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On existence and multiplicity for Schrödinger–Poisson systems involving weighted sublinear nonlinearities

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Abstract. We deal with existence and multiplicity for the following class of nonhomogeneous Schrödinger–Poisson systems

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi(x)u = f(x,u) + g(x) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $V, K : \mathbb{R}^3 \to \mathbb{R}^+$ are suitable potentials and $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ satisfies sublinear growth assumptions involving a finite number of positive weights W_i, $i = 1, \dots, r$ with $r \geq 1$. By exploiting compact embeddings of the functional space on which we work in every weighted space $L^{w_i}_{W_i}(\mathbb{R}^3)$, $w_i \in (1,2)$, we establish existence by means of a generalized Weierstrass theorem. Moreover, we prove multiplicity of solutions if f is odd in u and $g(x) \equiv 0$ thanks to a variant of the symmetric mountain pass theorem stated by R. Kajikiya for subquadratic functionals.

Keywords: Schrödinger–Poisson systems, sublinear nonlinearities, variational methods, compact embeddings.

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1 Introduction

In this paper we consider the following class of Schrödinger–Poisson systems (also called Schrödinger–Maxwell systems) in both nonhomogeneous case $g(x) \not\equiv 0$, namely

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi(x)u = f(x,u) + g(x) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{$\mathcal{P}_g$}$$

and in the homogeneous case $g(x) \equiv 0$, that is

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi(x)u = f(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{$\mathcal{P}_0$}$$

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This class of systems has a strong physical meaning since it arises in several applications from mathematical physics, in particular in quantum mechanics models where it describes the mutual interactions of charged particles in the electrostatic case (see e.g. [5, 6] and references therein for more detailed physical aspects). For this reason, many authors have devoted their attention to systems of this type and they have widely studied them by using variational methods under various conditions on the potentials $V(x)$ and $K(x)$ and the nonlinearity $f(x,u)$ especially when it is superlinear or asymptotically linear at infinity in u. On the contrary, up to now, there is no extensive literature dealing with the case of nonlinearities $f(x,u)$ sublinear at infinity especially involving suitable weights and this motivates our work. Let us start with the homogeneous case $g(x) \equiv 0$.

In 2012, Sun [12] proved the existence of infinitely many small negative energy solutions to $(\mathcal{P}_0)$ in the case $K(x) \equiv 1$ by a variant fountain theorem established in [16] under the following assumptions

- (V') $V \in C(\mathbb{R}^3, \mathbb{R})$ satisfies $\inf_{x\in\mathbb{R}^3} V(x) \geq a > 0$ with a a real constant;
- (V'') for any $M > 0$, $\text{meas}\{x \in \mathbb{R}^3 : V(x) \leq M\} < +\infty$ where meas denotes the Lebesgue measure on $\mathbb{R}^3$;
- (F') $F(x,u) = W_1(x)|u|^{w_1}$ where $F(x,u) = \int_0^u f(x,t)\,dt$, $W_1 : \mathbb{R}^3 \to \mathbb{R}$ is a positive continuous function such that $W_1 \in L^{\frac{2}{2-w_1}}(\mathbb{R}^3)$ with $w_1 \in (1,2)$.

In particular, conditions (V')–(V'') imply a coercive condition on V which was first introduced by Bartsch and Wang [4] in order to overcome the loss of compactness due to the unboundedness of the domain $\mathbb{R}^3$. Clearly, thanks to (F') only the one-weight nonlinearity $f(x,u) = w_1\, W_1(x)|u|^{w_1-1}$ is allowed.

In 2013, Liu, Guo and Zhang [8] generalized the results in Sun [12] since they showed for $(\mathcal{P}_0)$ with $K(x) \equiv 1$ the existence of a nontrivial solution by minimization arguments [10] and the multiplicity of solutions with negative energy which goes to zero by a symmetric mountain pass lemma based on genus properties in critical point theory (see Salvatore [11]) by removing assumption (V'') and relaxing assumption (F') with the following

$$|f(x,u)| \leq w_1\, W_1(x)|u|^{w_1-1} + w_2\, W_2(x)|u|^{w_2-1} \text{ for a.e. } x \in \mathbb{R}^3 \text{ and for all } u \in \mathbb{R}$$

with $W_1 \in L^{\frac{2}{2-w_1}}(\mathbb{R}^3)$ and $W_1 > 0$, $W_2 \in L^3(\mathbb{R}^3)$ and $W_2 \geq 0$ where $w_1 \in (1,2)$ and $w_2 \in [4/3, 2)$. This assumption makes indefinite nonlinearities $f(x,u)$ can be also considered and the presence of the weights W_1 and W_2 ensures a good property of compactness for these $f(x,u)$.

In 2013, Lv [9] also generalized the result in Sun [12] by showing existence of a nontrivial solution by minimization arguments [10] and multiplicity of solutions with vanishing and negative energy levels by the dual fountain theorem [14] to $(\mathcal{P}_0)$ in the case $K(x) \equiv 1$ without the coercive assumption (V'') and under only (V') on V. This has been possible since the odd nonlinearity $f(x,u)$ is supposed to satisfy suitable sublinear growth hypotheses which imply the existence of three weights $W_i \in L^{\frac{2}{2-w_i}}(\mathbb{R}^3)$, $W_i > 0$, $i \in \{1,2,3\}$ with $w_i \in (1,2)$ and allow to recover compact embeddings of the functional space in the weighted space $L^{w_i}_{W_i}(\mathbb{R}^3)$, $i = 1, 2, 3$. Precisely,

$$|f(x,u)| \leq \sum_{i=1}^{3} W_i(x)|u|^{w_i-1} \quad \text{for a.e. } x \in \mathbb{R}^3 \text{ and for all } u \in \mathbb{R}$$

so that the case of indefinite nonlinearities $f(x,u)$ is also covered. This result improves and completes the paper by Liu, Guo and Zhang [8].

Few years later, in 2015 Ye and Tang [15] improved the results in Sun [12] since they showed the existence of infinitely many small solutions with small negative energy to ($\mathcal{P}_0$) by means of a new version of symmetric mountain pass lemma developed by Kajikiya [7] with the non-negative potential $K \in L^2(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3)$ (see condition (K) below), under the weaker hypotheses

$(\overline{V'})$ $V \in C(\mathbb{R}^3, \mathbb{R})$ verifies $V(x) \geq 0$ for every $x \in \mathbb{R}^3$;

(V''') there exists $M > 0$ such that $\operatorname{meas}\{x \in \mathbb{R}^3 : V(x) \leq M\} < +\infty$.

Besides a suitable local assumption, on the continuous and odd nonlinearity $f(x,u)$ is assumed in particular the following sublinear growth condition

$$|f(x,u)| \leq W_1(x)|u|^{w_1-1} + W_2(x)|u|^{w_2-1} \quad \text{for a.e. } x \in \mathbb{R}^3 \text{ and for all } u \in \mathbb{R}$$

with weights $W_1 \in L^{\frac{2}{2-w_1}}(\mathbb{R}^3)$, $W_2 \in L^{\frac{2}{2-w_2}}(\mathbb{R}^3)$, $W_1, W_2 > 0$ and $w_1, w_2 \in (1,2)$.

Regarding the nonhomogeneous case $g(x) \not\equiv 0$, Wang, Ma and Wang [13] in 2016 established only existence to ($\mathcal{P}_g$) with non-negative $g \in L^2(\mathbb{R}^3)$, for a class of potentials $K(x) \geq 0$ with $K \in L^2(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3)$ (as in hypothesis (K) below), under conditions (V')–(V'') and the same sublinear growth condition assumed in [15] with two weights W_1 and W_2. In this case the authors work without using for compactness this last condition.

The aim of this paper is to study ($\mathcal{P}_g$) (resp. ($\mathcal{P}_0$)) under more generic conditions in order to generalize or to give complementary results to the ones listed above. More precisely, we investigate existence (resp. multiplicity) of solutions to ($\mathcal{P}_g$) (resp. ($\mathcal{P}_0$)) under the following assumptions:

(V) $V : \mathbb{R}^3 \to \mathbb{R}$ is a Lebesgue measurable function with $\operatorname{ess\,inf}_{\mathbb{R}^3} V(x) \geq a > 0$ where a is a real constant;

(K) $K \in L^2(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3)$ and $K(x) \geq 0$ for a.e. $x \in \mathbb{R}^3$;

(f_1) $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (i.e., $f(\cdot, s)$ is measurable on $\mathbb{R}^3$ for all $s \in \mathbb{R}$ and $f(x, \cdot)$ is continuous on $\mathbb{R}$ for a.e. $x \in \mathbb{R}^3$);

(f_2) there exists $W_i \in L^{\frac{2}{2-w_i}}(\mathbb{R}^3)$, $W_i > 0$ ($i \in \{1, \ldots, r\}$) with constant $w_i \in (1,2)$ such that

$$|f(x,s)| \leq \sum_{i=1}^{r} W_i(x)|s|^{w_i-1} \quad \text{for a.e. } x \in \mathbb{R}^3 \text{ and for all } s \in \mathbb{R};$$

(f_3) there exist $\Omega \subset \mathbb{R}^3$ with $\operatorname{meas}(\Omega) > 0$, $w_{r+1} \in (1,2)$, $\eta > 0$ and $\delta > 0$ such that

$$F(x,s) \geq \eta|s|^{w_{r+1}} \quad \text{for a.e. } x \in \Omega \text{ and for all } s \in \mathbb{R},\ |s| \leq \delta$$

where $F(x,s) = \int_0^s f(x,t)\,dt$;

(f_4) $f(x,s) = -f(x,-s)$ for a.e. $x \in \mathbb{R}^3$ and for all $s \in \mathbb{R}$;

(G) $g \in L^2(\mathbb{R}^3)$.

Thus, we obtain the following results. For the definition of the functional spaces E_V and $D^{1,2}(\mathbb{R}^3)$ and of the energy functional J_0 which appear in next theorems, see Section 2.

First, let us state the existence result for the nonhomogeneous case and for the homogeneous case.

Theorem 1.1 (Existence). *Suppose that (V), (K), (f_1) and (f_2) hold. Then, we get the following:*

- (i) *(nonhomogeneous case $g(x) \not\equiv 0$) if in addition (G) holds, problem $(\mathcal{P}_g)$ admits at least a non-trivial weak solution $(\overline{u}, \phi_{\overline{u}}) \in E_V \times D^{1,2}(\mathbb{R}^3)$;*
- (ii) *(homogeneous case $g(x) \equiv 0$) if (f_3) is also assumed, problem $(\mathcal{P}_0)$ possesses both a trivial weak solution and at least a non-trivial weak solution $(\overline{u}, \phi_{\overline{u}}) \in E_V \times D^{1,2}(\mathbb{R}^3)$.*

Now, let us provide the multiplicity result obtained in the case $g(x) \equiv 0$.

Theorem 1.2 (Multiplicity). *Assume that (V), (K), (f_1), (f_2), (f_3), (f_4) hold. Then, problem $(\mathcal{P}_0)$ has a sequence $\{(\overline{u}_k, \phi_{\overline{u}_k})\} \subset E_V \times D^{1,2}(\mathbb{R}^3)$ of non-trivial weak solutions such that*

$$J_0(\overline{u}_k, \phi_{\overline{u}_k}) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla \overline{u}_k|^2 + V(x)|\overline{u}_k|^2)\, dx - \frac{1}{4}\int_{\mathbb{R}^3} |\nabla \phi_{\overline{u}_k}|^2\, dx + \frac{1}{2}\int_{\mathbb{R}^3} K(x)\phi_{\overline{u}_k}\overline{u}_k^2\, dx - \int_{\mathbb{R}^3} F(x, \overline{u}_k)\, dx \to 0$$

as $k \to +\infty$.

Remark 1.3. Thanks to Remark 2.8 and the properties of J_0 and ϕ_u stated in Section 2, we remark that Theorem 1.2 gives in particular the existence of a sequence $\{(\overline{u}_k, \phi_{\overline{u}_k})\}$ of critical points of J_0 such that $J_0(\overline{u}_k, \phi_{\overline{u}_k}) \le 0$, $\overline{u}_k \neq 0$ and then $\phi_{\overline{u}_k} \neq 0$, $\lim_k \overline{u}_k = 0$ from which we get $\lim_k \phi_{\overline{u}_k} = 0$; consequently, $\lim_k J_0(\overline{u}_k, \phi_{\overline{u}_k}) = 0^-$.

Let us observe that, as concerns the existence result in the homogeneous case $g(x) \equiv 0$, we complete the papers by Sun [12] and by Ye and Tang [15] where no existence result has been stated. Moreover, we improve the existence of solutions to $(\mathcal{P}_0)$ for not necessarily constant potentials $K(x)$ by relaxing (V') with (V) in Lv [9] and in Liu, Guo and Zhang [8].

Moreover, we generalize the existence of multiple solutions obtained in Sun [12], Liu, Guo and Zhang [8] and Lv [9] to $(\mathcal{P}_0)$ for $K(x) \equiv 1$ to a more general class of potentials satisfying (K) thus providing the existence of infinitely many small solutions with small negative energy.

In the nonhomogeneous case $g(x) \not\equiv 0$, we improve the existence result established by Wang, Ma and Wang [13] since we relax condition (V') by (V), skip (V'') and recover compactness by the different requirement (f_2) involving r weights. Furthermore, we do not impose any sign condition on g.

Remark 1.4. Let us observe that, from (f_2) by integration it follows that

$$|F(x,s)| \le \sum_{i=1}^{r} \frac{1}{w_i} W_i(x)|s|^{w_i} \quad \text{for a.e. } x \in \mathbb{R}^3 \text{ and for all } s \in \mathbb{R}. \tag{1.1}$$

The paper is organized as follows: in Section 2 we introduce the variational formulation of the problem and we recall a generalized version of Weierstrass theorem, Mazur theorem and a convexity criterion. Moreover, we recall a variant of the symmetric mountain pass theorem for "subquadratic" problems stated in [7]. In Section 3 we prove Theorem 1.1 and in Section 4 we show Theorem 1.2.

2 Variational tools

In order to introduce the variational structure of the problem, let $E = H^1(\mathbb{R}^3)$ be the usual Sobolev space endowed with the standard scalar product

$$(u, v)_E = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + uv)\, dx$$

and the corresponding norm

$$\|u\|_E = (u, u)_E^{1/2} = \left(\int_{\mathbb{R}^3} \left(|\nabla u|^2 + |u|^2\right) dx \right)^{\frac{1}{2}}$$

with dual space $(E', \|\cdot\|_{E'})$. Moreover, let $D^{1,2}(\mathbb{R}^3)$ be the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_D = \|u\|_{D^{1,2}(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} |\nabla u|^2\, dx \right)^{\frac{1}{2}}.$$

We denote by $L^s(\mathbb{R}^3)$, $1 < s < +\infty$, the Lebesgue space endowed with the norm

$$|u|_s = |u|_{L^s(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} |u|^s\, dx \right)^{\frac{1}{s}}.$$

Moreover, let us introduce

$$E_V = \left\{ u \in E : \int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)|u|^2\right) dx < \infty \right\}.$$

By assumption (V), E_V is a Hilbert space endowed with the scalar product

$$(u, v)_V = (u, v)_{E_V} = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + V(x) uv)\, dx$$

and the related norm

$$\|u\|_V = (u, v)_V^{1/2} = \left(\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)|u|^2\right) dx \right)^{\frac{1}{2}}$$

with dual space $(E'_V, \|\cdot\|_{E'_V})$. From now on, let $1 < s < \infty$ and

$$L^s_V(\mathbb{R}^3) = \left\{ u \in L^s(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)|u|^s\, dx < \infty \right\}$$

endowed with the norm

$$|u|_{s,V} = \left(\int_{\mathbb{R}^3} V(x)|u|^s\, dx \right)^{\frac{1}{s}}.$$

Clearly, $E_V = E \cap L^2_V(\mathbb{R}^3)$ and by (V) we have that $E_V \hookrightarrow E$. Moreover, the following continuous embeddings hold

$$E_V \hookrightarrow L^s(\mathbb{R}^3) \quad \text{for any } s \in [2, 6] \quad \text{and} \quad D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$$

being $2^* = 2N/(N-2) = 6$ for $N = 3$.

From now on, c and C will denote real positive constants changing line from line.

At this point, we prove the following result which allows us to state the compact embedding of E_V in a weighted Lebesgue space with a specific weight $W(x)$; the result will be applied to the Lebesgue measurable weight W_i and to the constant $w = w_i$ for any $i \in \{1, \ldots, r\}$ in assumption (f_2).

Proposition 2.1. *Let $1 < w < 2$. Suppose that W is a positive function belonging to $L^\mu(\mathbb{R}^3)$ with $\mu = \left(\frac{2}{w}\right)' = \frac{2}{2-w}$. Under assumption (V), we get the following compact embedding*

$$E_V \hookrightarrow\hookrightarrow L^w_W(\mathbb{R}^3),$$

where

$$L^w_W(\mathbb{R}^3) = \left\{ u \in L^w(\mathbb{R}^3) : \int_{\mathbb{R}^3} W(x)|u|^w \, dx < \infty \right\}$$

endowed with the norm

$$|u|_{w,W} = \left(\int_{\mathbb{R}^3} W(x)|u|^w \, dx \right)^{\frac{1}{w}}.$$

Proof. We adapt the arguments used in [9, Lemma 2.1] (see also [2, Remark 2.3] and [3, Proposition 2.2]). Let $\{u_n\}$ be a sequence in E_V such that $u_n \rightharpoonup u$ in E_V. Clearly, $u_n - u$ is bounded in E_V, namely there exists a constant $M > 0$ such that $\|u_n - u\|_V \le M$. Since $W \in L^\mu(\mathbb{R}^3)$ we have

$$\text{for all } \varepsilon > 0 \text{ there exists } R_\varepsilon > 0 \text{ such that } \left(\int_{|x| \ge R_\varepsilon} |W(x)|^\mu \, dx \right)^{\frac{1}{\mu}} < \varepsilon.$$

Therefore, by Hölder's inequality and Sobolev embeddings we get

$$\begin{aligned} \int_{|x| \ge R_\varepsilon} W(x)|u_n - u|^w \, dx &\le \left(\int_{|x| \ge R_\varepsilon} |W(x)|^\mu \, dx \right)^{\frac{1}{\mu}} \left(\int_{|x| \ge R_\varepsilon} |u_n - u|^2 \, dx \right)^{\frac{w}{2}} \\ &\le \varepsilon \, |u_n - u|_2^w \le \varepsilon \, c^w \|u_n - u\|_V^w \le \varepsilon \, c^w \, M^w. \end{aligned} \tag{2.1}$$

Now, setting $E_V(B_{R_\varepsilon}(0)) = \{ u|_{B_{R_\varepsilon}(0)} : u \in E_V \}$, since

$$E_V(B_{R_\varepsilon}(0)) \hookrightarrow H^1(B_{R_\varepsilon}(0)) \hookrightarrow\hookrightarrow L^w_W(B_{R_\varepsilon}(0)),$$

from $u_n \rightharpoonup u$ in E_V we deduce $u_n|_{B_{R_\varepsilon}(0)} \rightharpoonup u|_{B_{R_\varepsilon}(0)}$ in $E_V(B_{R_\varepsilon}(0))$ and then $u_n \to u$ in $L^w_W(B_{R_\varepsilon}(0))$. Consequently,

$$\text{for every } \varepsilon > 0 \text{ there exists } n_\varepsilon \in \mathbb{N} \text{ such that for every } n > n_\varepsilon \text{ one has}$$

$$\int_{|x| \le R_\varepsilon} W(x)|u_n - u|^w \, dx < \varepsilon$$

which, together with (2.1), implies

$$\begin{aligned} \int_{\mathbb{R}^3} W(x)|u_n - u|^w \, dx &= \int_{|x| \le R_\varepsilon} W(x)|u_n - u|^w \, dx + \int_{|x| \ge R_\varepsilon} W(x)|u_n - u|^w \, dx \\ &< \varepsilon(1 + M^w c^w) \end{aligned}$$

and then $u_n \to u$ in $L^w_W(\mathbb{R}^3)$. □

Let us point out that in Sun [12] and Wang, Ma and Wang [13] potential V satisfies stronger assumptions (V')–(V''). These conditions allow to prove that $E_V \hookrightarrow\hookrightarrow L^s(\mathbb{R}^3)$ for all $2 \le s < 6$. Differently, here above in Proposition 2.1 we show that $E_V \hookrightarrow\hookrightarrow L^w_W(\mathbb{R}^3)$ with $w \in (1, 2)$. In the following (see Proposition 2.3 and Proposition 4.1) we will exploit only this weaker result in order to overcome the lack of compactness due to the unboundedness of the domain $\mathbb{R}^3$.

Under our assumptions, it is not difficult to see that system ($\mathcal{P}_g$) (resp. ($\mathcal{P}_0$)) has a variational structure, that is, it is possible to find its solutions by looking for critical points of the functional $J_g \in C^1(E_V \times D^{1,2}(\mathbb{R}^3), \mathbb{R})$ (resp. $J_0 \in C^1(E_V \times D^{1,2}(\mathbb{R}^3), \mathbb{R})$) defined by

$$J_g(u,\phi) = \frac{1}{2}\|u\|_V^2 - \frac{1}{4}\int_{\mathbb{R}^3} |\nabla\phi|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3} K(x)\phi u^2\,dx - \int_{\mathbb{R}^3} F(x,u)\,dx - \int_{\mathbb{R}^3} g(x)u\,dx$$

$$(\text{resp. } J_0(u,\phi) = \frac{1}{2}\|u\|_V^2 - \frac{1}{4}\int_{\mathbb{R}^3} |\nabla\phi|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3} K(x)\phi u^2\,dx - \int_{\mathbb{R}^3} F(x,u)\,dx)$$

for every $(u,\phi) \in E_V \times D^{1,2}(\mathbb{R}^3)$. But the functional J_g (resp. J_0) is strongly indefinite, namely it is unbounded from below and above on infinite dimensional subspaces. In order to remove its indefiniteness and to reduce to study a not strongly indefinite functional, we can use the following reduction method introduced in [5] (see also [6]). This method relies on the fact that, for every $u \in E_V$, the Lax–Milgram theorem implies the existence of a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ satisfying in the weak sense

$$-\Delta\phi_u = K(x)u^2 \quad \text{in } \mathbb{R}^3.$$

It is well known that ϕ_u can be written with the following integral formula

$$\phi_u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{K(y)u^2(y)}{|x-y|}\,dy.$$

So, substituting $\phi = \phi_u$ in J_g (resp. J_0) it is possible to consider the functional $I_g : E_V \to \mathbb{R}$ (resp. $I_0 : E_V \to \mathbb{R}$) defined by $I_g(u) = J_g(u,\phi_u)$ (resp. $I_0(u) = J_0(u,\phi_u)$) for every $u \in E_V$. Now, by multiplying $-\Delta\phi_u = K(x)u^2$ by ϕ_u and integrating by parts we get

$$\int_{\mathbb{R}^3} K(x)\phi_u u^2\,dx = \int_{\mathbb{R}^3} -(\Delta\phi_u)\phi_u\,dx = \int_{\mathbb{R}^3} |\nabla\phi_u|^2\,dx, \tag{2.2}$$

then the reduced functional I_g (resp. I_0) takes the form for every $u \in E_V$

$$I_g(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)|u|^2\right)dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2\,dx - \int_{\mathbb{R}^3} F(x,u)\,dx - \int_{\mathbb{R}^3} g(x)u\,dx$$

$$(\text{resp. } I_0(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)|u|^2\right)dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2\,dx - \int_{\mathbb{R}^3} F(x,u)\,dx).$$

At the same time, problem ($\mathcal{P}_g$) (resp. problem ($\mathcal{P}_0$)) can be reduced to an equivalent single Schrödinger equation with a nonlocal term. Indeed, substituting $\phi = \phi_u$ in ($\mathcal{P}_g$) (resp. ($\mathcal{P}_0$)) we get the following equation

$$-\Delta u + V(x)u + K(x)\phi_u(x)u = f(x,u) + g(x) \quad \text{in } \mathbb{R}^3 \tag{$\mathcal{S}_g$}$$

$$(\text{resp. } -\Delta u + V(x)u + K(x)\phi_u(x)u = f(x,u) \quad \text{in } \mathbb{R}^3.) \tag{$\mathcal{S}_0$}$$

As we will prove in Proposition 2.3, $I_g \in C^1(E_V, \mathbb{R})$ (resp. $I_0 \in C^1(E_V, \mathbb{R})$) and every critical point of I_g (resp. I_0) corresponds to a solution $u \in E_V$ to ($\mathcal{S}_g$) (resp. ($\mathcal{S}_0$)) and provides a solution $(u,\phi) \in E_V \times D^{1,2}(\mathbb{R}^3)$ to ($\mathcal{P}_g$) (resp. ($\mathcal{P}_0$)).

Remark 2.2. Since by (K), it is $K(x) \geq 0$ for a.e. $x \in \mathbb{R}^3$, we get $\phi_u \geq 0$ for any $u \in E_V$.

Now, as just noticed, by hypothesis (V) it is $E_V \hookrightarrow H^1(\mathbb{R}^3)$; this fact together with the well known continuity of $\phi_u : H^1(\mathbb{R}^3) \to D^{1,2}(\mathbb{R}^3)$ implies also $\phi_u : E_V \to D^{1,2}(\mathbb{R}^3)$ is continuous.

Furthermore, let us observe that, if $K \in L^2(\mathbb{R}^3)$ or $K \in L^\infty(\mathbb{R}^3)$, by (2.2), Hölder's inequality and Sobolev embeddings we obtain

$$\begin{aligned}\|\phi_u\|_D^2 &= \int_{\mathbb{R}^3} K(x)\phi_u u^2\,dx \\ &\le \left(\int_{\mathbb{R}^3} (K(x))^2\,dx\right)^{1/2} \left(\int_{\mathbb{R}^3} (\phi_u)^6\,dx\right)^{1/6} \left(\int_{\mathbb{R}^3} (u^2)^3\,dx\right)^{1/3} \\ &\le |K|_2\, c\|\phi_u\|_D\, |u|_6^2\end{aligned}$$

or

$$\begin{aligned}\|\phi_u\|_D^2 = \int_{\mathbb{R}^3} K(x)\phi_u u^2\,dx &\le |K|_\infty \left(\int_{\mathbb{R}^3} (\phi_u)^6\,dx\right)^{1/6} \left(\int_{\mathbb{R}^3} (u^2)^{6/5}\,dx\right)^{5/6} \\ &\le |K|_\infty\, c\|\phi_u\|_D\, |u|_{12/5}^2.\end{aligned}$$

Therefore, in the first case we get

$$\|\phi_u\|_D \le |K|_2\, c\, |u|_6^2 \tag{2.3}$$

while in the second

$$\|\phi_u\|_D \le |K|_\infty\, c\, |u|_{12/5}^2. \tag{2.4}$$

At this point we can state the following variational principle and recover the compactness of the problem.

Proposition 2.3. *Assume that (V), (K), (f_1), (f_2) and (G) hold. Then, the weak solutions of ($\mathcal{P}_g$) (resp. ($\mathcal{P}_0$)) are the critical points of the energy functional $I_g : E_V \to \mathbb{R}$ (resp. $I_0 : E_V \to \mathbb{R}$) defined by*

$$I_g(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)|u|^2\right) dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2\,dx - \int_{\mathbb{R}^3} F(x,u)\,dx - \int_{\mathbb{R}^3} g(x)u\,dx$$

$$(\text{resp. } I_0(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)|u|^2\right) dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2\,dx - \int_{\mathbb{R}^3} F(x,u)\,dx)$$

for every $u \in E_V$. More precisely, $I_g \in C^1(E_V,\mathbb{R})$ (resp. $I_0 \in C^1(E_V,\mathbb{R})$) and its derivative $dI_g : E_V \to E_V'$ (resp. $dI_0 : E_V \to E_V'$) is defined as

$$\begin{aligned}dI_g(u)[\zeta] &= \int_{\mathbb{R}^3} [\nabla u \cdot \nabla \zeta + V(x)u\,\zeta + K(x)\phi_u u\,\zeta - f(x,u)\,\zeta - g(x)\,\zeta]\,dx \\ (\text{resp. } dI_0(u)[\zeta] &= \int_{\mathbb{R}^3} [\nabla u \cdot \nabla \zeta + V(x)u\,\zeta + K(x)\phi_u u\,\zeta - f(x,u)\,\zeta]\,dx)\end{aligned} \tag{2.5}$$

for all $u, \zeta \in E_V$. Consequently, the pair $(u,\phi) \in E_V \times D^{1,2}(\mathbb{R}^3)$ is a solution of problem ($\mathcal{P}_g$) (resp. ($\mathcal{P}_0$)) if and only if $u \in E_V$ is a critical point of I_g (resp. I_0) and $\phi = \phi_u$.

Moreover, the function $u \longmapsto f(\cdot, u(\cdot))$ is compact from E_V to E_V'.

Proof. Let us start by showing that the functional I_g (resp. I_0) is well defined and its Fréchet derivative given in (2.5) is a continuous operator from E_V to E_V'. For the sake of completeness, we give here all the details of the proof. We define and study separately the following maps

$$\varphi_V(u) = \frac{1}{2}\,\|u\|_V^2, \qquad \varphi_K(u) = \int_{\mathbb{R}^3} K(x)\phi_u u^2\,dx$$

$$\varphi_F(u) = \int_{\mathbb{R}^3} F(x,u)\,dx \quad \text{and} \quad \varphi_g(u) = \int_{\mathbb{R}^3} g(x)u\,dx.$$

Clearly, $\varphi_V \in C^1(E_V, \mathbb{R})$ since φ_V is continuous from E_V to $\mathbb{R}$ and its Gâteaux differential at u

$$d\varphi_V(u)[\zeta] = \int_{\mathbb{R}^3} \nabla u \cdot \nabla \zeta \, dx + \int_{\mathbb{R}^3} V(x) u \zeta \, dx$$

is a linear continuous map on E_V.

Concerning the map φ_K, we need to show that $\varphi_K \in C^1(E_V, \mathbb{R})$ with

$$d\varphi_K(u)[\zeta] \;=\; \int_{\mathbb{R}^3} K(x)\phi_u u\, \zeta \, dx \quad \text{for all } u, \zeta \in E_V. \tag{2.6}$$

By Remark 2.2, if $K \in L^2(\mathbb{R}^3)$ it results

$$|\varphi_K(u)| \le \int_{\mathbb{R}^3} K(x)\phi_u u^2 \, dx \le |K|_2^2 \, c^2 \, |u|_6^4$$

and, if $K \in L^\infty(\mathbb{R}^3)$ one has

$$|\varphi_K(u)| \le \int_{\mathbb{R}^3} K(x)\phi_u u^2 \, dx \le |K|_\infty^2 \, c^2 \, |u|_{12/5}^4,$$

then by Sobolev embeddings $\varphi_K(u) \in \mathbb{R}$ for any $u \in E_V$.

Now we prove that

$$\int_{\mathbb{R}^3} K(x)\phi_u u\, \zeta \, dx \in \mathbb{R} \qquad \text{for all } u, \zeta \in E_V.$$

Indeed, if $K \in L^2(\mathbb{R}^3)$, by Hölder's inequality and (2.3) we get the following

$$\begin{aligned}\left| \int_{\mathbb{R}^3} K(x)\phi_u u\, \zeta \, dx \right| &\le \int_{\mathbb{R}^3} K(x)\phi_u |u|\, |\zeta| \, dx \le |K|_2 \, |\phi_u|_6 \left(\int_{\mathbb{R}^3} (|u|\, |\zeta|)^3 \, dx \right)^{1/3} \\ &\le |K|_2 \, c \, \|\phi_u\|_D \left(\int_{\mathbb{R}^3} ((|u|)^3)^2 \, dx \right)^{1/6} \left(\int_{\mathbb{R}^3} ((|\zeta|)^3)^2 \, dx \right)^{1/6} \\ &\le |K|_2^2 \, c^2 \, |u|_6 \, |\zeta|_6.\end{aligned}$$

Similarly, if $K \in L^\infty(\mathbb{R}^3)$ by Hölder's inequality and (2.4) we obtain

$$\begin{aligned}\left| \int_{\mathbb{R}^3} K(x)\phi_u u\, \zeta \, dx \right| &\le \int_{\mathbb{R}^3} K(x)\phi_u |u|\, |\zeta| \, dx \le |K|_\infty \, |\phi_u|_6 \left(\int_{\mathbb{R}^3} (|u|\, |\zeta|)^{6/5} \, dx \right)^{5/6} \\ &\le |K|_\infty \, c \, \|\phi_u\|_D \left(\int_{\mathbb{R}^3} ((|u|^{6/5}))^2 \, dx \right)^{5/12} \left(\int_{\mathbb{R}^3} ((|\zeta|^{6/5}))^2 \, dx \right)^{5/12} \\ &\le |K|_\infty^2 \, c^2 \, |u|_{12/5} \, |\zeta|_{12/5}.\end{aligned}$$

By Sobolev embeddings in both cases we have done. It is not difficult to find that the Gâteaux derivative of φ_K at u is as in (2.6) and it is linear and continuous from E_V to $\mathbb{R}$. It remains to prove that $d\varphi_K$ is continuous from E_V to E_V', i.e.

$$\|d\varphi_K(u_n) - d\varphi_K(u)\|_{E_V'} \to 0 \quad \text{if } u_n \to u \text{ in } E_V. \tag{2.7}$$

First, observe that by adding and subtracting $K(x)\phi_{u_n} u\, \zeta$ in the integral we have

$$\begin{aligned}|(d\varphi_K(u_n) - d\varphi_K(u))[\zeta]| &\le \int_{\mathbb{R}^3} |K(x)\phi_{u_n} u_n \, \zeta - K(x)\phi_u u\, \zeta| \, dx \\ &\le \int_{\mathbb{R}^3} K(x)|u_n - u|\phi_{u_n} |\zeta| \, dx + \int_{\mathbb{R}^3} K(x)|\phi_{u_n} - \phi_u||u|\, |\zeta| \, dx.\end{aligned} \tag{2.8}$$

Now, if $K \in L^2(\mathbb{R}^3)$, by Hölder's inequality and Sobolev embeddings it follows

$$\int_{\mathbb{R}^3} K(x)|u_n - u|\phi_{u_n}|\zeta|\,dx \le |K|_2\, C\,\|\phi_{u_n}\|_D\,\|u_n - u\|_V\,|\zeta|_V$$
$$\int_{\mathbb{R}^3} K(x)|\phi_{u_n} - \phi_u||u|\,|\zeta|\,dx \le |K|_2\, C\,\|\phi_{u_n} - \phi_u\|_D\,\|u\|_V\,|\zeta|_V.$$

Similarly, if $K \in L^\infty(\mathbb{R}^3)$ we get

$$\int_{\mathbb{R}^3} K(x)|u_n - u|\phi_{u_n}|\zeta|\,dx \le |K|_\infty\, C\,\|\phi_{u_n}\|_D\,\|u_n - u\|_V\,|\zeta|_V$$
$$\int_{\mathbb{R}^3} K(x)|\phi_{u_n} - \phi_u||u|\,|\zeta|\,dx \le |K|_\infty\, C\,\|\phi_{u_n} - \phi_u\|_D\,\|u\|_V\,|\zeta|_V.$$

As $u_n \to u$ in E_V, by the continuity of ϕ_u from E_V in $D^{1,2}(\mathbb{R}^3)$ ensured in Remark 2.2 we get $\phi_{u_n} \to \phi_u$ as $n \to +\infty$ and consequently the boundedness of ϕ_{u_n} in $D^{1,2}(\mathbb{R}^3)$; therefore, the right terms in these four inequalities above go to zero and by (2.8) the convergence in (2.7) follows.

Now, we have to prove that also $\varphi_F \in C^1(E_V, \mathbb{R})$ with

$$d\varphi_F(u)[\zeta] \;=\; \int_{\mathbb{R}^3} f(x,u)\,\zeta\,dx \qquad \text{for all } u, \zeta \in E_V. \tag{2.9}$$

Let us point out that, by (1.1) in Remark 1.4 and Hölder's inequality, we have

$$|\varphi_F(u)| \le \int_{\mathbb{R}^3} |F(x,u)|\,dx \le \sum_{i=1}^{r} \frac{1}{w_i} \int_{\mathbb{R}^3} W_i(x)|u|^{w_i}\,dx \le \sum_{i=1}^{r} \frac{1}{w_i} |W_i|_{\mu_i}\,|u|_2^{w_i}$$

where $\mu_i = \left(\frac{2}{w_i}\right)' = \frac{2}{2-w_i}$ and similarly by (f_2) we obtain

$$\int_{\mathbb{R}^3} |f(x,u)||\zeta|\,dx \le \sum_{i=1}^{r} \int_{\mathbb{R}^3} W_i(x)|u|^{w_i-1}|\zeta|\,dx \le \sum_{i=1}^{r} |W_i|_{\mu_i}\,|u|_2^{w_i-1}\,|\zeta|_2.$$

Hence, by Sobolev embeddings it follows that $\varphi_F(u) \in \mathbb{R}$ and $d\varphi_F(u)[\zeta] \in \mathbb{R}$ for all $u, \zeta \in E_V$. Moreover, standard tools imply that the Gâteaux derivative of φ_F at u is as in (2.9) and it is linear and continuous from E_V to $\mathbb{R}$.

At this point, we have to prove that $d\varphi_F$ is continuous from E_V to E'_V, i.e.

$$\|d\varphi_F(u_n) - d\varphi_F(u)\|_{E'_V} \to 0 \quad \text{if } u_n \to u \text{ in } E_V. \tag{2.10}$$

Indeed, by Hölder's inequality and Sobolev embeddings,

$$\begin{aligned} |(d\varphi_F(u_n) - d\varphi_F(u))[\zeta]| &\le \int_{\mathbb{R}^3} |f(x,u_n) - f(x,u)||\zeta|\,dx \\ &\le |f(\cdot, u_n(\cdot)) - f(\cdot, u(\cdot))|_2\,|\zeta|_2. \end{aligned}$$

Now, by (f_2) we get for a.e. $x \in \mathbb{R}^3$

$$\begin{aligned}
|f(x,u_n) - f(x,u)|^2 &\le 2\left(|f(x,u_n)|^2 + |f(x,u)|^2\right) \\
&\le 2\left(\left(\sum_{i=1}^r W_i(x)|u_n|^{(w_i-1)}\right)^2 + \left(\sum_{i=1}^r W_i(x)|u|^{(w_i-1)}\right)^2\right) \\
&\le 2\left(2\sum_{i=1}^r (W_i(x))^2|u_n|^{2(w_i-1)} + 2\sum_{i=1}^r (W_i(x))^2|u|^{2(w_i-1)}\right) \\
&\le 2^2\left(\sum_{i=1}^r (W_i(x))^2|u_n|^{2(w_i-1)} + \sum_{i=1}^r (W_i(x))^2|u|^{2(w_i-1)}\right) \\
&\le 2^2\left(\sum_{i=1}^r 2^{2(w_i-1)-1}(W_i(x))^2|u_n - u|^{2(w_i-1)}\right. \\
&\qquad \left. + \sum_{i=1}^r (2^{2(w_i-1)-1}+1)(W_i(x))^2|u|^{2(w_i-1)}\right).
\end{aligned}$$

By Fatou's lemma, it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \liminf_{n\to+\infty}\left(c\left(\sum_{i=1}^r (W_i(x))^2|u_n-u|^{2(w_i-1)} + \sum_{i=1}^r (W_i(x))^2|u|^{2(w_i-1)}\right) - |f(x,u_n)-f(x,u)|^2\right) dx \\
&\quad \le \liminf_{n\to+\infty}\int_{\mathbb{R}^3}\left(c\left(\sum_{i=1}^r (W_i(x))^2|u_n-u|^{2(w_i-1)} + \sum_{i=1}^r (W_i(x))^2|u|^{2(w_i-1)}\right)\right. \\
&\qquad\qquad \left. - |f(x,u_n)-f(x,u)|^2\right) dx. \qquad (2.11)
\end{aligned}$$

Now, we observe that, since $u_n \to u$ in E_V it is $u_n(x) \to u(x)$ a.e. $x \in \mathbb{R}^3$, therefore

$$(W_i(x))^2|u_n(x) - u(x)|^{2(w_i-1)} \to 0 \quad \text{a.e. } x \in \mathbb{R}^3 \text{ and for all } i = 1,\dots,r$$

and also by (f_1)

$$|f(x,u_n(x)) - f(x,u(x))|^2 \to 0 \quad \text{a.e. } x \in \mathbb{R}^3.$$

On the other hand, by Hölder's inequality and Sobolev embeddings we get

$$\int_{\mathbb{R}^3} (W_i(x))^2|u_n-u|^{2(w_i-1)}\,dx \le |W_i|^2_{\mu_i}\,|u_n-u|_2^{2(w_i-1)} \quad \text{for all } i = 1,\dots,r$$

and, since $u_n \to u$ in $L^2(\mathbb{R}^3)$ by continuous embeddings, also the left-hand side term goes to zero as $n \to +\infty$ for every $i = 1,\dots,r$. Consequently, (2.11) implies

$$\begin{aligned}
c\int_{\mathbb{R}^3}\sum_{i=1}^r (W_i(x))^2|u|^{2(w_i-1)}\,dx &\le c\int_{\mathbb{R}^3}\sum_{i=1}^r (W_i(x))^2|u|^{2(w_i-1)}\,dx \\
&\quad + \liminf_{n\to+\infty}\left(-\int_{\mathbb{R}^3}|f(x,u_n)-f(x,u)|^2\,dx\right)
\end{aligned}$$

from which it follows that

$$0 \le -\limsup_{n\to+\infty}\left(\int_{\mathbb{R}^3}|f(x,u_n)-f(x,u)|^2\,dx\right)$$

and therefore

$$0 \leq \liminf_{n\to+\infty} \left(\int_{\mathbb{R}^3} |f(x,u_n) - f(x,u)|^2 \, dx \right) \leq \limsup_{n\to+\infty} \left(\int_{\mathbb{R}^3} |f(x,u_n) - f(x,u)|^2 \, dx \right) \leq 0.$$

Hence,

$$|f(\cdot,u_n(\cdot)) - f(\cdot,u(\cdot))|_2 \to 0 \quad \text{as } n \to +\infty$$

and (2.10) is proved.

By exploiting the arguments carried out in [5,6], we get that the pair $(u,\phi) \in E_V \times D^{1,2}(\mathbb{R}^3)$ is a solution of problem ($\mathcal{P}_g$) (resp. ($\mathcal{P}_0$)) if and only if $u \in E_V$ is a critical point of I_g (resp. I_0) and $\phi = \phi_u$.

Finally, we prove that $d\varphi_F$ is compact from E_V to E_V'. Let $\{u_n\}$ be a sequence in E_V such that $u_n \rightharpoonup u$ in E_V. By Proposition 2.1, for all $i = 1,\dots,r$ it is $u_n \to u$ in $L^{w_i}_{W_i}(\mathbb{R}^3)$ namely

$$\int_{\mathbb{R}^3} W_i(x)|u_n - u|^{w_i} \, dx \to 0 \quad \text{as } n \to +\infty. \tag{2.12}$$

Fixed $i = 1,\dots,r$ and taken $\alpha_i = \frac{2}{w_i} \in (0,2)$, by Hölder's inequality we get

$$\begin{aligned}
&\int_{\mathbb{R}^3} (W_i(x))^2 |u_n - u|^{2(w_i-1)} \, dx \\
&= \int_{\mathbb{R}^3} (W_i(x))^{\alpha_i} (W_i(x))^{2-\alpha_i} |u_n - u|^{2(w_i-1)} \, dx \\
&\leq \left(\int_{\mathbb{R}^3} (W_i(x))^{\mu_i} \right)^{\frac{\alpha_i}{\mu_i}} \left(\int_{\mathbb{R}^3} \left((W_i(x))^{2-\alpha_i} |u_n - u|^{2(w_i-1)} \right)^{(\frac{\mu_i}{\alpha_i})'} dx \right)^{\frac{1}{(\frac{\mu_i}{\alpha_i})'}} \\
&= |W_i|^{\alpha_i}_{\mu_i} |u_n - u|^{\frac{1}{(\frac{\mu_i}{\alpha_i})'}}_{w_i,W_i},
\end{aligned}$$

hence, by (2.12) it follows that

$$\int_{\mathbb{R}^3} (W_i(x))^2 |u_n - u|^{2(w_i-1)} \, dx \to 0 \quad \text{as } n \to +\infty.$$

Then, arguing as in the proof of the continuity of $d\varphi_F$, as soon as $u_n \rightharpoonup u$ in E_V we get $f(\cdot,u_n(\cdot)) \to f(\cdot,u(\cdot))$ in $L^2(\mathbb{R}^3)$ so $d\varphi_F(u_n) \to d\varphi_F(u)$ in E_V' as $n \to +\infty$ and we conclude that $d\varphi_F(u)$ is compact from E_V to E_V'.

Finally it is standard to prove that $\varphi_g \in C^1(E_V,\mathbb{R})$ with derivative

$$d\varphi_g(u))[\zeta] = \int_{\mathbb{R}^3} g(x)\,\zeta \, dx \quad \text{for every } u,\zeta \in E_V$$

and the proof is completed. □

Now, in order to prove in next Section 3 the existence result by minimization arguments, we will exploit the following generalized version of the Weierstrass theorem.

Theorem 2.4. *Let $(X,\|\cdot\|)$ be a reflexive Banach space and $M \subseteq X$ be a weakly closed subset of X. Suppose that the functional $I : M \to \mathbb{R}$ is coercive and (sequentially) weak lower semi-continuous on M.*

Then, I is bounded from below on M and

$$\textit{there exists } u_0 \in M \textit{ such that } I(u_0) = \min_{u\in M} I(u).$$

In order to prove the weak lower semi-continuity of the energy functional I_g (resp. I_0), it will be useful to apply the following Mazur theorem.

Theorem 2.5 (Mazur). *Let X be an infinite dimensional reflexive Banach space and $I : X \to \mathbb{R}$ a continuous and convex functional on X.*

Then, I is weak lower semi-continuous on X.

We will exploit also the following convexity criterion.

Proposition 2.6 (Convexity criterion). *Let X be an infinite dimensional Banach space and $I : X \to \mathbb{R}$ a C^1 functional on X. If*

$$(dI(u) - dI(v))\,[u - v] \geq 0 \quad \textit{for every } u, v \in X$$

then I is convex on X.

In addition, in order to show the multiplicity result, we recall a suitable version stated by R. Kajikiya in [7] of the classical symmetric mountain pass theorem (see [1]).

Let X be an infinite dimensional Banach space, X' its dual space and $I : X \to \mathbb{R}$ be a C^1 functional. Let us recall that I satisfies the Palais–Smale, briefly (PS), condition if any (PS) sequence, i.e. any sequence $\{u_k\}$ in X such that $\{I(u_k)\}$ is bounded and $dI(u_k) \to 0$ in X' as $k \to +\infty$, has a convergent subsequence.

For all integer k, let

$$\Gamma_k = \{A \subset X - \{0\} \mid A \text{ closed and symmetric, } \gamma(A) \geq k\},$$

where, as usual, $\gamma(A)$ denotes the genus of the set A (for the definition and relative properties see e.g. [10]).

The following result was proved in [7, Theorem 1].

Theorem 2.7 (Kajikiya). *Let $I \in C^1(X, \mathbb{R})$ satisfying*

(A_1) *I is even, bounded from below, $I(0) = 0$ and I satisfies the (PS) condition;*

(A_2) *for every $k \in \mathbb{N}$ there exists $A_k \in \Gamma_k$ such that $\sup_{A_k} I(u) < 0$.*

Then,

(B_1) *either there exists a sequence $\{u_k\}$ such that $dI(u_k) = 0$, $I(u_k) < 0$ and $\{u_k\}$ converges to zero;*

(B_2) *or there exist two sequences $\{u_k\}$ and $\{v_k\}$ such that $dI(u_k) = 0$, $I(u_k) = 0$, $u_k \neq 0$, $\lim_k u_k = 0$, $dI(v_k) = 0$, $I(v_k) < 0$, $\lim_k I(v_k) = 0$ and $\{v_k\}$ converges to a non-zero limit.*

Remark 2.8. In any case (B_1) or (B_2), Theorem 2.7 gives the existence of a sequence $\{u_k\}$ of critical points such that $I(u_k) \leq 0$, $u_k \neq 0$, $\lim_k u_k = 0$ and, consequently, $\lim_k I(u_k) = 0$.

3 Proof of Theorem 1.1

First, let us recall the next useful results.

Lemma 3.1. *Assume (V) and (K) hold. Then,*

$$\int_{\mathbb{R}^3} K(x)\,(\phi_u u - \phi_v v)\,(u - v)\,dx \geq 0 \quad \textit{for every } u, v \in E_V.$$

Proof. Since

$$\int_{\mathbb{R}^3} K(x)\,(\phi_u u - \phi_v v)\,(u-v)\,dx = \int_{\mathbb{R}^3} K(x)\,\left(\phi_u u^2 + \phi_v v^2\right)\,dx - \int_{\mathbb{R}^3} K(x)\,(\phi_u uv + \phi_v uv)\,dx,$$

in order to get the thesis, it is sufficient to prove that

$$\int_{\mathbb{R}^3} K(x)\,(\phi_u uv + \phi_v uv)\,dx \le \int_{\mathbb{R}^3} K(x)\,\left(\phi_u u^2 + \phi_v v^2\right)\,dx \tag{3.1}$$

for every $u, v \in E_V$. By Hölder's inequality we get

$$\begin{aligned}
\int_{\mathbb{R}^3} K(x)\,(\phi_u uv + \phi_v uv)\,dx &= \int_{\mathbb{R}^3} K(x)\phi_u uv\,dx + \int_{\mathbb{R}^3} K(x)\phi_v uv\,dx \\
&= \int_{\mathbb{R}^3} \left((K(x)\phi_u)^{1/2} u\right)\left((K(x)\phi_u)^{1/2} v\right)dx \\
&\quad + \int_{\mathbb{R}^3} \left((K(x)\phi_v)^{1/2} u\right)\left((K(x)\phi_v)^{1/2} v\right)dx \\
&\le \left(\int_{\mathbb{R}^3} K(x)\phi_u u^2\,dx\right)^{1/2}\left(\int_{\mathbb{R}^3} K(x)\phi_u v^2\,dx\right)^{1/2} \\
&\quad + \left(\int_{\mathbb{R}^3} K(x)\phi_v u^2\,dx\right)^{1/2}\left(\int_{\mathbb{R}^3} K(x)\phi_v v^2 dx\right)^{1/2}.
\end{aligned} \tag{3.2}$$

Now, if we multiply first by ϕ_u then by ϕ_v the following two equations

$$-\Delta\phi_u = K(x)u^2 \quad \text{and} \quad -\Delta\phi_v = K(x)v^2,$$

by integration by parts we get

$$\int_{\mathbb{R}^3} K(x)\phi_u u^2\,dx = \int_{\mathbb{R}^3} |\nabla\phi_u|^2\,dx, \tag{3.3}$$

$$\int_{\mathbb{R}^3} K(x)\phi_u v^2\,dx = \int_{\mathbb{R}^3} \nabla\phi_v \cdot \nabla\phi_u\,dx, \tag{3.4}$$

and

$$\int_{\mathbb{R}^3} K(x)\phi_v u^2\,dx = \int_{\mathbb{R}^3} \nabla\phi_u \cdot \nabla\phi_v\,dx, \tag{3.5}$$

$$\int_{\mathbb{R}^3} K(x)\phi_v v^2\,dx = \int_{\mathbb{R}^3} |\nabla\phi_v|^2\,dx. \tag{3.6}$$

By substituting equalities (3.3)–(3.6) in the last line of (3.2) and by applying again Hölder's inequality we get

$$\begin{aligned}
\int_{\mathbb{R}^3} K(x)\,(\phi_u uv + \phi_v uv)\,dx &\le \left(\int_{\mathbb{R}^3} |\nabla\phi_u|^2\,dx\right)^{1/2}\left(\int_{\mathbb{R}^3} \nabla\phi_v\cdot\nabla\phi_u\,dx\right)^{1/2} \\
&\quad + \left(\int_{\mathbb{R}^3} \nabla\phi_u\cdot\nabla\phi_v\,dx\right)^{1/2}\left(\int_{\mathbb{R}^3} |\nabla\phi_v|^2\,dx\right)^{1/2} \\
&= \left(\int_{\mathbb{R}^3} \nabla\phi_u\cdot\nabla\phi_v\,dx\right)^{1/2}\left(\|\phi_u\|_D + \|\phi_v\|_D\right) \\
&\le \left(\int_{\mathbb{R}^3} |\nabla\phi_u|^2\,dx\right)^{1/4}\left(\int_{\mathbb{R}^3} |\nabla\phi_v|^2\,dx\right)^{1/4}\left(\|\phi_u\|_D + \|\phi_v\|_D\right) \\
&= \|\phi_u\|_D^{1/2}\,\|\phi_v\|_D^{1/2}\,\left(\|\phi_u\|_D + \|\phi_v\|_D\right).
\end{aligned} \tag{3.7}$$

Recall that the following inequality holds

$$(xy)^{1/2}(x+y) \le x^2 + y^2 \quad \text{for every } x, y \ge 0.$$

By applying it to the last line of (3.7) and by exploiting equalities (3.3) and (3.6) we obtain

$$\begin{aligned}\int_{\mathbb{R}^3} K(x)\,(\phi_u uv + \phi_v uv)\,dx &\le \|\phi_u\|_D^2 + \|\phi_v\|_D^2 \\ &= \int_{\mathbb{R}^3} K(x)\phi_u u^2\,dx + \int_{\mathbb{R}^3} K(x)\phi_v v^2\,dx \\ &= \int_{\mathbb{R}^3} K(x)\,(\phi_u u^2 + \phi_v v^2)\,dx,\end{aligned}$$

thus we get (3.1) and this completes the proof. □

Thanks to Lemma 3.1, we can prove the next proposition.

Proposition 3.2. *Suppose that* (V) *and* (K) *are satisfied. Then, the functional* $\varphi_K \in C^1(E_V, \mathbb{R})$ *defined by*

$$\varphi_K(u) = \int_{\mathbb{R}^3} K(x)\phi_u u^2\,dx \quad \text{for every } u \in E_V,$$

is convex on E_V.

Proof. By Proposition 2.3 we already know $\varphi_K \in C^1(E_V, \mathbb{R})$ and

$$(d\varphi_K(u) - d\varphi_K(v))\,[u-v] = \int_{\mathbb{R}^3} K(x)\,(\phi_u u - \phi_v v)\,(u-v)\,dx$$

for any $u, v \in E_V$. Then, by Lemma 3.1 we can apply Proposition 2.6 to the functional $I = \varphi_K$ and to the Banach space $X = E_V$ and we obtain the thesis. □

Consequently, we get the following result.

Proposition 3.3. *Under assumptions* (V) *and* (K)*, the* C^1 *functional* φ_K *is weak lower semicontinuous on* E_V.

Proof. Since $\varphi_K \in C^1(E_V, \mathbb{R})$ is convex by Proposition 3.2, the thesis is a direct consequence of Theorem 2.5. □

Proof of Theorem 1.1.
(i) First, let us consider the nonhomogeneous case $g(x) \not\equiv 0$. Observe that, since $-\Delta\phi_u = K(x)u^2$, by multiplying by ϕ_u and integrating by parts we get $\int_{\mathbb{R}^3} K(x)\phi_u u^2\,dx = \|\phi_u\|_D^2 \ge 0$. Therefore, from (1.1), Hölder's inequality and Sobolev embeddings, we obtain

$$\begin{aligned}I_g(u) &= \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + V(x)|u|^2\right)dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2\,dx \\ &\quad - \int_{\mathbb{R}^3} F(x,u)\,dx - \int_{\mathbb{R}^3} g(x)u\,dx \\ &\ge \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + V(x)|u|^2\right)dx - \int_{\mathbb{R}^3} F(x,u)\,dx - \int_{\mathbb{R}^3} g(x)u\,dx \\ &\ge \frac{1}{2}\|u\|_V^2 - \sum_{i=1}^{r}\frac{1}{w_i}\int_{\mathbb{R}^3} W_i(x)|u|^{w_i}\,dx - |g|_2|u|_2 \\ &\ge \frac{1}{2}\|u\|_V^2 - \sum_{i=1}^{r}\frac{1}{w_i}|W_i|_{\mu_i}\,|u|_2^{w_i} - c_2|g|_2\|u\|_V \\ &\ge \frac{1}{2}\|u\|_V^2 - c\sum_{i=1}^{r}|W_i|_{\mu_i}\,\|u\|_V^{w_i} - c_2|g|_2\|u\|_V.\end{aligned} \tag{3.8}$$

Then, since $w_i \in (1,2)$ for any $i \in \{1,\dots,r\}$, it follows that I_g is coercive and bounded from below on the reflexive Banach space E_V.

Moreover, the functional I_g is weak lower semicontinuous on E_V. In order to show it, it is useful to write it again as $I_g = \varphi_V + \frac{1}{4}\varphi_K - \varphi_F - \varphi_g$ by using the notations introduced in Proposition 2.3. Clearly, φ_V is weak lower semicontinuous by the norm properties while φ_K is weak lower semicontinuous on E_V by Proposition 3.3. In addition, φ_F is weak continuous as it is of class C^1 on E_V and its derivative $d\varphi_F$ is compact by Proposition 2.3. Moreover, φ_g is linear continuous then it is weak continuous on E_V.

Consequently, by the generalized Weierstrass theorem stated in Theorem 2.4 there exists $\overline{u} \in E_V$ such that $I_g(\overline{u}) = \min_{u\in E_V} I_g(u)$. Hence, $\overline{u}$ is a critical point of I_g and, by applying Proposition 2.3 we get $\overline{u}$ is a solution of problem $(\mathcal{S}_g)$ and then $(\overline{u}, \phi_{\overline{u}})$ is a solution to $(\mathcal{P}_g)$.

Now, since $g(x) \not\equiv 0$, equation $(\mathcal{S}_g)$ does not admit the trivial solution. Therefore, $\overline{u}$ is a not trivial solution to $(\mathcal{S}_g)$ and we obtain that $(\overline{u}, \phi_{\overline{u}})$ is a not trivial solution to $(\mathcal{P}_g)$.

(ii) Now we consider the homogeneous case $g(x) \equiv 0$. By (f_2), equation $(\mathcal{S}_0)$ admits always the trivial solution $u = 0$ with $I_0(0) = 0$. Clearly, the generalized Weierstrass theorem also applies to I_0 by adapting previous arguments with $g(x) \equiv 0$. In any case, since we assume also (f_3) holds, the solution $\overline{u}$ to $(\mathcal{S}_0)$ is not trivial; indeed, recall that $\phi_{tu} = t^2\phi_u$ for every $t > 0$ and any $u \in E_V$ and let us fix $u_1 \in E_V \cap C_c(\mathbb{R}^3)$ with $u_1 \neq 0$ and $\mathrm{supp}(u_1) \subseteq \Omega$; by (f_3) and $1 < w_{r+1} < 2$ we get

$$\begin{aligned} I_0(\varepsilon u_1) &= \frac{\varepsilon^2}{2}\|u_1\|_V^2 + \frac{1}{4}\int_\Omega K(x)\phi_{\varepsilon u_1}(\varepsilon u_1)^2\,dx - \int_\Omega F(x,\varepsilon u_1)\,dx \\ &\le \frac{\varepsilon^2}{2}\|u_1\|_V^2 + \frac{\varepsilon^4}{4}\int_\Omega K(x)\phi_{u_1}u_1^2\,dx - \eta\varepsilon^{w_{r+1}}|u_1|_{w_{r+1}}^{w_{r+1}} \\ &< 0 = I_0(0) \end{aligned}$$

for $\varepsilon > 0$ small enough. Therefore, system $(\mathcal{P}_0)$ has a non-trivial weak solution $(\overline{u}, \phi_{\overline{u}})$. □

Remark 3.4. Here above we have exploited the weak lower semicontinuity of φ_K in particular and then of the functional I_g (resp. I_0). Really, the existence result can be also found by applying [10, Theorem 2.7] (see also [14, Corollary 2.5]) since $I_g \in C^1(E_V,\mathbb{R})$ (resp. $I_0 \in C^1(E_V,\mathbb{R})$) is bounded from below on E_V and, as proved in the next Proposition 4.1, satisfies (PS) condition by neglecting the presence of $d\varphi_K$ thanks to Lemma 3.1.

4 Proof of Theorem 1.2

Let us take in mind from now on we treat only the homogeneous case $g(x) \equiv 0$.

As proved in Proposition 2.3, compact embeddings stated in Proposition 2.1 allow us to recover the compactness of $d\varphi_F$.

On the contrary, Proposition 2.1 which is weaker with respect compactness results $E_V \hookrightarrow\hookrightarrow L^s(\mathbb{R}^3)$ for all $2 \le s < 6$ obtained by assumptions (V')–(V'') does not enable us to show that $d\varphi_K$ is compact. Fortunately, this problem is overcome thanks to Lemma 3.1 and we can state the following proposition.

Proposition 4.1. *Suppose (V), (K), (f_1) and (f_2) hold. Then, the functional I_0 satisfies (PS) condition.*

Proof. Let $\{u_k\} \subset E_V$ be a (PS) sequence of I_0, namely $\{I_0(u_k)\}$ is bounded and $dI_0(u_k) \to 0$ in E'_V as $k \to +\infty$. As observed in the proof of Theorem 1.1, by (3.8) we get that I_0 is coercive on E_V and this implies that $\{u_k\}$ is bounded in E_V. Thus, up to subsequence, there exists $u \in E_V$ such that $u_k \rightharpoonup u$ as $k \to +\infty$.

By (2.5), Lemma 3.1 and Hölder's inequality we get

$$\begin{aligned}\|u_k - u\|_V^2 = & \ (dI_0(u_k) - dI_0(u))\,[u_k - u] - \int_{\mathbb{R}^3} K(x)\,(\phi_{u_k}u_k - \phi_u u)\,(u_k - u)\,dx \\ & + \int_{\mathbb{R}^3} (f(x,u_k) - f(x,u))\,(u_k - u)\,dx \\ \le & \ \|dI_0(u_k)\|_{E'_V}\,\|u_k - u\|_V - dI_0(u)[u_k - u] \\ & + \left(\int_{\mathbb{R}^3} |f(x,u_k) - f(x,u)|^2\,dx\right)^{1/2} |u_k - u|_2.\end{aligned}$$

The first term in the last line goes to zero since $dI_0(u_k) \to 0$ in E'_V; the second one also tends to zero because $dI_0(u)$ is linear and continuous from E_V to E'_V and $u_k \rightharpoonup u$ as $k \to +\infty$. The same occurs for the third term since, by Proposition 2.3, we have that the function $u \to f(\cdot, u(\cdot))$ is compact from E_V to E'_V.

Then, we can conclude that $u_k \to u$ in E_V and (PS) condition is proved. □

Now, we prove the following result which allows us to show (A_2) in Theorem 2.7.

Proposition 4.2. *Assume that (V), (K), (f_1), (f_2) and (f_3) hold. Then,*

$$\text{for every } k \in \mathbb{N} \text{ there exists } A_k \in \Gamma_k \text{ such that } \sup_{A_k} I_0 < 0.$$

Proof. Fixed $k \in \mathbb{N}$, let us consider k disjoint open sets $\Omega_1, \dots, \Omega_k$ such that $\bigcup_{j=1}^k \Omega_j \subset \Omega$ with Ω as in assumption (f_3). For every $\varepsilon > 0$ and for every $j = 1, \dots, k$ there exist a closed set H_j and an open set G_j such that $H_j \subset \Omega_j \subset G_j$, $\text{meas}(G_j \setminus \Omega_j) < \varepsilon$ and $\text{meas}(\Omega_j \setminus H_j) < \varepsilon$. Without loss of generality, we can assume $\bigcap_{j=1}^k G_j = \emptyset$. Moreover, for every G_j there exists $\varphi_j \in C_0^\infty(G_j, \mathbb{R})$ such that $\varphi_j|_{H_j} = 1$ and $0 \le \varphi_j \le 1$.

Now, let us consider $\nu_j = \frac{\varphi_j}{\|\varphi_j\|_V}$ and denote by ν_j again its null extension on $\mathbb{R}^N \setminus G_j$. Clearly, $\nu_1, \dots, \nu_k$ are linearly independent functions in E_V.

Denoted by $E_{V,k}$ the k-dimensional vector space generated by $\nu_1, \dots, \nu_k$, for every $u \in E_{V,k}$ we get $u = \sum_{j=1}^k \lambda_j \nu_j$ with $\lambda_j \in \mathbb{R}$, $j = 1, \dots, k$.

Therefore, for all $u \in E_{V,k}$ it results

$$(a)\quad |u|_{w_{r+1}}^{w_{r+1}} \le \sum_{j=1}^k |\lambda_j|^{w_{r+1}} \int_{\Omega_j} |\nu_j|^{w_{r+1}}\,dx + \sum_{j=1}^k \frac{|\lambda_j|^{w_{r+1}}}{\|\varphi_j\|_V^{w_{r+1}}}\,\varepsilon.$$

Indeed,

$$\begin{aligned}|u|_{w_{r+1}}^{w_{r+1}} &= \int_{\mathbb{R}^3} |u|^{w_{r+1}}\,dx \\ &= \sum_{j=1}^k |\lambda_j|^{w_{r+1}} \int_{\Omega_j} |\nu_j|^{w_{r+1}}\,dx + \sum_{j=1}^k |\lambda_j|^{w_{r+1}} \int_{G_j\setminus\Omega_j} |\nu_j|^{w_{r+1}}\,dx \\ &= \sum_{j=1}^k |\lambda_j|^{w_{r+1}} \int_{\Omega_j} |\nu_j|^{w_{r+1}}\,dx + \sum_{j=1}^k |\lambda_j|^{w_{r+1}} \int_{G_j\setminus\Omega_j} \frac{\varphi_j^{w_{r+1}}}{\|\varphi_j\|_V^{w_{r+1}}}\,dx \\ &\le \sum_{j=1}^k |\lambda_j|^{w_{r+1}} \int_{\Omega_j} |\nu_j|^{w_{r+1}}\,dx + \sum_{j=1}^k \frac{|\lambda_j|^{w_{r+1}}}{\|\varphi_j\|_V^{w_{r+1}}}\,\text{meas}(G_j \setminus \Omega_j)\end{aligned}$$

and, since $\text{meas}(G_j \setminus \Omega_j) < \varepsilon$, (a) is proved.

(b) $\|u\|_V^2 = \sum_{j=1}^{k} |\lambda_j|^2$.

Since $\|\nu_j\|_V = 1$ for every $j = 1, \dots, k$ by the definition of ν_j we get

$$\begin{aligned}
\|u\|_V^2 &= \int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)|u|^2\right) dx \\
&= \sum_{j=1}^{k} |\lambda_j|^2 \int_{G_j} \left(|\nabla \nu_j|^2 + V(x)|\nu_j|^2\right) dx \\
&= \sum_{j=1}^{k} |\lambda_j|^2 \|\nu_j\|_V^2 = \sum_{j=1}^{k} |\lambda_j|^2.
\end{aligned}$$

(c) There exists $\widetilde{c}_k > 0$ such that $\widetilde{c}_k \|u\|_V \leq |u|_{w_{r+1}}$.

(c) follows since $E_{V,k}$ has finite dimension and then all norms are equivalent in $E_{V,k}$.

From (1.1) in Remark 1.4, we get

$$F(x, \lambda_j \nu_j(x)) \geq -\sum_{i=1}^{r} \frac{W_i(x)}{w_i} |\lambda_j \nu_j(x)|^{w_i} \quad \text{for a.e } x \in \mathbb{R}^3;$$

therefore, it is

$$\begin{aligned}
\sum_{j=1}^{k} \int_{G_j \setminus \Omega_j} F(x, \lambda_j \nu_j)\, dx &\geq -\sum_{j=1}^{k} \int_{G_j \setminus \Omega_j} \sum_{i=1}^{r} \frac{W_i(x)}{w_i} |\lambda_j \nu_j|^{w_i}\, dx \\
&= -\sum_{j=1}^{k} \sum_{i=1}^{r} \frac{|\lambda_j|^{w_i}}{w_i} \int_{G_j \setminus \Omega_j} W_i(x) |\nu_j|^{w_i}\, dx
\end{aligned} \tag{4.1}$$

and, by Hölder's inequality and $\text{meas}(G_j \setminus \Omega_j) < \varepsilon$, we obtain

$$\begin{aligned}
\sum_{j=1}^{k} \int_{G_j \setminus \Omega_j} F(x, \lambda_j \nu_j)\, dx &\geq -\sum_{j=1}^{k} \sum_{i=1}^{r} \frac{|\lambda_j|^{w_i}}{w_i} |W_i|_{\mu_i} \left(\int_{G_j \setminus \Omega_j} |\nu_j|^2\, dx \right)^{\frac{w_i}{2}} \\
&\geq -\sum_{j=1}^{k} \sum_{i=1}^{r} \frac{|\lambda_j|^{w_i}}{w_i} |W_i|_{\mu_i} \frac{1}{\|\varphi_j\|_V^{w_i}} \left(\text{meas}(G_j \setminus \Omega_j)\right)^{\frac{w_i}{2}} \\
&\geq -\sum_{j=1}^{k} \left(\sum_{i=1}^{r} \frac{|\lambda_j|^{w_i}}{w_i} |W_i|_{\mu_i} \frac{1}{\|\varphi_j\|_V^{w_i}} \varepsilon^{\frac{w_i}{2}} \right).
\end{aligned} \tag{4.2}$$

At this point, by (b) taken any $u \in E_{V,k}$ with $\|u\|_V^2 = \sum_{j=1}^{k} |\lambda_j|^2 = r_k^2$, we can choose r_k small enough such that, by exploiting the equivalence of the norms $|\cdot|_\infty$ and $\|\cdot\|_V$ in $E_{V,k}$, it is

$$|u|_\infty^2 \leq \bar{c}_k \|u\|_V^2 = \bar{c}_k r_k^2 < \delta^2$$

with δ the constant which appears in (f_3). Moreover, by Remark 2.2 and continuous embeddings we have

$$\frac{1}{4} \int_{\mathbb{R}^3} K(x) \phi_u u^2\, dx \leq C \|u\|_V^4 = C r_k^4.$$

Therefore, since $w_{r+1} < 2 < 4$, passing to a smaller r_k we suppose that

$$r_k^2 + C\, r_k^4 - \eta\, (\widetilde{c}_k)^{w_{r+1}}\, r_k^{w_{r+1}} < 0, \tag{4.3}$$

with η and $\widetilde{c}_k$ as in (f_3) and (c). So, since $|\lambda_j \nu_j|_\infty \leq |u|_\infty < \delta$ for every $j = 1, \ldots, k$, by (f_3) we get

$$F(x, \lambda_j \nu_j(x)) \geq \eta\, |\lambda_j \nu_j(x)|^{w_{r+1}} \quad \text{for a.e. } x \in \Omega_j, \text{ for any } j = 1, \ldots, k. \tag{4.4}$$

By (4.1), (4.2) and (4.4) it is

$$\begin{aligned}
I_0(u) &= \frac{1}{2}\|u\|_V^2 + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u\, u^2\, dx - \int_{\mathbb{R}^3} F(x,u)\, dx \\
&= \frac{1}{2}\|u\|_V^2 + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u\, u^2\, dx - \sum_{j=1}^{k}\int_{G_j} F(x, \lambda_j \nu_j)\, dx \\
&= \frac{1}{2}\|u\|_V^2 + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u\, u^2\, dx - \sum_{j=1}^{k}\left(\int_{G_j\setminus\Omega_j} F(x, \lambda_j \nu_j)\, dx + \int_{\Omega_j} F(x, \lambda_j \nu_j)\, dx\right) \\
&\leq \frac{1}{2}\|u\|_V^2 + C\,\|u\|_V^4 + \sum_{j=1}^{k}\left(\sum_{i=1}^{r}\frac{|\lambda_j|^{w_i}}{w_i}|W_i|_{\mu_i}\frac{1}{\|\varphi_j\|_V^{w_i}}\varepsilon^{\frac{w_i}{2}}\right) - \eta\sum_{j=1}^{k}|\lambda_j|^{w_{r+1}}\int_{\Omega_j}|\nu_j|^{w_{r+1}}\, dx
\end{aligned}$$

which joint to (a) and (c) gives

$$\begin{aligned}
I_0(u) &\leq \frac{1}{2}\|u\|_V^2 + C\,\|u\|_V^4 + \sum_{j=1}^{k}\left(\sum_{i=1}^{r}\frac{|\lambda_j|^{w_i}}{w_i}|W_i|_{\mu_i}\frac{1}{\|\varphi_j\|_V^{w_i}}\varepsilon^{\frac{w_i}{2}}\right) \\
&\quad - \eta\left(|u|_{w_{r+1}}^{w_{r+1}} - \sum_{j=1}^{k}\frac{|\lambda_j|^{w_{r+1}}}{\|\varphi_j\|_V^{w_{r+1}}}\,\varepsilon\right) \\
&\leq \frac{1}{2}r_k^2 + C\, r_k^4 - \eta(\widetilde{c}_k\, r_k)^{w_{r+1}} + \sum_{j=1}^{k}\left(\sum_{i=1}^{r}\frac{|\lambda_j|^{w_i}}{w_i}|W_i|_{\mu_i}\frac{1}{\|\varphi_j\|_V^{w_i}}\varepsilon^{\frac{w_i}{2}}\right) + \eta\varepsilon\sum_{j=1}^{k}\frac{|\lambda_j|^{w_{r+1}}}{\|\varphi_j\|_V^{w_{r+1}}}.
\end{aligned} \tag{4.5}$$

Fix $\varepsilon > 0$ small enough such that

$$\sum_{j=1}^{k}\left(\sum_{i=1}^{r}\frac{|\lambda_j|^{w_i}}{w_i}|W_i|_{\mu_i}\frac{1}{\|\varphi_j\|_V^{w_i}}\varepsilon^{\frac{w_i}{2}}\right) + \eta\varepsilon\sum_{j=1}^{k}\frac{|\lambda_j|^{w_{r+1}}}{\|\varphi_j\|_V^{w_{r+1}}} < \frac{1}{2}\, r_k^2,$$

then by (4.3) and (4.5), $I_0(u) < 0$ holds for every $u \in E_k \cap S_{r_k}$, $S_{r_k} = \{u \in E_V : \|u\|_\rho = r_k\}$. Consequently, for every $k \in \mathbb{N}$ it results

$$\sup_{u\in E_k\cap S_{r_k}} I_0(u) < 0,$$

then, by well known properties of the genus, the thesis follows with $A_k = E_k \cap S_{r_k}$. □

Now, we are ready to prove the multiplicity result stated in Theorem 1.2.

Proof of Theorem 1.2. Since (f_4) holds, the functional I_0 is even. As proved in the proof of Theorem 1.1, thanks to (3.8) we get I_0 is bounded from below on E_V; moreover, from (f_2) it is $I_0(0) = 0$. By Proposition 4.1, I_0 satisfies (PS) condition. Hence, I_0 satisfies assumption (A_1) in Theorem 2.7.

Furthermore, (A_2) holds by Proposition 4.2. By Theorem 2.7 (see also Remark 2.8), there exists a sequence $\{\overline{u}_k\}$ in E_V of critical points of I_0 such that $\overline{u}_k \neq 0$, $\lim_k \overline{u}_k = 0$ in E_V and $\lim_k I_0(\overline{u}_k) = 0$. Therefore, by Proposition 2.3, $\{\overline{u}_k\}$ is a sequence of non-trivial solutions to $(\mathcal{S}_0)$ such that $\overline{u}_k \to 0$ in E_V and $I_0(\overline{u}_k) \to 0$ as $k \to +\infty$; hence, by the continuity of ϕ_u and J_0 we obtain that $\{(\overline{u}_k, \phi_{\overline{u}_k})\}$ is a sequence of solutions to system $(\mathcal{P}_0)$ with $\overline{u}_k \to 0$ in E_V, $\phi_{\overline{u}_k} \to 0$ in $D^{1,2}(\mathbb{R}^3)$ and $J_0(\overline{u}_k, \phi_{\overline{u}_k}) = I_0(\overline{u}_k) \to 0$ as $k \to +\infty$. □

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