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New solvable class of product-type systems of difference equations on the complex domain and a new method for proving the solvability

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Abstract. This paper continues the investigation of solvability of product-type systems of difference equations, by studying the following system with two variables:

$$z_n = \alpha z_{n-1}^a w_{n-2}^b, \quad w_n = \beta w_{n-3}^c z_{n-2}^d, \quad n \in \mathbb{N}_0,$$

where $a, b, c, d \in \mathbb{Z}$, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, $w_{-3}, w_{-2}, w_{-1}, z_{-2}, z_{-1} \in \mathbb{C} \setminus \{0\}$. It is shown that there are some important cases such that the system cannot be solved by using our previous methods. Hence, we also present a method different from the previous ones by which the solvability of the system is shown also in the cases.

Keywords: system of difference equations, product-type system, solvable in closed form.

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1 Introduction

Various classes of nonlinear difference equations and systems have attracted interest of numerous mathematicians (see, for example, [1–4, 8–43] and the references therein). After some starting results on the long-term behavior of solutions to concrete systems, which were usually natural extensions of some scalar equations and whose study has been essentially initiated by Papaschinopoulos and Schinas [10–12], several authors have continued their investigation in a few different directions (see, for example, [3, 4, 8, 9, 13, 14, 16, 17, 21–28, 30–43]). One of the directions is the classical problem of solving the equations and system and their applications [1, 5–7], a topic which has regained some popularity recently (see, for example, [2, 3, 15, 18, 20–33, 35–43] and the references therein). The main ideas in our paper [18] have

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attracted some interest and have been used frequently in the last decade (see, for example, [2, 3, 15, 20, 22, 24, 25, 29, 33, 35–39] and numerous related references therein). An interesting system, which has been treated in another way and is motivated by the system in [23], can be found in our recent paper [31].

Having studied the equations/systems which are obtained from the product-type ones by acting on their right-hand sides by some standard operators (usually the translation or max-type ones, see, for example, [19] and [34] and the references therein), we have turned to the product-type systems, but on the complex domain. The case of positive initial values is essentially known since the corresponding theory is based on the theory of linear difference equations with constant coefficients which is one of the basic and classical ones [1, 5–7]. An interesting max-type system of difference equations has been reduced to a product-type system of this type in [21] and solved essentially by using the theory. However, there are several problems in dealing with difference equations and systems on the complex domain. One of them is that many basic complex functions are multi-valued. Hence, the product-type systems whose initial values are real or complex are of some interest. Another problem is that the transformation methods similar to those ones in [3, 15, 18, 20, 22, 24, 25, 29, 33, 35–39], cannot be easily used for the case of product-type systems on the complex domain, unlike the case of positive initial values. Recently, we have noticed that some product-type systems are solvable on the complex domain (see [28, 32, 40]). It can be immediately noticed that the systems in these three papers do not have arbitrary multipliers. Soon after that it was shown that two multipliers can be added to the system in [32] so that such obtained system is also solvable [41]. The motivation for adding multipliers stemmed from previously published paper [26]. Three other related product-type systems have been studied recently in [30], [42] and [43]. Product-type equations have appeared recently also in [29], where can be found several methods and tricks for solving difference equations. What is quite interesting in the research of product-type systems of difference equations of the form in [26, 30, 41–43] on the complex domain, is the fact that there are just a few cases of solvable ones in closed form, which is connected to the impossibility of solving polynomial equations which are of degree five or more. Of course, there are many product-type systems which are *theoretically* solvable. However, for these systems we only know the form of the formulas for their general solutions, but do not have explicit (closed form) formulas for them. Thus, the problem of finding all *practically* solvable product-type systems is important.

If k and l are two integers such that $k \le l$, then $j = \overline{k,l}$ denotes the set of all $j \in \mathbb{Z}$ such that $k \le j \le l$. Also, as usual, we regard that $\sum_{i=s}^{t} \zeta_i = 0$, if $t < s$, and where ζ_i are some real or complex numbers.

Continuing our previous investigations on product-type systems of difference equations, especially the ones in [26, 30, 41–43], here we will consider the following system:

$$z_n = \alpha z_{n-1}^a w_{n-2}^b, \quad w_n = \beta w_{n-3}^c z_{n-2}^d, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where parameters a, b, c, d are integers, while parameters α and β and initial values w_{-3}, w_{-2}, w_{-1}, z_{-2}, z_{-1} are complex numbers.

If some of the initial values w_{-i}, $i = \overline{1,3}$, z_{-j}, $j = 1, 2$, are equal to zero and $\min\{a, b, c, d\} < 0$, then such solutions are not defined. Hence, of a much greater interest is the case when for the initial values of system (1.1) the following relations hold:

$$w_{-i} \neq 0, \quad i = \overline{1,3} \quad \text{and} \quad z_{-j} \neq 0, \quad j = 1, 2. \tag{1.2}$$

Therefore, the case will be considered in this paper. If $\alpha = 0$ or $\beta = 0$, then in the case

$\min\{a, b, c, d\} < 0$, appears the same problem. Hence, we will also assume that $\alpha \neq 0$ and $\beta \neq 0$.

Our aim is to prove the (practical) solvability of system (1.1) under above posed conditions. A bit surprisingly, we show that the methods applied in papers [26,30,41–43] cannot be used in dealing with the problem of solvability of the system in all the cases. For this reason we devise another method which will help in solving the problem.

2 Main results

The main results concerning the solvability of system (1.1) are presented in this section. Five theorems are proved. Some results give closed form formulas for solutions to system (1.1), whereas the others present methods for getting the corresponding closed form formulas. The first theorem deals with the case $c = bd = 0$, the second with the case $b = 0 \neq c$, the third with the case $d = 0 \neq c$, the fourth with the case $ac \neq bd \neq 0$, and the fifth with the case $ac = bd \neq 0$.

Before we state the results, note that

$$\begin{aligned} z_0 &= \alpha z_{-1}^a w_{-2}^b, & w_0 &= \beta w_{-3}^c z_{-2}^d, \\ z_1 &= \alpha^{1+a} z_{-1}^{a^2} w_{-2}^{ab} w_{-1}^b, & w_1 &= \beta w_{-2}^c z_{-1}^d, \\ z_2 &= \alpha^{1+a+a^2} \beta^b z_{-2}^{bd} z_{-1}^{a^3} w_{-3}^{bc} w_{-2}^{a^2 b} w_{-1}^{ab}, & w_2 &= \alpha^d \beta z_{-1}^{ad} w_{-2}^{bd} w_{-1}^c, \end{aligned} \tag{2.1}$$

and that from $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, $w_{-3}, w_{-2}, w_{-1}, z_{-2}, z_{-1} \in \mathbb{C} \setminus \{0\}$ and the equations in (1.1) we get

$$z_n \neq 0, \ n \geq -2, \quad \text{and} \quad w_n \neq 0, \ n \geq -3. \tag{2.2}$$

Theorem 2.1. *Assume that $a, b, d \in \mathbb{Z}$, $c = 0$, $bd = 0$, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ and $w_{-3}, w_{-2}, w_{-1}, z_{-2}, z_{-1} \in \mathbb{C} \setminus \{0\}$. Then the following statements hold.*

(a) If $a \neq 1$, then the general solution to system (1.1) is given by the following formulas

$$z_n = \alpha^{\frac{1-a^{n+1}}{1-a}} \beta^{b\frac{1-a^{n-1}}{1-a}} z_{-1}^{a^{n+1}} w_{-2}^{ba^n} w_{-1}^{ba^{n-1}}, \tag{2.3}$$

and

$$w_n = \alpha^{d\frac{1-a^{n-1}}{1-a}} \beta z_{-1}^{da^{n-1}}, \tag{2.4}$$

for $n \geq 2$.

(b) If $a = 1$, then the general solution to system (1.1) is given by the following formulas

$$z_n = \alpha^{n+1} \beta^{b(n-1)} z_{-1} w_{-2}^b w_{-1}^b, \tag{2.5}$$

and

$$w_n = \alpha^{d(n-1)} \beta z_{-1}^d, \tag{2.6}$$

for $n \in \mathbb{N}$.

Proof. First note that in this case (1.1) is

$$z_n = \alpha z_{n-1}^a w_{n-2}^b, \quad w_n = \beta z_{n-2}^d, \tag{2.7}$$

for $n \in \mathbb{N}_0$.

Using (2.7) and condition $bd = 0$, we get

$$z_n = \alpha\beta^b z_{n-1}^a,$$

for $n \geq 2$.

Thus

$$z_n = (\alpha\beta^b)^{\sum_{j=0}^{n-2} a^j} z_1^{a^{n-1}}, \tag{2.8}$$

for $n \geq 2$, which along with (2.1) yields

$$\begin{aligned} z_n &= (\alpha\beta^b)^{\sum_{j=0}^{n-2} a^j} (\alpha^{1+a} z_{-1}^{a^2} w_{-2}^{ab} w_{-1}^b)^{a^{n-1}} \\ &= \alpha^{\sum_{j=0}^{n} a^j} \beta^{b\sum_{j=0}^{n-2} a^j} z_{-1}^{a^{n+1}} w_{-2}^{ba^n} w_{-1}^{ba^{n-1}}, \end{aligned} \tag{2.9}$$

for $n \geq 2$. Note that (2.9) holds for $n = 1$ and $a \neq 0$, too.

From (2.9) it easily follows that (2.3) holds if $a \neq 1$, whereas (2.5) follows immediately by taking $a = 1$.

From (2.7), (2.9) and $bd = 0$, it follows that

$$\begin{aligned} w_n &= \alpha^{d\sum_{j=0}^{n-2} a^j} \beta^{1+bd\sum_{j=0}^{n-4} a^j} z_{-1}^{da^{n-1}} w_{-2}^{bda^{n-2}} w_{-1}^{bda^{n-3}} \\ &= \alpha^{d\sum_{j=0}^{n-2} a^j} \beta z_{-1}^{da^{n-1}}, \end{aligned} \tag{2.10}$$

for $n \geq 4$. In fact, (2.10) holds for $n \geq 2$, and even for $n = 1$, if $a \neq 0$ (see (2.1)).

From (2.10) is easily get that (2.4) holds if $a \neq 1$, whereas (2.6) immediately follows by taking $a = 1$ in (2.10). □

Theorem 2.2. *Assume that $a, c, d \in \mathbb{Z}$, $b = 0 \neq c$, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ and $w_{-3}, w_{-2}, w_{-1}, z_{-2}, z_{-1} \in \mathbb{C} \setminus \{0\}$. Then system (1.1) is solvable in closed form.*

Proof. First note that in this case (1.1) is

$$z_n = \alpha z_{n-1}^a, \quad w_n = \beta w_{n-3}^c z_{n-2}^d, \tag{2.11}$$

for $n \in \mathbb{N}_0$.

From the first equation in (2.11), we get

$$z_n = \alpha^{\sum_{j=0}^{n} a^j} z_{-1}^{a^{n+1}}, \tag{2.12}$$

for $n \in \mathbb{N}_0$, from which we have that

$$z_n = \alpha^{\frac{1-a^{n+1}}{1-a}} z_{-1}^{a^{n+1}}, \tag{2.13}$$

if $a \neq 1$, and

$$z_n = \alpha^{n+1} z_{-1}, \tag{2.14}$$

if $a = 1$. Note that formula (2.12) also holds for $n = -1$ if $a \neq 0$.

Employing (2.12) in the second equation in (2.11), we get

$$w_n = \beta\alpha^{d\sum_{j=0}^{n-2} a^j} z_{-1}^{da^{n-1}} w_{n-3}^{c}, \tag{2.15}$$

for $n \geq 2$, and consequently

$$w_{3n+i} = \beta\alpha^{d\sum_{j=0}^{3n+i-2} a^j} z_{-1}^{da^{3n+i-1}} w_{3(n-1)+i}^{c}, \tag{2.16}$$

for $n \in \mathbb{N}$ and $i = -1, 0, 1$.

From (2.16) we get

$$\begin{aligned} w_{3n+i} &= \beta\alpha^{d\sum_{j=0}^{3n+i-2} a^j} z_{-1}^{da^{3n+i-1}} \left(\beta\alpha^{d\sum_{j=0}^{3n+i-5} a^j} z_{-1}^{da^{3n+i-4}} w_{3(n-2)+i}^{c}\right)^c \\ &= \beta^{1+c}\alpha^{d\sum_{j=0}^{3n+i-2} a^j + dc\sum_{j=0}^{3n+i-5} a^j} z_{-1}^{da^{3n+i-1}+dca^{3n+i-4}} w_{3(n-2)+i}^{c^2}, \end{aligned} \tag{2.17}$$

for $n \geq 2$ and $i = -1, 0, 1$.

An inductive argument along with (2.16) shows that

$$w_{3n+i} = \beta^{\sum_{j=0}^{k-1} c^j} \alpha^{d\sum_{j=0}^{k-1} c^j \sum_{j=0}^{3(n-j)+i-2} a^j} z_{-1}^{d\sum_{j=0}^{k-1} c^j a^{3(n-j)+i-1}} w_{3(n-k)+i}^{c^k}, \tag{2.18}$$

for every $n \geq k$ and $i = -1, 0, 1$.

By taking $n = k$ in (2.18) we get

$$w_{3n+i} = \beta^{\sum_{j=0}^{n-1} c^j} \alpha^{d\sum_{j=0}^{n-1} c^j \sum_{j=0}^{3(n-j)+i-2} a^j} z_{-1}^{d\sum_{j=0}^{n-1} c^j a^{3(n-j)+i-1}} w_{i}^{c^n}, \tag{2.19}$$

for every $n \in \mathbb{N}$ and $i = -1, 0, 1$.

From (2.19) and (2.1) it follows that

$$w_{3n-1} = \beta^{\sum_{j=0}^{n-1} c^j} \alpha^{d\sum_{j=0}^{n-1} c^j \sum_{l=0}^{3(n-j)-3} a^l} z_{-1}^{d\sum_{j=0}^{n-1} c^j a^{3(n-j)-2}} w_{-1}^{c^n}, \tag{2.20}$$

$$\begin{aligned} w_{3n} &= \beta^{\sum_{j=0}^{n-1} c^j} \alpha^{d\sum_{j=0}^{n-1} c^j \sum_{l=0}^{3(n-j)-2} a^l} z_{-1}^{d\sum_{j=0}^{n-1} c^j a^{3(n-j)-1}} \left(\beta w_{-3}^{c} z_{-2}^{d}\right)^{c^n} \\ &= \beta^{\sum_{j=0}^{n} c^j} \alpha^{d\sum_{j=0}^{n-1} c^j \sum_{l=0}^{3(n-j)-2} a^l} z_{-1}^{d\sum_{j=0}^{n-1} c^j a^{3(n-j)-1}} z_{-2}^{dc^n} w_{-3}^{c^{n+1}}, \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} w_{3n+1} &= \beta^{\sum_{j=0}^{n-1} c^j} \alpha^{d\sum_{j=0}^{n-1} c^j \sum_{l=0}^{3(n-j)-1} a^l} z_{-1}^{d\sum_{j=0}^{n-1} c^j a^{3(n-j)}} \left(\beta w_{-2}^{c} z_{-1}^{d}\right)^{c^n} \\ &= \beta^{\sum_{j=0}^{n} c^j} \alpha^{d\sum_{j=0}^{n-1} c^j \sum_{l=0}^{3(n-j)-1} a^l} z_{-1}^{d\sum_{j=0}^{n} c^j a^{3(n-j)}} w_{-2}^{c^{n+1}}, \end{aligned} \tag{2.22}$$

for $n \in \mathbb{N}$.

Case $c \neq 1 \neq a^3 \neq c$. From (2.20)–(2.22) we have

$$\begin{aligned} w_{3n-1} &= \beta^{\frac{1-c^n}{1-c}} \alpha^{d\sum_{j=0}^{n-1} c^j \frac{1-a^{3n-3j-2}}{1-a}} z_{-1}^{d\sum_{j=0}^{n-1} c^j a^{3(n-j)-2}} w_{-1}^{c^n} \\ &= \beta^{\frac{1-c^n}{1-c}} \alpha^{\frac{d}{1-a}\left(\frac{1-c^n}{1-c} - a\frac{a^{3n}-c^n}{a^3-c}\right)} z_{-1}^{ad\frac{a^{3n}-c^n}{a^3-c}} w_{-1}^{c^n} \\ &= \beta^{\frac{1-c^n}{1-c}} \alpha^{\frac{d(a^3-c+(c+a+a^2)(1-a)c^n+(c-1)a^{3n+1})}{(1-a)(1-c)(a^3-c)}} z_{-1}^{ad\frac{a^{3n}-c^n}{a^3-c}} w_{-1}^{c^n}, \end{aligned} \tag{2.23}$$

$$
\begin{aligned}
w_{3n} &= \beta^{\frac{1-c^{n+1}}{1-c}} \alpha^{d\sum_{j=0}^{n-1} c^j \frac{1-a^{3n-3j-1}}{1-a}} z_{-1}^{d\sum_{j=0}^{n-1} c^j a^{3(n-j)-1}} z_{-2}^{dc^n} w_{-3}^{c^{n+1}} \\
&= \beta^{\frac{1-c^{n+1}}{1-c}} \alpha^{\frac{d}{1-a}\left(\frac{1-c^n}{1-c} - a^2 \frac{a^{3n}-c^n}{a^3-c}\right)} z_{-1}^{a^2 d \frac{a^{3n}-c^n}{a^3-c}} z_{-2}^{dc^n} w_{-3}^{c^{n+1}} \\
&= \beta^{\frac{1-c^{n+1}}{1-c}} \alpha^{\frac{d(a^3-c+(c+ac+a^2)(1-a)c^n+(c-1)a^{3n+2})}{(1-a)(1-c)(a^3-c)}} z_{-1}^{a^2 d \frac{a^{3n}-c^n}{a^3-c}} z_{-2}^{dc^n} w_{-3}^{c^{n+1}},
\end{aligned} \tag{2.24}
$$

and

$$
\begin{aligned}
w_{3n+1} &= \beta^{\frac{1-c^{n+1}}{1-c}} \alpha^{d\sum_{j=0}^{n-1} c^j \frac{1-a^{3n-3j}}{1-a}} z_{-1}^{d\sum_{j=0}^{n} c^j a^{3(n-j)}} w_{-2}^{c^{n+1}} \\
&= \beta^{\frac{1-c^{n+1}}{1-c}} \alpha^{\frac{d}{1-a}\left(\frac{1-c^n}{1-c} - a^3 \frac{a^{3n}-c^n}{a^3-c}\right)} z_{-1}^{d \frac{a^{3n+3}-c^{n+1}}{a^3-c}} w_{-2}^{c^{n+1}} \\
&= \beta^{\frac{1-c^{n+1}}{1-c}} \alpha^{\frac{d(a^3-c+(1-a^3)c^{n+1}+(c-1)a^{3n+3})}{(1-a)(1-c)(a^3-c)}} z_{-1}^{d \frac{a^{3n+3}-c^{n+1}}{a^3-c}} w_{-2}^{c^{n+1}},
\end{aligned} \tag{2.25}
$$

for $n \in \mathbb{N}$.

Case $c = a^3 \neq 1$. From (2.20)–(2.22) we have

$$
\begin{aligned}
w_{3n-1} &= \beta^{\frac{1-a^{3n}}{1-a^3}} \alpha^{d\sum_{j=0}^{n-1} a^{3j} \frac{1-a^{3n-3j-2}}{1-a}} z_{-1}^{d\sum_{j=0}^{n-1} a^{3j} a^{3(n-j)-2}} w_{-1}^{a^{3n}} \\
&= \beta^{\frac{1-a^{3n}}{1-a^3}} \alpha^{\frac{d(1-a^{3n}-n(1-a^3)a^{3n-2})}{(1-a)(1-a^3)}} z_{-1}^{dna^{3n-2}} w_{-1}^{a^{3n}},
\end{aligned} \tag{2.26}
$$

$$
\begin{aligned}
w_{3n} &= \beta^{\frac{1-a^{3n+3}}{1-a^3}} \alpha^{d\sum_{j=0}^{n-1} a^{3j} \frac{1-a^{3n-3j-1}}{1-a}} z_{-1}^{d\sum_{j=0}^{n-1} a^{3j} a^{3(n-j)-1}} z_{-2}^{da^{3n}} w_{-3}^{a^{3n+3}} \\
&= \beta^{\frac{1-a^{3n+3}}{1-a^3}} \alpha^{\frac{d(1-a^{3n}-n(1-a^3)a^{3n-1})}{(1-a)(1-a^3)}} z_{-1}^{dna^{3n-1}} z_{-2}^{da^{3n}} w_{-3}^{a^{3n+3}},
\end{aligned} \tag{2.27}
$$

and

$$
\begin{aligned}
w_{3n+1} &= \beta^{\frac{1-a^{3n+3}}{1-a^3}} \alpha^{d\sum_{j=0}^{n-1} a^{3j} \frac{1-a^{3n-3j}}{1-a}} z_{-1}^{d\sum_{j=0}^{n} a^{3j} a^{3(n-j)}} w_{-2}^{a^{3n+3}} \\
&= \beta^{\frac{1-a^{3n+3}}{1-a^3}} \alpha^{\frac{d(1-(n+1)a^{3n}+na^{3n+3})}{(1-a)(1-a^3)}} z_{-1}^{d(n+1)a^{3n}} w_{-2}^{a^{3n+3}}, \quad n \in \mathbb{N}.
\end{aligned} \tag{2.28}
$$

Case $c \neq 1 = a$. From (2.20)–(2.22) we have

$$
\begin{aligned}
w_{3n-1} &= \beta^{\frac{1-c^n}{1-c}} \alpha^{d\sum_{j=0}^{n-1} c^j (3(n-j)-2)} z_{-1}^{d\frac{1-c^n}{1-c}} w_{-1}^{c^n} \\
&= \beta^{\frac{1-c^n}{1-c}} \alpha^{d\left((3n-2)\frac{1-c^n}{1-c} - 3c\frac{1-nc^{n-1}+(n-1)c^n}{(1-c)^2}\right)} z_{-1}^{d\frac{1-c^n}{1-c}} w_{-1}^{c^n} \\
&= \beta^{\frac{1-c^n}{1-c}} \alpha^{\frac{d(3n-2-(3n+1)c+2c^n+c^{n+1})}{(1-c)^2}} z_{-1}^{d\frac{1-c^n}{1-c}} w_{-1}^{c^n},
\end{aligned} \tag{2.29}
$$

$$
\begin{aligned}
w_{3n} &= \beta^{\frac{1-c^{n+1}}{1-c}} \alpha^{d\sum_{j=0}^{n-1} c^j (3(n-j)-1)} z_{-1}^{d\frac{1-c^n}{1-c}} z_{-2}^{dc^n} w_{-3}^{c^{n+1}} \\
&= \beta^{\frac{1-c^{n+1}}{1-c}} \alpha^{d\left((3n-1)\frac{1-c^n}{1-c} - 3c\frac{1-nc^{n-1}+(n-1)c^n}{(1-c)^2}\right)} z_{-1}^{d\frac{1-c^n}{1-c}} z_{-2}^{dc^n} w_{-3}^{c^{n+1}} \\
&= \beta^{\frac{1-c^{n+1}}{1-c}} \alpha^{\frac{d(3n-1-(3n+2)c+c^n+2c^{n+1})}{(1-c)^2}} z_{-1}^{d\frac{1-c^n}{1-c}} z_{-2}^{dc^n} w_{-3}^{c^{n+1}},
\end{aligned} \tag{2.30}
$$

and

$$
\begin{aligned}
w_{3n+1} &= \beta^{\frac{1-c^{n+1}}{1-c}} \alpha^{3d\sum_{j=0}^{n-1} c^j (n-j)} z_{-1}^{d\frac{1-c^{n+1}}{1-c}} w_{-2}^{c^{n+1}} \\
&= \beta^{\frac{1-c^{n+1}}{1-c}} \alpha^{3d\left(n\frac{1-c^n}{1-c} - c\frac{1-nc^{n-1}+(n-1)c^n}{(1-c)^2}\right)} z_{-1}^{d\frac{1-c^{n+1}}{1-c}} w_{-2}^{c^{n+1}} \\
&= \beta^{\frac{1-c^{n+1}}{1-c}} \alpha^{3d\frac{n-(n+1)c+c^{n+1}}{(1-c)^2}} z_{-1}^{d\frac{1-c^{n+1}}{1-c}} w_{-2}^{c^{n+1}}, \quad n \in \mathbb{N}.
\end{aligned} \tag{2.31}
$$

Case $1 \neq c \neq -1 = a$. From (2.20)–(2.22) we have

$$
\begin{aligned}
w_{3n-1} &= \beta^{\sum_{j=0}^{n-1} c^j} \alpha^{d\sum_{j=0}^{n-1} c^j \sum_{l=0}^{3(n-j)-3} (-1)^l} z_{-1}^{d\sum_{j=0}^{n-1} c^j (-1)^{3(n-j)-2}} w_{-1}^{c^n} \\
&= \beta^{\frac{1-c^n}{1-c}} \alpha^{d\sum_{j=0}^{n-1} c^j \frac{1-(-1)^{3(n-j)-2}}{2}} z_{-1}^{d(-1)^n \frac{1-(-c)^n}{1+c}} w_{-1}^{c^n} \\
&= \beta^{\frac{1-c^n}{1-c}} \alpha^{\frac{d}{2}\left(\frac{1-c^n}{1-c} - (-1)^n \frac{1-(-c)^n}{1+c}\right)} z_{-1}^{d\frac{(-1)^n - c^n}{1+c}} w_{-1}^{c^n} \\
&= \beta^{\frac{1-c^n}{1-c}} \alpha^{\frac{d(1+c-(1-c)(-1)^n - 2c^{n+1})}{2(1-c^2)}} z_{-1}^{d\frac{(-1)^n - c^n}{1+c}} w_{-1}^{c^n},
\end{aligned} \tag{2.32}
$$

$$
\begin{aligned}
w_{3n} &= \beta^{\sum_{j=0}^{n} c^j} \alpha^{d\sum_{j=0}^{n-1} c^j \sum_{l=0}^{3(n-j)-2} (-1)^l} z_{-1}^{d\sum_{j=0}^{n-1} c^j (-1)^{3(n-j)-1}} z_{-2}^{dc^n} w_{-3}^{c^{n+1}} \\
&= \beta^{\frac{1-c^{n+1}}{1-c}} \alpha^{d\sum_{j=0}^{n-1} c^j \frac{1-(-1)^{3(n-j)-1}}{2}} z_{-1}^{d(-1)^{n-1} \frac{1-(-c)^n}{1+c}} z_{-2}^{dc^n} w_{-3}^{c^{n+1}} \\
&= \beta^{\frac{1-c^{n+1}}{1-c}} \alpha^{\frac{d}{2}\left(\frac{1-c^n}{1-c} + (-1)^n \frac{1-(-c)^n}{1+c}\right)} z_{-1}^{d\frac{c^n - (-1)^n}{1+c}} z_{-2}^{dc^n} w_{-3}^{c^{n+1}} \\
&= \beta^{\frac{1-c^{n+1}}{1-c}} \alpha^{\frac{d(1+c+(1-c)(-1)^n - 2c^n)}{2(1-c^2)}} z_{-1}^{d\frac{c^n - (-1)^n}{1+c}} z_{-2}^{dc^n} w_{-3}^{c^{n+1}},
\end{aligned} \tag{2.33}
$$

and

$$
\begin{aligned}
w_{3n+1} &= \beta^{\sum_{j=0}^{n} c^j} \alpha^{d\sum_{j=0}^{n-1} c^j \sum_{l=0}^{3(n-j)-1} (-1)^l} z_{-1}^{d\sum_{j=0}^{n} c^j (-1)^{3(n-j)}} w_{-2}^{c^{n+1}} \\
&= \beta^{\frac{1-c^{n+1}}{1-c}} \alpha^{d\sum_{j=0}^{n-1} c^j \frac{1-(-1)^{3(n-j)}}{2}} z_{-1}^{d(-1)^n \sum_{j=0}^{n} (-c)^j} w_{-2}^{c^{n+1}} \\
&= \beta^{\frac{1-c^{n+1}}{1-c}} \alpha^{\frac{d(1+c-(1-c)(-1)^n - 2c^{n+1})}{2(1-c^2)}} z_{-1}^{d\frac{c^{n+1} - (-1)^{n+1}}{1+c}} w_{-2}^{c^{n+1}},
\end{aligned} \tag{2.34}
$$

for $n \in \mathbb{N}$.

Case $c = 1 \neq a$. From (2.20)–(2.22) we have

$$
\begin{aligned}
w_{3n-1} &= \beta^n \alpha^{d\sum_{j=0}^{n-1} \frac{1-a^{3(n-j)-2}}{1-a}} z_{-1}^{d\sum_{j=0}^{n-1} a^{3(n-j)-2}} w_{-1} \\
&= \beta^n \alpha^{\frac{d(a^{3n+1} - a + n(1-a^3))}{(1-a)(1-a^3)}} z_{-1}^{ad\frac{a^{3n}-1}{a^3-1}} w_{-1},
\end{aligned} \tag{2.35}
$$

$$
\begin{aligned}
w_{3n} &= \beta^{n+1} \alpha^{d\sum_{j=0}^{n-1} \frac{1-a^{3(n-j)-1}}{1-a}} z_{-1}^{d\sum_{j=0}^{n-1} a^{3(n-j)-1}} z_{-2}^d w_{-3} \\
&= \beta^{n+1} \alpha^{\frac{d(a^{3n+2} - a^2 + n(1-a^3))}{(1-a)(1-a^3)}} z_{-1}^{a^2 d\frac{a^{3n}-1}{a^3-1}} z_{-2}^d w_{-3},
\end{aligned} \tag{2.36}
$$

and

$$
\begin{aligned}
w_{3n+1} &= \beta^{n+1} \alpha^{d\sum_{j=0}^{n-1} \frac{1-a^{3(n-j)}}{1-a}} z_{-1}^{d\sum_{j=0}^{n} a^{3(n-j)}} w_{-2} \\
&= \beta^{n+1} \alpha^{\frac{d(a^{3n+3} - (n+1)a^3 + n)}{(1-a)(1-a^3)}} z_{-1}^{d\frac{a^{3n+3}-1}{a^3-1}} w_{-3},
\end{aligned} \tag{2.37}
$$

for $n \in \mathbb{N}$.

Case $c = a = 1$. From (2.20)–(2.22) we have

$$\begin{aligned} w_{3n-1} &= \beta^n \alpha^{d\sum_{j=0}^{n-1}(3(n-j)-2)} z_{-1}^{dn} w_{-1} \\ &= \beta^n \alpha^{d\frac{n(3n-1)}{2}} z_{-1}^{dn} w_{-1}, \end{aligned} \tag{2.38}$$

$$\begin{aligned} w_{3n} &= \beta^{n+1} \alpha^{d\sum_{j=0}^{n-1}(3(n-j)-1)} z_{-1}^{dn} z_{-2}^{d} w_{-3} \\ &= \beta^{n+1} \alpha^{d\frac{n(3n+1)}{2}} z_{-1}^{dn} z_{-2}^{d} w_{-3}, \end{aligned} \tag{2.39}$$

and

$$\begin{aligned} w_{3n+1} &= \beta^{n+1} \alpha^{3d\sum_{j=0}^{n-1}(n-j)} z_{-1}^{d(n+1)} w_{-2} \\ &= \beta^{n+1} \alpha^{3d\frac{n(n+1)}{2}} z_{-1}^{d(n+1)} w_{-2}, \end{aligned} \tag{2.40}$$

for $n \in \mathbb{N}$. □

From the proof of Theorem 2.2 we see that the following corollary holds.

Corollary 2.3. *Consider system of difference equation* (1.1) *where* $a, c, d \in \mathbb{Z}$, $b = 0 \neq c$, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ *and* $w_{-3}, w_{-2}, w_{-1}, z_{-2}, z_{-1} \in \mathbb{C} \setminus \{0\}$. *Then the following statements hold.*

(a) If $c \neq 1 \neq a^3 \neq c$, *then the general solution to system* (1.1) *is given by formulas* (2.13), (2.23)–(2.25).

(b) If $c = a^3 \neq 1$, *then the general solution to system* (1.1) *is given by formulas* (2.13), (2.26)–(2.28).

(c) If $c \neq 1 = a$, *then the general solution to system* (1.1) *is given by formulas* (2.14), (2.29)–(2.31).

(d) If $1 \neq c \neq -1 = a$, *then the general solution to system* (1.1) *is given by formulas* (2.13), (2.32)–(2.34).

(e) If $c = 1 \neq a$, *then the general solution to system* (1.1) *is given by formulas* (2.13), (2.35)–(2.37).

(f) If $c = a = 1$, *then the general solution to system* (1.1) *is given by formulas* (2.14), (2.38)–(2.40).

Theorem 2.4. *Assume that* $a, b, c \in \mathbb{Z}$, $d = 0 \neq c$, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ *and* $w_{-3}, w_{-2}, w_{-1}, z_{-2}, z_{-1} \in \mathbb{C} \setminus \{0\}$. *Then system* (1.1) *is solvable in closed form.*

Proof. Since $d = 0$, system (1.1) is

$$z_n = \alpha z_{n-1}^a w_{n-2}^b, \quad w_n = \beta w_{n-3}^c, \tag{2.41}$$

for $n \in \mathbb{N}_0$.

Second equation in (2.41) yields

$$w_{3n+i} = \beta w_{3(n-1)+i}^c, \tag{2.42}$$

for $n \in \mathbb{N}_0$ and $i = \overline{0,2}$, from which it follows that

$$w_{3n+i} = \beta^{\sum_{j=0}^{n} c^j} w_{i-3}^{c^{n+1}}, \tag{2.43}$$

for $n \in \mathbb{N}_0$ and $i = \overline{0,2}$.

Hence, from (2.43) we have that

$$w_{3n+i} = \beta^{\frac{1-c^{n+1}}{1-c}} w_{i-3}^{c^{n+1}}, \tag{2.44}$$

for $n \in \mathbb{N}_0$ and $i = \overline{0,2}$, when $c \neq 1$, whereas

$$w_{3n+i} = \beta^{n+1} w_{i-3}, \tag{2.45}$$

for $n \in \mathbb{N}_0$ and $i = \overline{0,2}$, when $c = 1$.

By iterating the first equation in (2.41) in variable z twice, we get

$$\begin{aligned} z_n &= \alpha(\alpha z_{n-2}^a w_{n-3}^b)^a w_{n-2}^b \\ &= \alpha^{1+a} z_{n-2}^{a^2} w_{n-3}^{ba} w_{n-2}^b \\ &= \alpha^{1+a}(\alpha z_{n-3}^a w_{n-4}^b)^{a^2} w_{n-3}^{ba} w_{n-2}^b \\ &= \alpha^{1+a+a^2} w_{n-4}^{ba^2} w_{n-3}^{ba} w_{n-2}^b z_{n-3}^{a^3}, \end{aligned}$$

for $n \geq 2$, from which it follows that

$$z_{3n} = \alpha^{1+a+a^2} w_{3n-4}^{ba^2} w_{3n-3}^{ba} w_{3n-2}^b z_{3(n-1)}^{a^3} \tag{2.46}$$

$$z_{3n+1} = \alpha^{1+a+a^2} w_{3n-3}^{ba^2} w_{3n-2}^{ba} w_{3n-1}^b z_{3(n-1)+1}^{a^3} \tag{2.47}$$

$$z_{3n+2} = \alpha^{1+a+a^2} w_{3n-2}^{ba^2} w_{3n-1}^{ba} w_{3n}^b z_{3(n-1)+2}^{a^3}. \tag{2.48}$$

By iterating relations (2.46)–(2.48) in variable z and using an inductive argument similar to the one in the proof of Theorem 2.2, is obtained

$$z_{3n} = z_0^{a^{3n}} \prod_{j=0}^{n-1} \left(\alpha^{1+a+a^2} w_{3j-1}^{ba^2} w_{3j}^{ba} w_{3j+1}^b \right)^{a^{3(n-j-1)}} \tag{2.49}$$

$$z_{3n+1} = z_1^{a^{3n}} \prod_{j=0}^{n-1} \left(\alpha^{1+a+a^2} w_{3j}^{ba^2} w_{3j+1}^{ba} w_{3j+2}^b \right)^{a^{3(n-j-1)}} \tag{2.50}$$

$$z_{3n+2} = z_{-1}^{a^{3(n+1)}} \prod_{j=0}^{n} \left(\alpha^{1+a+a^2} w_{3j-2}^{ba^2} w_{3j-1}^{ba} w_{3j}^b \right)^{a^{3(n-j)}}, \tag{2.51}$$

for $n \in \mathbb{N}_0$.

Using formula (2.43), along with the first equation in (2.41) with $n = 0, 1$, into (2.49)–(2.51), we get

$$\begin{aligned} z_{3n} = {} & \alpha^{\sum_{j=0}^{3n} a^j} w_{-1}^{ba^2 \sum_{j=0}^{n-1} c^j a^{3(n-j-1)}} w_{-2}^{b \sum_{j=0}^{n} c^j a^{3(n-j)}} w_{-3}^{bac \sum_{j=0}^{n-1} c^j a^{3(n-j-1)}} \\ & \times \beta^{b(1+a) \sum_{j=0}^{n-1} c^j a^{3(n-j-1)} + b(1+a+a^2) \sum_{j=0}^{n-1} a^{3(n-j-1)} \sum_{i=0}^{j-1} c^i} z_{-1}^{a^{3n+1}}, \end{aligned} \tag{2.52}$$

$$\begin{aligned} z_{3n+1} = {} & \alpha^{\sum_{j=0}^{3n+1} a^j} w_{-1}^{b \sum_{j=0}^{n} c^j a^{3(n-j)}} w_{-2}^{ba \sum_{j=0}^{n} c^j a^{3(n-j)}} w_{-3}^{ba^2 c \sum_{j=0}^{n-1} c^j a^{3(n-j-1)}} \\ & \times \beta^{b(1+a+a^2) \sum_{j=0}^{n-1} a^{3(n-j-1)} \sum_{i=0}^{j} c^i} z_{-1}^{a^{3n+2}}, \end{aligned} \tag{2.53}$$

$$
\begin{aligned}
z_{3n+2} = {} & \alpha^{\sum_{j=0}^{3n+2} a^j} w_{-1}^{ba\sum_{j=0}^{n} c^j a^{3(n-j)}} w_{-2}^{ba^2\sum_{j=0}^{n} c^j a^{3(n-j)}} w_{-3}^{bc\sum_{j=0}^{n} c^j a^{3(n-j)}} \\
& \times \beta^{b\sum_{j=0}^{n} c^j a^{3(n-j)} + b(1+a+a^2)\sum_{j=0}^{n} a^{3(n-j)} \sum_{i=0}^{j-1} c^i} z_{-1}^{a^{3(n+1)}},
\end{aligned} \tag{2.54}
$$

for $n \in \mathbb{N}_0$.

From formulas (2.52)–(2.54), similar to the proof of Theorem 2.2, are obtained closed form formulas for subsequences $(z_{3n+i})_{n\in\mathbb{N}_0}$, $i = \overline{0,2}$, in all possible cases, which is omitted. □

Theorem 2.5. *Assume that $a, b, c, d \in \mathbb{Z}$, $ac \neq bd \neq 0$, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ and $w_{-3}, w_{-2}, w_{-1}, z_{-2}, z_{-1} \in \mathbb{C} \setminus \{0\}$. Then system (1.1) is solvable in closed form.*

Remark 2.6. Before we prove Theorem 2.5 we want to explain why the method which was successfully used in [26, 41–43] fails for the case of system (1.1). Namely, note that (2.2) holds so that (1.1) yields

$$w_{n-2}^{b} = \frac{z_n}{\alpha z_{n-1}^{a}}, \tag{2.55}$$

and

$$w_n^b = \beta^b w_{n-3}^{bc} z_{n-2}^{bd}, \tag{2.56}$$

for $n \in \mathbb{N}_0$.

Combining (2.55) and (2.56) we get

$$z_{n+2} = \alpha^{1-c} \beta^b z_{n+1}^{a} z_{n-1}^{c} z_{n-2}^{bd-ac}, \tag{2.57}$$

for $n \in \mathbb{N}_0$.

Let $\delta = \alpha^{1-c}\beta^b$,

$$a_1 = a, \quad b_1 = 0, \quad c_1 = c, \quad d_1 = bd - ac, \tag{2.58}$$

$$y_1 = 1. \tag{2.59}$$

Then by using the procedure in [26, 41–43], it is obtained that

$$z_{n+2} = \delta^{y_k} z_{n+2-k}^{a_k} z_{n+1-k}^{b_k} z_{n-k}^{c_k} z_{n-k-1}^{d_k}, \tag{2.60}$$

for $k, n \in \mathbb{N}$, where $n \geq k-1$, and

$$
\begin{aligned}
a_k &= a_1 a_{k-1} + b_{k-1}, & b_k &= b_1 a_{k-1} + c_{k-1}, \\
c_k &= c_1 a_{k-1} + d_{k-1}, & d_k &= d_1 a_{k-1},
\end{aligned} \tag{2.61}
$$

$$y_k := y_{k-1} + a_{k-1}. \tag{2.62}$$

Setting $k = n+1$ in (2.60), using (2.1), (2.61), (2.62) and by some calculation, is obtained

$$
\begin{aligned}
z_{n+2} &= \delta^{y_{n+1}} z_1^{a_{n+1}} z_0^{b_{n+1}} z_{-1}^{c_{n+1}} z_{-2}^{d_{n+1}} \\
&= \alpha^{y_{n+3} - c y_{n+1}} \beta^{b y_{n+1}} z_{-2}^{(bd-ac)a_n} z_{-1}^{a_{n+3}} w_{-2}^{ba_{n+2}} w_{-1}^{ba_{n+1}},
\end{aligned} \tag{2.63}
$$

for $n \in \mathbb{N}_0$.

From (2.61) it easily follows that a_k satisfies the following difference equation

$$a_k = a_1 a_{k-1} + b_1 a_{k-2} + c_1 a_{k-3} + d_1 a_{k-4}, \tag{2.64}$$

for $k \geq 5$. Since equation (2.64) is a linear difference equation of the fourth order it can be solved in closed form. From this, since $y_k = 1 + \sum_{j=1}^{k-1} a_j$, and since the last sum can be calculated in closed form (due to the special form of a_k, see, e.g. [1,5,7]), using (2.63) is proved the solvability of equation (2.57).

We also have

$$z_{n-2}^d = \frac{w_n}{\beta w_{n-3}^c}, \tag{2.65}$$

and

$$z_n^d = \alpha^d z_{n-1}^{ad} w_{n-2}^{bd}, \tag{2.66}$$

for $n \in \mathbb{N}_0$.

Combining (2.65) and (2.66) we get

$$w_{n+2} = \alpha^d \beta^{1-a} w_{n+1}^a w_{n-1}^c w_{n-2}^{bd-ac}, \tag{2.67}$$

for $n \in \mathbb{N}_0$.

As above, from (2.67) we get

$$w_{n+2} = \eta^{y_k} w_{n+2-k}^{a_k} w_{n+1-k}^{b_k} w_{n-k}^{c_k} w_{n-k-1}^{d_k}, \tag{2.68}$$

for every $k, n \in \mathbb{N}$ such that $1 \leq k \leq n+1$, where $\eta = \alpha^d \beta^{1-a}$, sequences $(a_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}}$, $(c_k)_{k\in\mathbb{N}}$, $(d_k)_{k\in\mathbb{N}}$ satisfy (2.58) and (2.61), whereas sequence $(y_k)_{k\in\mathbb{N}}$ satisfies (2.59) and (2.62).

For $k = n+1$, equation (2.68) along with (2.1), (2.61) and (2.62) yields

$$\begin{aligned} w_{n+2} &= \eta^{y_{n+1}} w_1^{a_{n+1}} w_0^{b_{n+1}} w_{-1}^{c_{n+1}} w_{-2}^{d_{n+1}} \\ &= \alpha^{dy_{n+1}} \beta^{y_{n+3}-ay_{n+2}} z_{-2}^{d(a_{n+2}-aa_{n+1})} z_{-1}^{da_{n+1}} w_{-3}^{c(a_{n+2}-aa_{n+1})} \\ &\quad \times w_{-2}^{a_{n+4}-aa_{n+3}} w_{-1}^{a_{n+3}-aa_{n+2}}, \end{aligned} \tag{2.69}$$

for every $n \in \mathbb{N}_0$, from which the solvability of equation (2.67) follows.

However, although (2.63) and (2.69) are solutions to equations (2.57) and (2.67) respectively, a direct check easily shows that they are not solutions to system (1.1). Namely, note that w_n depends, among others, on initial value w_{-3} if $c \neq 0$, which is not the case with z_n, from which the claim easily follows along with the equations in (1.1). Such a situation has not appeared in our previous papers on the topic so far (see [26, 41–43]). Hence we need an alternative approach in dealing with the solvability of system (1.1).

Proof of Theorem 2.5. From the first equation in (1.1) and since z_2 depends on all the initial values $w_{-3}, w_{-2}, w_{-1}, z_{-2}, z_{-1}$ (see (2.1)), a simple inductive argument shows that z_n depends on these initial values for each $n \geq 2$, while from the second equation in (1.1) and by a simple inductive argument is obtained that w_n also depends on all these initial values for each $n \geq 4$. Hence, z_n and w_n can be written in the following form:

$$z_n = \alpha^{x_n} \beta^{y_n} z_{-2}^{a_n} z_{-1}^{b_n} w_{-3}^{c_n} w_{-2}^{d_n} w_{-1}^{e_n}, \tag{2.70}$$

for $n \geq -2$, and

$$w_n = \alpha^{u_n}\beta^{v_n}z_{-2}^{\alpha_n}z_{-1}^{\beta_n}w_{-3}^{\gamma_n}w_{-2}^{\delta_n}w_{-1}^{\eta_n}, \tag{2.71}$$

for $n \geq -3$.

Employing (2.70) and (2.71) in both equations in (1.1) we get

$$\begin{aligned} z_n &= \alpha(\alpha^{x_{n-1}}\beta^{y_{n-1}}z_{-2}^{a_{n-1}}z_{-1}^{b_{n-1}}w_{-3}^{c_{n-1}}w_{-2}^{d_{n-1}}w_{-1}^{e_{n-1}})^a \\ &\quad\times(\alpha^{u_{n-2}}\beta^{v_{n-2}}z_{-2}^{\alpha_{n-2}}z_{-1}^{\beta_{n-2}}w_{-3}^{\gamma_{n-2}}w_{-2}^{\delta_{n-2}}w_{-1}^{\eta_{n-2}})^b \\ &= \alpha^{ax_{n-1}+bu_{n-2}+1}\beta^{ay_{n-1}+bv_{n-2}}z_{-2}^{aa_{n-1}+b\alpha_{n-2}}z_{-1}^{ab_{n-1}+b\beta_{n-2}} \\ &\quad\times w_{-3}^{ac_{n-1}+b\gamma_{n-2}}w_{-2}^{ad_{n-1}+b\delta_{n-2}}w_{-1}^{ae_{n-1}+b\eta_{n-2}}, \end{aligned} \tag{2.72}$$

for $n \geq -1$, and

$$\begin{aligned} w_n &= \beta(\alpha^{u_{n-3}}\beta^{v_{n-3}}z_{-2}^{\alpha_{n-3}}z_{-1}^{\beta_{n-3}}w_{-3}^{\gamma_{n-3}}w_{-2}^{\delta_{n-3}}w_{-1}^{\eta_{n-3}})^c \\ &\quad\times(\alpha^{x_{n-2}}\beta^{y_{n-2}}z_{-2}^{a_{n-2}}z_{-1}^{b_{n-2}}w_{-3}^{c_{n-2}}w_{-2}^{d_{n-2}}w_{-1}^{e_{n-2}})^d \\ &= \alpha^{dx_{n-2}+cu_{n-3}}\beta^{dy_{n-2}+cv_{n-3}+1}z_{-2}^{da_{n-2}+c\alpha_{n-3}}z_{-1}^{db_{n-2}+c\beta_{n-3}} \\ &\quad\times w_{-3}^{dc_{n-2}+c\gamma_{n-3}}w_{-2}^{dd_{n-2}+c\delta_{n-3}}w_{-1}^{de_{n-2}+c\eta_{n-3}}, \end{aligned} \tag{2.73}$$

for $n \in \mathbb{N}_0$.

From (2.70)-(2.73) it follows that sequences x_n, y_n, a_n, b_n, c_n, d_n, e_n, u_n, v_n, α_n, β_n, γ_n, δ_n, η_n, satisfy the following systems of difference equations

$$x_n = ax_{n-1} + bu_{n-2} + 1, \quad u_n = dx_{n-2} + cu_{n-3}, \tag{2.74}$$
$$y_n = ay_{n-1} + bv_{n-2}, \quad v_n = dy_{n-2} + cv_{n-3} + 1, \tag{2.75}$$
$$a_n = aa_{n-1} + b\alpha_{n-2}, \quad \alpha_n = da_{n-2} + c\alpha_{n-3}, \tag{2.76}$$
$$b_n = ab_{n-1} + b\beta_{n-2}, \quad \beta_n = db_{n-2} + c\beta_{n-3}, \tag{2.77}$$
$$c_n = ac_{n-1} + b\gamma_{n-2}, \quad \gamma_n = dc_{n-2} + c\gamma_{n-3}, \tag{2.78}$$
$$d_n = ad_{n-1} + b\delta_{n-2}, \quad \delta_n = dd_{n-2} + c\delta_{n-3}, \tag{2.79}$$
$$e_n = ae_{n-1} + b\eta_{n-2}, \quad \eta_n = de_{n-2} + c\eta_{n-3}, \tag{2.80}$$

for $n \in \mathbb{N}_0$, and the following initial conditions

$$u_{-3} = 0, \quad v_{-3} = 0, \quad \alpha_{-3} = 0, \quad \beta_{-3} = 0, \quad \gamma_{-3} = 1, \quad \delta_{-3} = 0, \quad \eta_{-3} = 0, \tag{2.81}$$
$$x_{-2} = 0, \quad y_{-2} = 0, \quad a_{-2} = 1, \quad b_{-2} = 0, \quad c_{-2} = 0, \quad d_{-2} = 0, \quad e_{-2} = 0, \tag{2.82}$$
$$u_{-2} = 0, \quad v_{-2} = 0, \quad \alpha_{-2} = 0, \quad \beta_{-2} = 0, \quad \gamma_{-2} = 0, \quad \delta_{-2} = 1, \quad \eta_{-2} = 0, \tag{2.83}$$
$$x_{-1} = 0, \quad y_{-1} = 0, \quad a_{-1} = 0, \quad b_{-1} = 1, \quad c_{-1} = 0, \quad d_{-1} = 0, \quad e_{-1} = 0, \tag{2.84}$$
$$u_{-1} = 0, \quad v_{-1} = 0, \quad \alpha_{-1} = 0, \quad \beta_{-1} = 0, \quad \gamma_{-1} = 0, \quad \delta_{-1} = 0, \quad \eta_{-1} = 1. \tag{2.85}$$

Note also that from (2.74)–(2.85) it follows that

$$x_0 = 1, \quad y_0 = 0, \quad a_0 = 0, \quad b_0 = a, \quad c_0 = 0, \quad d_0 = b, \quad e_0 = 0, \tag{2.86}$$
$$u_0 = 0, \quad v_0 = 1, \quad \alpha_0 = d, \quad \beta_0 = 0, \quad \gamma_0 = c, \quad \delta_0 = 0, \quad \eta_0 = 0, \tag{2.87}$$
$$x_1 = 1 + a, \quad y_1 = 0, \quad a_1 = 0, \quad b_1 = a^2, \quad c_1 = 0, \quad d_1 = ab, \quad e_1 = b, \tag{2.88}$$
$$u_1 = 0, \quad v_1 = 1, \quad \alpha_1 = 0, \quad \beta_1 = d, \quad \gamma_1 = 0, \quad \delta_1 = c, \quad \eta_1 = 0, \tag{2.89}$$

which matches with (2.1).

Since $d \neq 0$, then from the second equation in (2.74) is obtained

$$x_{n-2} = \frac{u_n - cu_{n-3}}{d}, \quad n \in \mathbb{N}_0. \tag{2.90}$$

By using (2.90) in the first equation in (2.74) we easily get

$$u_{n+2} = au_{n+1} + cu_{n-1} + (bd - ac)u_{n-2} + d, \tag{2.91}$$

for $n \in \mathbb{N}_0$

Also from the second equation in (2.75) is obtained

$$y_{n-2} = \frac{v_n - cv_{n-3} - 1}{d}, \quad n \in \mathbb{N}_0. \tag{2.92}$$

By using (2.92) in the first equation in (2.75) we easily get

$$v_{n+2} = av_{n+1} + cv_{n-1} + (bd - ac)v_{n-2} + 1 - a, \tag{2.93}$$

for $n \in \mathbb{N}_0$.

From (2.76)–(2.80) is similarly obtained

$$a_{n-2} = \frac{\alpha_n - c\alpha_{n-3}}{d}, \tag{2.94}$$

$$b_{n-2} = \frac{\beta_n - c\beta_{n-3}}{d}, \tag{2.95}$$

$$c_{n-2} = \frac{\gamma_n - c\gamma_{n-3}}{d}, \tag{2.96}$$

$$d_{n-2} = \frac{\delta_n - c\delta_{n-3}}{d}, \tag{2.97}$$

$$e_{n-2} = \frac{\eta_n - c\eta_{n-3}}{d}, \tag{2.98}$$

for $n \in \mathbb{N}_0$, and

$$\alpha_{n+2} = a\alpha_{n+1} + c\alpha_{n-1} + (bd - ac)\alpha_{n-2}, \tag{2.99}$$

$$\beta_{n+2} = a\beta_{n+1} + c\beta_{n-1} + (bd - ac)\beta_{n-2}, \tag{2.100}$$

$$\gamma_{n+2} = a\gamma_{n+1} + c\gamma_{n-1} + (bd - ac)\gamma_{n-2}, \tag{2.101}$$

$$\delta_{n+2} = a\delta_{n+1} + c\delta_{n-1} + (bd - ac)\delta_{n-2}, \tag{2.102}$$

$$\eta_{n+2} = a\eta_{n+1} + c\eta_{n-1} + (bd - ac)\eta_{n-2}, \tag{2.103}$$

for $n \in \mathbb{N}_0$. Note that (2.91), (2.93), (2.99)–(2.103) hold also for $n = -1$.

At this point it is important to note that all the transformations that we have just done transform systems of difference equations (2.74)–(2.80) into equivalent ones. Therefore, the sets of solutions of the original and transformed systems are the same.

Let on the space of all sequences $(t_n)_{n \geq -2}$ be defined the following linear operator

$$L(t_n) = t_{n+2} - at_{n+1} - ct_{n-1} + (ac - bd)t_{n-2}, \tag{2.104}$$

for $n \in \mathbb{N}_0$.

Then from (2.91) we have that $L(u_n) = d$, $n \in \mathbb{N}_0$, from (2.93) we have that $L(v_n) = 1 - a$, $n \in \mathbb{N}_0$, while from (2.99)–(2.103) we have that

$$L(\alpha_n) = L(\beta_n) = L(\gamma_n) = L(\delta_n) = L(\eta_n) = 0,$$

for $n \in \mathbb{N}_0$.

Since the linear difference equation

$$L(t_n) = 0, \tag{2.105}$$

is of the fourth order it is solvable in closed form, so that the sequences α_n, β_n, γ_n, δ_n, η_n, can be calculated explicitly, by using the corresponding initial conditions in (2.82)–(2.89). Using such obtained formulas for α_n, β_n, γ_n, δ_n, η_n, in equations (2.94)–(2.98) respectively the closed form formulas for a_n, b_n, c_n, d_n, e_n, are found. Beside this, the following nonhomogeneous difference equation

$$L(t_n) = f, \tag{2.106}$$

where f is a constant is also solvable, since it is well-known that a particular solution to the equation can be found in the following form

$$t_n = \hat{c}_0 + \hat{c}_1 n + \hat{c}_2 n^2 + \hat{c}_3 n^3 + \hat{c}_4 n^4,$$

for some constants $\hat{c}_j$, $j = \overline{0,4}$ (see, e.g., [1,7]). Hence, equation (2.106) can be solved for $f = d$ and $f = 1 - a$ respectively. From this and by using the corresponding initial conditions in (2.82)–(2.89) are found the corresponding closed form formulas for the sequences u_n and v_n respectively. Using such obtained formulas for u_n and v_n in (2.90) and (2.92) respectively, we get closed form formulas for x_n and y_n respectively. Finally, using the obtained formulas for these fourteen sequences in formulas (2.72) and (2.73) are obtained closed form formulas for solutions to system (1.1), finishing the proof of the theorem. □

Remark 2.7. Since $ac \neq bd$ equations (2.105) and (2.106) produce solutions not only for $n \geq -2$. Namely, from (2.105) we have that

$$t_{n-2} = \frac{t_{n+2} - at_{n+1} - ct_{n-1}}{bd - ac}, \tag{2.107}$$

while from (2.106) we have that

$$t_{n-2} = \frac{t_{n+2} - at_{n+1} - ct_{n-1} - f}{bd - ac}, \tag{2.108}$$

from which t_n can be calculated for all negative indices if initial values t_{-2}, t_{-1}, t_0 and t_1 are given. Consequently, sequences x_n, y_n, a_n, b_n, c_n, d_n, e_n, u_n, v_n, α_n, β_n, γ_n, δ_n, η_n appearing in the proof of Theorem 2.5 can be also calculated for every $n \in \mathbb{Z}$. Hence, formulas (2.72) and (2.73), among others, hold also for $n \geq -2$, that is, $n \geq -3$, respectively.

Remark 2.8. The case $b \neq 0$ is treated similarly/dually. Namely, from the first equation in (2.74) is obtained

$$u_{n-2} = \frac{x_n - ax_{n-1} - 1}{b}, \quad n \in \mathbb{N}_0. \tag{2.109}$$

By using (2.109) in the second equation in (2.74) we easily get

$$x_{n+2} = ax_{n+1} + cx_{n-1} + (bd - ac)x_{n-2} + 1 - c, \tag{2.110}$$

for $n \in \mathbb{N}_0$.

Also from the first equation in (2.75) is obtained

$$v_{n-2} = \frac{y_n - ay_{n-1}}{b}, \quad n \in \mathbb{N}_0. \tag{2.111}$$

By using (2.111) in the second equation in (2.75) we easily get

$$y_{n+2} = ay_{n+1} + cy_{n-1} + (bd - ac)y_{n-2} + b, \quad n \in \mathbb{N}_0. \tag{2.112}$$

From (2.76)–(2.80) is similarly obtained

$$\alpha_{n-2} = \frac{a_n - aa_{n-1}}{b}, \tag{2.113}$$

$$\beta_{n-2} = \frac{b_n - ab_{n-1}}{b}, \tag{2.114}$$

$$\gamma_{n-2} = \frac{c_n - ac_{n-1}}{b}, \tag{2.115}$$

$$\delta_{n-2} = \frac{d_n - ad_{n-1}}{b}, \tag{2.116}$$

$$\eta_{n-2} = \frac{e_n - ae_{n-1}}{b}, \tag{2.117}$$

$$a_{n+2} = aa_{n+1} + ca_{n-1} + (bd - ac)a_{n-2}, \tag{2.118}$$

$$b_{n+2} = ab_{n+1} + cb_{n-1} + (bd - ac)b_{n-2}, \tag{2.119}$$

$$c_{n+2} = ac_{n+1} + cc_{n-1} + (bd - ac)c_{n-2}, \tag{2.120}$$

$$d_{n+2} = ad_{n+1} + cd_{n-1} + (bd - ac)d_{n-2}, \tag{2.121}$$

$$e_{n+2} = ae_{n+1} + ce_{n-1} + (bd - ac)e_{n-2}, \tag{2.122}$$

for $n \in \mathbb{N}_0$.

From (2.110) we have that $L(x_n) = 1 - c$, $n \in \mathbb{N}_0$, from (2.112) we have that $L(y_n) = b$, $n \in \mathbb{N}_0$, while from (2.118)–(2.122) we have

$$L(a_n) = L(b_n) = L(c_n) = L(d_n) = L(e_n) = 0, \quad n \in \mathbb{N}_0.$$

From these equations as in the proof of Theorem 2.5 it is showed that closed form formulas for x_n, y_n, a_n, b_n, c_n, d_n, e_n, can be found, from which along with (2.109), (2.111), (2.113)–(2.117) are obtained closed for formulas for u_n, v_n, α_n, β_n, γ_n, δ_n, η_n, from which the solvability of (1.1) also follows under the conditions of the theorem.

The next theorem deals with the case $ac = bd \neq 0$.

Theorem 2.9. *Assume that $a, b, c, d \in \mathbb{Z}$, $ac = bd \neq 0$, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ and $w_{-3}, w_{-2}, w_{-1}, z_{-2}, z_{-1} \in \mathbb{C} \setminus \{0\}$. Then system (1.1) is solvable in closed form.*

Proof. First note that in this case equalities (2.74)–(2.103) also hold. Since $ac = bd$ and $c \neq 0$, we specially obtain

$$u_{n+2} = au_{n+1} + cu_{n-1} + d, \tag{2.123}$$
$$v_{n+2} = av_{n+1} + cv_{n-1} + 1 - a, \tag{2.124}$$
$$\alpha_{n+2} = a\alpha_{n+1} + c\alpha_{n-1}, \tag{2.125}$$
$$\beta_{n+2} = a\beta_{n+1} + c\beta_{n-1}, \tag{2.126}$$
$$\gamma_{n+2} = a\gamma_{n+1} + c\gamma_{n-1}, \tag{2.127}$$
$$\delta_{n+2} = a\delta_{n+1} + c\delta_{n-1}, \tag{2.128}$$
$$\eta_{n+2} = a\eta_{n+1} + c\eta_{n-1}, \tag{2.129}$$

for $n \in \mathbb{N}_0$, while operator L defined in (2.104) becomes

$$L(t_n) = t_{n+2} - at_{n+1} - ct_{n-1}.$$

Now difference equation (2.105) is of the third order, so it is solvable in closed form. Hence equations (2.125)–(2.129) can be solved, so that the sequences α_n, β_n, γ_n, δ_n, η_n, can be calculated explicitly, using the corresponding conditions in (2.82)–(2.89). These formulas along with (2.94)–(2.98) yield formulas for a_n, b_n, c_n, d_n, e_n. Further, difference equation (2.106) is also solvable, since a particular solution to the equation can be found in the following form

$$t_n = (\hat{d}_0 + \hat{d}_1 n + \hat{d}_2 n^2 + \hat{d}_3 n^3),$$

for some constants $\hat{d}_j$, $j = \overline{0,3}$. Specially, it is solved for $f = d$ and $f = 1 - a$, which gives formulas for the sequences u_n and v_n, and consequently closed-form formulas for x_n and y_n. Using such obtained formulas in (2.72) and (2.73) we get formulas for solutions to system (1.1), in this case. □

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