575360; url

[6] J. Eckhardt, F. Gesztesy, R. Nichols, G. Teschl, Inverse spectral theory for Sturm–Liouville operators with distributed potentials, *J. London Math. Soc.* **88**(2013), 801–828. MR3229765; url

[7] J. Eckhardt, Two inverse spectral problems for a class of singular Krein strings, *Int. Math. Res. Not.* **13**(2014), 3692–3713. MR3145132; url

[8] G. Freiling, V. Yurko, *Inverse Sturm–Liouville problems and their applications*, NOVA Science Publishers, New York, 2001. MR2094651

[9] F. Gesztesy, B. Simon, On the determination of a potential from three spectra, in: *Differential operators and spectral theory*, Amer. Math. Soc. Transl. Ser. 2, Vol. 189, Adv. Math. Sci., 41, Amer. Math. Soc., Providence, RI, 1999, pp. 85–92. MR1730505

[10] X. He, H. Volkmer, Riesz bases of solutions of Sturm–Liouville equations, *J. Fourier Anal. Appl.* **7**(2001), 297–307. MR1835284; url

[11] B. Ja. Levin, Ju. I. Lyubarskii, Interpolation by entire functions of special classes and related expansions in series of exponents, *Izv. Akad. Nauk USSR* **39**(1975), 657–702. MR0454019

[12] B. M. Levitan, *Inverse Sturm–Liouville Problems*, VNU Science Press, Utrecht, 1987. MR0933088

[13] V. A. Marchenko, *Sturm–Liouville operators and applications*, Operator Theory: Advances and Applications, Vol. 22, Birkhäuser Verlag, Basel, 1986. MR0897106

[14] O. Martinyuk, V. Pivovarchik, On the Hochstadt–Lieberman theorem, *Inverse Problems* **26**(2010), No. 3, 035011, 6 pp. MR2594381; url

[15] V. N. Pivovarchik, An inverse Sturm–Liouville problem by three spectra, *Integral Equ. Oper. Theory* **34**(1999), 234–243. MR1694710; url

[16] G. Wei, X. Wei, A generalization of three spectra theorem for inverse Sturm–Liouville problems, *App. Math. Letters* **35**(2014), 41–45. MR3212843; url