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On multiple sign-changing solutions for some second-order integral boundary value problems *

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Abstract In this paper, by employing fixed point index theory and Leray-Schauder degree theory, we obtain the existence and multiplicity of sign-changing solutions for nonlinear second-order differential equations with integral boundary value conditions.

Keywords Integral boundary value problem; sign-changing solution; fixed point index; Leray-Schauder degree

MSC 34B10; 34B18

1 Introduction

In this paper, we are concerned with the existence of sign-changing solutions for the following nonlinear second-order integral boundary-value problem (BVP for short)

$$\begin{cases} x''(t) + f(x(t)) = 0, \quad 0 \le t \le 1, \\ x(0) = 0, \quad x(1) = \displaystyle\int_0^1 a(s)x(s)\mathrm{d}s, \end{cases} \tag{1}$$

where $f \in C(\mathbb{R}, \mathbb{R}), a \in L[0,1]$ is nonnegative with $\int_0^1 a^2(s)\mathrm{d}s < 1$.

Nonlocal boundary value problems have been well studied especially on a compact interval. For example, Gupta et al. have made an extensive study of multi-point boundary value problems in [4, 5, 6]. Many researchers have studied positive solutions for multi-point boundary value problems, and obtained sufficient conditions for existence, see [4, 5, 6, 15, 28] and the references therein. Nodal solutions for multi-point boundary value problems have also been paid much attention by some authors, see [13, 14, 17, 19] for reference.

Boundary-value problems with integral boundary conditions for ordinary differential equations arise in different fields of applied mathematics and physics such as heat conduction, chemical engineering, underground water flow, thermo-elasticity, and plasma physics. Moreover, boundary-value problems with Riemann-Stieltjes integral conditions constitute a very interesting and important class of problems. They include two, three, multi-point and integral boundary-value problems as

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special cases, see [7, 8, 21, 22]. For boundary value problems with other integral boundary conditions and comments on their importance, we refer the reader to the papers by Karakostas and Tsamatos [7, 8], Yuhua Li and Fuyi Li [11], Lomtatidze and Malaguti [12], Webb and Infante [21, 22] and Yang [26, 27] and the references therein.

In [21, 22], Webb and Infante used fixed point index theory and formulated a general method for solving problems with integral boundary conditions of Riemann-Stieltjes type. In [22] they studied the existence of multiple positive solutions of nonlinear differential equations of the form

$$-u''(t) = g(t)f(t, u(t)), \quad t \in (0, 1)$$

with boundary conditions including the following

$$\begin{gathered} u(0) = \alpha[u],\ u(1) = \beta[u], \\ u(0) = \alpha[u],\ u'(1) = \beta[u], \\ u(0) = \alpha[u],\ u'(1) + \beta[u] = 0, \\ u'(0) = \alpha[u],\ u(1) = \beta[u], \\ u'(0) + \alpha[u] = 0,\ u(1) = \beta[u], \end{gathered}$$

where α, β are linear functionals on $C[0, 1]$ given by

$$\alpha[u] = \int_0^1 u(s)\mathrm{d}A(s),\ \beta[u] = \int_0^1 u(s)\mathrm{d}B(s),$$

with A, B functions of bounded variation. By giving a general approach to cover all of these boundary conditions (and others) in a unified way, their work included much previous work as special cases and improved the corresponding results. We should note that the work of Webb and Infante does not require the functionals $\alpha[u], \beta[u]$ to be positive for all positive u.

Recently, Xu [24] considered the following second-order multi-point boundary value problem

$$\begin{cases} y''(t) + f(y) = 0, \quad 0 \le t \le 1, \\ y(0) = 0, \quad y(1) = \displaystyle\sum_{i=1}^{m-2} \alpha_i y(\eta_i), \end{cases} \tag{2}$$

where $f \in C(\mathbb{R}, \mathbb{R}), 0 < \alpha_i, i = 1, 2, \cdots, m-2, 0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$ with $\sum_{i-1}^{m-2} \alpha_i < 1$. By using fixed point index theory, under some suitable conditions, they obtained some existence results for multiple solutions including sign-changing solutions. In [17], Rynne carefully investigated the spectral properties of the linearisation of BVP (2) which are used to prove a Rabinowitz-type global bifurcation theorem for BVP (2). Then nodal solutions of the above problem are obtained by using this global bifurcation theorem. Thereafter, employing similar method to [24], Pang, Dong and Wei [16], Wei and Pang [23], Li and Liu [10] proved the existence of multiple solutions to some fourth two-point and multi-point boundary value problems, respectively.

Recently, utilizing the fixed point index theory and computing eigenvalues and the algebraic multiplicity of the corresponding linear operator, Li and Li [11] obtained some existence results for sign-changing solutions for the following integral boundary-value problem

$$\begin{cases} x''(t) + f(x(t)) = 0, \quad 0 \le t \le 1, \\ x(0) = 0, \quad x(1) = g\Big(\displaystyle\int_0^1 x(s)\mathrm{d}s\Big), \end{cases} \tag{3}$$

where $f, g \in C(\mathbb{R}, \mathbb{R})$.

To the authors' knowledge, there are few papers that have considered the existence of multiple sign-changing solutions for integral boundary value problems. Motivated by [11], [17], [19], [21], [22], the purpose of this paper is to investigate sign-changing solutions for BVP (1) following the method formulated by Xu in [24]. Obviously, BVP (1) can not be included in BVP (3). We will show that BVP (1) has at least six different nontrivial solutions when f satisfies certain conditions: two positive solutions, two negative solutions and two sign-changing solutions. Moreover, if f is also odd, then the BVP (1) has at least eight different nontrivial solutions, which are two positive, two negative and four sign-changing solutions.

We shall organize this paper as follows. In Section 2, some preliminaries and lemmas are given including the study of the eigenvalues of the linear operator $A'(\theta)$ and $A'(\infty)$. The main results are proved by using the fixed point index and Leray-Schauder degree method in section 3. A concrete example is given to illustrate the application of the main results in Section 4.

2 Preliminaries and several lemmas

Denote $\sigma_1 = \int_0^1 sa(s)\mathrm{d}s, \sigma_1^* = \int_0^1 s(1-s)a(s)\mathrm{d}s, \gamma = \frac{1+\int_0^1(1-s)a(s)\mathrm{d}s}{1-\sigma_1}$. Let

$$E = \left\{x \in C^1[0,1] : x(0) = 0,\ x(1) = \int_0^1 a(s)x(s)\mathrm{d}s\right\},$$

$$P = \{x \in E : x(t) \geq 0 \text{ for } t \in [0,1]\}.$$

For $x \in E$, let $\|x\| = \|x\|_0 + \|x'\|_0$, where $\|x\|_0 = \max_{t\in[0,1]} \|x(t)\|$ and $\|x'\|_0 = \max_{t\in[0,1]} \|x'(t)\|$. Then, $(E, \|\cdot\|)$ is a Banach space and P is a cone of E. Let

$$\alpha_0 = \lim_{x\to 0} \frac{f(x)}{x}, \quad \beta_0 = \lim_{|x|\to\infty} \frac{f(x)}{x}. \tag{4}$$

Define operators K, F and A as follows

$$(Kx)(t) = \int_0^1 \kappa(t,s)x(s)\mathrm{d}s, \quad (Fx)(t) = f(x(t)),\ t \in [0,1],\ \forall\, x \in E, \quad A = KF, \tag{5}$$

where

$$\kappa(t,s) = G(t,s) + \frac{t}{1-\sigma_1}\int_0^1 G(\tau,s)a(\tau)\mathrm{d}\tau, \tag{6}$$

$$G(t,s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases} \tag{7}$$

Denote the set of eigenvalues of the operator K by $\sigma(K)$, since K is a completely continuous operator, $\sigma(K)$ is countable. Throughout this paper, we adopt the following hypotheses.

(H_1) $f \in C(\mathbb{R}, \mathbb{R})$ and $f(x)x > 0$ for all $x \in \mathbb{R}\backslash\{0\}$;

(H_2) For each of the sets $\alpha_0\sigma(K)$ and $\beta_0\sigma(K)$, the number of elements greater than 1 is even, where $\alpha_0\sigma(K) = \{y : y = \alpha_0 x,\ x \in \sigma(K)\}$;

(H_3) There exists $r>0$ such that

$$|f(x)| < \frac{6(1-\sigma_1)}{(1-\sigma_1)(3+\gamma)+6\sigma_1^*} r,$$

for all x with $|x|<r$.

The main results of this paper are the following.

Theorem 1 *Suppose that* (H_1) $-$ (H_3) *hold. Then integral boundary-value problem (1) has at least two sign-changing solutions. Moreover, the integral boundary-value problem (1) also has at least two positive solutions and two negative solutions.*

Theorem 2 *Suppose that* (H_1) $-$ (H_3) *hold, and, f is odd, i.e.,* $f(-x)=-f(x)$ *for all* $x\in\mathbb{R}$. *Then integral boundary-value problem (1) has at least eight different nontrivial solutions, which are four sign-changing solutions, two positive solutions and two negative solutions.*

Before giving the proofs of Theorems 1 and 2, we list some preliminary lemmas.

Lemma 1 *For any* $u\in C[0,1]$, $x\in C^2[0,1]$ *is a solution of the following problem*

$$\begin{cases} -x''(t)=u(t), \quad 0<t<1, \\ x(0)=0, \quad x(1)=\displaystyle\int_0^1 a(s)x(s)\mathrm{d}s, \end{cases} \tag{8}$$

if and only if $x\in C[0,1]$ *is a solution of the integral equation*

$$x(t)=\int_0^1 \kappa(t,s)u(s)\mathrm{d}s. \tag{9}$$

Remark 1. Lemma 1 was shown in [1] by a direct calculation, a more general results for a Riemann-Stieltjes integral boundary value condition was shown in [21, 22] by a simple method.

Obviously, $A,K:E\to E$ are all completely continuous operators. By Lemma 1, we know that x is a solution of BVP (1) if and only if x is a fixed point of operator A.

Lemma 2 *([1])* *For* $t,s\in[0,1]$, *we have* $\kappa(t,s)\le\gamma e(s)$, *where* $e(s)=s(1-s)$.

Lemma 3 *Suppose that* (H_1) *and* (H_2) *hold. Then the operator* $A=KF$ *is Fréchet differentiable at* θ *and* ∞. *Moreover,* $A'(\theta)=\alpha_0 K$ *and* $A'(\infty)=\beta_0 K$.

Proof. For any $\varepsilon>0$, by (4), there exists $\delta>0$ such that for any $0<|x|<\delta$,

$$\left|\frac{f(x)}{x}-\alpha_0\right|<\varepsilon,$$

that is $|f(x)-\alpha_0 x|\le\varepsilon|x|$, for all $0\le|x|<\delta$. It is easy to see from (H_1) that $f(0)=0$. Then, for any $x\in E$ with $\|x\|<\delta$, by Lemma 2, we have

$$\begin{aligned} &|(Ax-A\theta-\alpha_0 Kx)(t)| = |(K(F(x)-\alpha_0 x))(t)| \\ &\le \gamma\int_0^1 \max_{s\in[0,1]}|f(x(s))-\alpha_0 x(s)|e(s)\mathrm{d}s \\ &\le \frac{1}{6}\gamma\|x\|_0\varepsilon \le \frac{1}{6}\gamma\|x\|\varepsilon, \quad t\in[0,1]. \end{aligned}$$

This implies

$$\|Ax - A\theta - \alpha_0 Kx\|_0 \le \frac{1}{6}\gamma\|x\|\varepsilon,\ x \in E,\ \|x\| < \delta. \tag{10}$$

On the other hand, for any $x \in E, \|x\| < \delta$,

$$\begin{aligned}
&|(Ax - A\theta - \alpha_0 Kx)'(t)| = \Big| \int_0^1 \partial_t \kappa(t,s)(f(x(s)) - \alpha_0 x(s))\mathrm{d}s \Big| \\
&\le \int_0^1 |\partial_t \kappa(t,s)||f(x(s)) - \alpha_0 x(s)|\mathrm{d}s \\
&\le \Big(\Big| -\int_0^t s\mathrm{d}s + \int_t^1 (1-s)\mathrm{d}s \Big| + \frac{\sigma_1^*}{1-\sigma_1} \Big) \max_{s\in[0,1]} |f(x(s)) - \alpha_0 x(s)|\mathrm{d}s \\
&\le \Big(\frac{1}{2} + \frac{\sigma_1^*}{1-\sigma_1} \Big) \max_{s\in[0,1]} |f(x(s)) - \alpha_0 x(s)| \\
&\le \Big(\frac{1}{2} + \frac{\sigma_1^*}{1-\sigma_1} \Big)\|x\|_0\varepsilon \le \Big(\frac{1}{2} + \frac{\sigma_1^*}{1-\sigma_1} \Big)\|x\|\varepsilon, \quad t \in [0,1].
\end{aligned}$$

Thus,

$$\|(Ax - A\theta - \alpha_0 Kx)'\|_0 \le \Big(\frac{1}{2} + \frac{\sigma_1^*}{1-\sigma_1} \Big)\|x\|\varepsilon,\ x \in E,\ \|x\| < \delta. \tag{11}$$

By (10) and (11),we have

$$\begin{aligned}
\|Ax - A\theta - \alpha_0 Kx\| &= \|Ax - A\theta - \alpha_0 Kx\|_0 + \|(Ax - A\theta - \alpha_0 Kx)'\|_0 \\
&\le \Big(\frac{1}{2} + \frac{\sigma_1^*}{1-\sigma_1} + \frac{\gamma}{6} \Big)\|x\|\varepsilon.
\end{aligned}$$

Therefore,

$$\lim_{\|x\|\to 0} \frac{\|Ax - A\theta - \alpha_0 Kx\|}{\|x\|} = 0,$$

which means that A is Fréchet differentiable at θ and $A'(\theta) = \alpha_0 K$.

For any $\varepsilon > 0$, by (4), there exists $R > 0$ such that

$$|f(x) - \beta_0(x)| < \varepsilon|x|$$

for any $|x| > R$. Let $b = \max_{\|x\|\le R} |f(x) - \beta_0(x)|$. Then, we have for any $x \in \mathbb{R}$

$$|f(x) - \beta_0(x)| \le \varepsilon|x| + b.$$

As a consequence,

$$\begin{aligned}
&|(Ax - \beta_0 Kx)(t)| = |(K(F(x) - \beta_0 x))(t)| \\
&\le \gamma \int_0^1 \max_{s\in[0,1]} |f(x(s)) - \beta_0 x(s)|e(s)\mathrm{d}s \\
&\le \frac{\gamma}{6}(\|x\|_0\varepsilon + b) \le \frac{\gamma}{6}(\|x\|\varepsilon + b), \quad t \in [0,1].
\end{aligned}$$

This implies that

$$\|Ax - \beta_0 Kx\|_0 \le \frac{\gamma}{6}(\|x\|\varepsilon + b),\ x \in E. \tag{12}$$

Similarly, we can show that

$$\|(Ax-\beta_0 Kx)'\|_0 \le \Big(\frac{1}{2}+\frac{\sigma_1^*}{1-\sigma_1}\Big)(\|x\|\varepsilon+b),\ x\in E. \tag{13}$$

By (12) and (13), we have

$$\begin{aligned}\|Ax-\beta_0 Kx\| &= \|Ax-\beta_0 Kx\|_0+\|(Ax-\beta_0 Kx)'\|_0\\ &\le \Big(\frac{1}{2}+\frac{\sigma_1^*}{1-\sigma_1}+\frac{\gamma}{6}\Big)(\|x\|\varepsilon+b).\end{aligned}$$

Consequently,

$$\lim_{\|x\|\to\infty}\frac{\|Ax-\beta_0 Kx\|}{\|x\|}=0,$$

which means that A is Fréchet differentiable at ∞ and $A'(\infty)=\beta_0 K$. $\square$

Lemma 4 *Let β be a positive number. Then the sequence of positive eigenvalues of the linear operator βK is countable. Moreover, the algebraic multiplicity of each positive eigenvalue of the operator βK is equal to 1.*

Proof. The set of positive eigenvalues of the linear operator βK is countable because $K: C[0,1]\to C[0,1]$ is a completely continuous operator.

To prove all eigenvalues are simple we can, without loss of generality, take $\beta=1$.

An eigenvector x of K corresponding to $\lambda>0$ is a nonzero solution of the problem

$$\begin{gathered}-\lambda x''(t)+x(t)=0,\ t\in(0,1),\\ x(0)=0,\ x(1)=\int_0^1 a(s)x(s)\mathrm{d}s.\end{gathered}$$

Therefore the eigenfunctions are scalar multiples of $\sin(kt)$ and $\lambda=1/k^2$ where k is one of the positive solutions of the equation

$$\sin(k)=\int_0^1 a(s)\sin(ks)\mathrm{d}s. \tag{14}$$

Next, we show that the algebraic multiplicity of each positive eigenvalue of the operator K is equal to 1. To do this we will show that for each eigenvalue $\lambda=1/k^2$, where k is a solution of (14), the following inclusion is valid,

$$\ker(\lambda I-K)^2\subset\ker(\lambda I-K).$$

Let $y\in\ker(\lambda I-K)^2$, then $(\lambda I-K)y\in\ker(\lambda I-K)$ so

$$(\lambda I-K)y=C\sin(kt),\quad \text{for some constant } C.$$

If $C=0$ the result is shown. Suppose that $C\ne 0$. Then, by Lemma 1, y is a solution of the problem

$$\lambda y''+y=-Ck^2\sin(kt),\ y(0)=0,\ y(1)=\int_0^1 a(s)y(s)\mathrm{d}s,$$

that is,

$$y'' + k^2 y = -Ck^4 \sin(kt),\ y(0) = 0,\ y(1) = \int_0^1 a(s)y(s)\mathrm{d}s.$$

The solution of this problem is of the form $c_1 \sin(kt) + c_2\, t\cos(kt)$, where $C \neq 0$ implies $c_2 \neq 0$. Since the term $\sin(kt)$ satisfies the boundary condition at $t = 1$ so must the term $t\cos(kt)$, that is,

$$\cos(k) = \int_0^1 a(s)s\cos(ks)\,\mathrm{d}s. \tag{15}$$

Noting that $\int_0^1 w(s)\mathrm{d}s \leq \left(\int_0^1 w^2(s)\mathrm{d}s\right)^{1/2}$, from (14) and (15) we have

$$\sin(k) \leq \left(\int_0^1 a^2(s)\sin^2(ks)\mathrm{d}s\right)^{1/2},\ \ \cos(k) \leq \left(\int_0^1 a^2(s)\cos^2(ks)\mathrm{d}s\right)^{1/2},$$

hence

$$1 = \sin^2(k) + \cos^2(k) \leq \int_0^1 a^2(s)(\sin^2(ks) + \cos^2(ks))\mathrm{d}s = \int_0^1 a^2(s)\mathrm{d}s < 1.$$

This contradiction proves that we must have $C = 0$ so the eigenvalue is simple. $\square$

Lemma 5 *Suppose that condition* (H_1) *holds. If $x \in P\backslash\{\theta\}$ is a solution of BVP (1), then $x \in \overset{\circ}{P}$. Similarly, if $x \in -P\backslash\{\theta\}$ is a solution of BVP (1), then $x \in -\overset{\circ}{P}$.*

Proof. Let $x \in P\backslash\{\theta\}$ is a solution of BVP (1). Then

$$x(t) = \int_0^1 \kappa(t,s)f(s,x(s))\mathrm{d}s = \int_0^1 G(t,s)f(x(s))\mathrm{d}s + \frac{t}{1-\sigma_1}\int_0^1\int_0^1 G(s,\tau)f(x(s))\mathrm{d}\tau\mathrm{d}s.$$

For all $t \in [0,1]$, we have

$$x(t) > 0,\quad t \in (0,1],$$

and

$$x'(0) = \int_0^1 (1-s)f(x(s))\mathrm{d}s + \frac{1}{1-\sigma_1}\int_0^1\int_0^1 G(s,\tau)f(x(s))\mathrm{d}\tau\mathrm{d}s > 0.$$

As the proof of Lemma 2.7 in [24], we can show that $x \in \overset{\circ}{P}$. $\square$

We recall three well known lemmas, they can be found, for example, in [2].

Lemma 6 *Let $\theta \in \Omega$ and $A : P \cap \overline{\Omega} \to P$ be completely continuous. Suppose that*

$$Ax \neq \mu x,\ \forall\ x \in \partial\Omega,\ \mu \geq 1.$$

Then $i(A, P\cap\Omega, P) = 1$.

Lemma 7 *Let Ω be an open set in E and $\theta \in \Omega, A : \overline{\Omega} \to E$ be completely continuous. Suppose that*

$$\|Ax\| \leq \|x\|,\ Ax \neq x,\ \forall\ x \in \partial\Omega.$$

Then $\deg(I - A, \Omega, \theta) = 1$.

Lemma 8 *Let $A : P \to P$ be completely continuous. Suppose that A is differentiable at θ and ∞ along P and 1 is not an eigenvalue of $A'_+(\theta)$ and $A'_+(\infty)$ corresponding to a positive eigenfunction.*

(i) *If $A'_+(\theta)$ has a positive eigenfunction corresponding to an eigenvalue greater than 1, and $A\theta = \theta$. Then there exists $\tau > 0$ such that $i(A, P \cap B(\theta, r), P) = 0$ for any $0 < r < \tau$.*

(ii) *If $A'_+(\infty)$ has a positive eigenfunction which corresponds to an eigenvalue greater than 1. Then there exists $\zeta > 0$ such that $i(A, P \cap B(\theta, R), P) = 0$ for any $R > \zeta$.*

We now prove an important result for our work.

Lemma 9 *Suppose that* $(\mathrm{H}_1) - (\mathrm{H}_3)$ *hold. Then*

(i) *There exists $r_0 \in (0, r)$ such that for all $\tau \in (0, r_0]$,*

$$i(A, P \cap B(\theta, \tau), P) = 0, \quad i(A, -P \cap B(\theta, \tau), -P) = 0.$$

(ii) *There exists $R_0 > r$ such that for all $R \geq R_0$,*

$$i(A, P \cap B(\theta, R), P) = 0, \quad i(A, -P \cap B(\theta, R), -P) = 0.$$

Proof. We prove only conclusion (i), conclusion (ii) can be proved in a similar way. By Lemma 3, $A'_+(0) = \alpha_0 K$. An eigenvalue of $\alpha_0 K$ is $\alpha_0 \lambda$ where λ is an eigenvalue of K, that is, $\lambda = 1/k^2$ where k is a solution of

$$\sin(k) = \int_0^1 a(s) \sin(ks) \mathrm{d}s, \tag{16}$$

and the corresponding eigenfunction is $\sin(kt)$. By assumption, there are an even number of eigenvalues of $\alpha_0 K$ greater than 1, in particular, $\alpha_0/k_1^2 > 1$ where k_1 is the smallest positive root of (16). We show that $k_1 \in (0, \pi)$ so that the corresponding eigenfunction is positive on $(0, 1)$.

Define a real function F by

$$F(k) := \sin(k) - \int_0^1 a(s) \sin(ks) \mathrm{d}s.$$

Then F is continuous and $F(\pi) = -\int_0^1 a(s) \sin(\pi s)\, ds < 0$. Also $F(0) = 0$ and

$$F'(k) = \cos(k) - \int_0^1 a(s) s \cos(ks) \mathrm{d}s,$$

so $F'(0) = 1 - \int_0^1 a(s) s \, \mathrm{d}s > 0$. Thus $F(\delta) > 0$ for $\delta > 0$ sufficiently small and so, by the Intermediate Value Theorem, there exists a root in (δ, π), in particular $k_1 \in (0, \pi)$.

It follows from Lemma 8 that there exists $\tau_0 > 0$ such that $i(A, P \cap B(\theta, \tau), P) = 0$ for any $0 < \tau \leq \tau_0$.

Similarly, we can show that there exists $\tau_1 > 0$ such that $i(A, -P \cap B(\theta, \tau), -P) = 0$ for any $0 < \tau \leq \tau_1$. Let $r_0 = \min\{\tau_0, \tau_1\}$. Then, the conclusion (i) holds. □

From [9, Theorem 21.6, 21.2], we have the following two lemmas.

Lemma 10 *Let A be a completely continuous operator, let $x_0 \in E$ be a fixed point of A and assume that A is defined in a neighborhood of x_0 and Fréchet differentiable at x_0. If 1 is not an* *eigenvalue of the linear operator $A'(x_0)$, then x_0 is an isolated singular point of the completely continuous vector field $I - A$ and for small enough $r > 0$*

$$\deg(I - A, B(x_0, r), \theta) = (-1)^k,$$

where k is the sum of the algebraic multiplicities of the real eigenvalues of $A'(x_0)$ in $(1, +\infty)$.

Lemma 11 *Let A be a completely continuous operator which is defined on a Banach space E. Assume that 1 is not an eigenvalue of the asymptotic derivative. The completely continuous vector field $I - A$ is then nonsingular on spheres $S_\rho = \{x : \|x\| = \rho\}$ of sufficiently large radius ρ and*

$$\deg(I - A, B(\theta, \rho), \theta) = (-1)^k,$$

where k is the sum of the algebraic multiplicities of the real eigenvalues of $A'(\infty)$ in $(1, +\infty)$.

To simplify the proof of the main results, we will use the following lemma.

Lemma 12 *([16]) Let P be a solid cone of a real Banach space E, Ω be a relatively bounded open subset of P, $A : P \to P$ be a completely continuous operator. If all fixed point of A in Ω is the interior point of P, there exists an open subset O of E such that $O \subset \Omega$ and*

$$\deg(I - A, O, 0) = i(A, \Omega, P).$$

3 Proof of main results

Proof of Theorem 1 By the condition (H_3) and Lemma 2, for any $x \in E$ with $\|x\| = r$, we have

$$\begin{aligned} |(Ax)(t)| &= \Big|\int_0^1 \kappa(t,s) f(x(s))\mathrm{d}s\Big| \le \gamma \int_0^1 e(s) \max_{s\in[0,1]} |f(x(s))|\mathrm{d}s \\ &< \frac{\gamma}{6} \times \frac{6(1-\sigma_1)}{(1-\sigma_1)(3+\gamma)+6\sigma_1^*} r = \frac{\gamma(1-\sigma_1)}{(1-\sigma_1)(3+\gamma)+6\sigma_1^*} r, \quad t \in [0,1]. \end{aligned}$$

Therefore,

$$\|Ax\|_0 < \frac{\gamma(1-\sigma_1)}{(1-\sigma_1)(3+\gamma)+6\sigma_1^*} r. \tag{17}$$

Similarly, we can show that for any $x \in E$ with $\|x\| = r$

$$\|(Ax)'\|_0 < \Big(\frac{1}{2} + \frac{\sigma_1^*}{1-\sigma_1}\Big) \times \frac{6(1-\sigma_1)}{(1-\sigma_1)(3+\gamma)+6\sigma_1^*} r = \frac{3(1-\sigma_1)+6\sigma_1^*}{(1-\sigma_1)(3+\gamma)+6\sigma_1^*} r. \tag{18}$$

It follows from (17) and (18) that $\|Ax\| < r$ for all $\|x\| = r$. Then by Lemma 6 and 7, we have

$$i(A, P \cap B(\theta, r), P) = 1, \tag{19}$$

$$i(A, -P \cap B(\theta, r), -P) = 1, \tag{20}$$

$$\deg(I - A, B(\theta, r), \theta) = 1. \tag{21}$$

By (H_2) and Lemma 4, the sum of the algebraic multiplicities of the real eigenvalues of the operator $A'(\theta) = \alpha_0 K$ in $(1, +\infty)$ is even. Therefore, there exists $0 < r_1 < r_0$ such that

$$\deg(I - A, B(\theta, r_1), \theta) = 1, \tag{22}$$

where r_0 has the same meaning as that in Lemma 9. Similarly, by Lemmas 4, 11 and (H_2), we have for some $R_1 \geq R_0$

$$\deg(I - A, B(\theta, R_1), \theta) = 1, \tag{23}$$

where R_0 has the same meaning as that in Lemma 9. By Lemma 9, we have

$$i(A, P \cap B(\theta, r_1), P) = 0, \tag{24}$$

$$i(A, -P \cap B(\theta, r_1), -P) = 0, \tag{25}$$

$$i(A, P \cap B(\theta, R_1), P) = 0, \tag{26}$$

$$i(A, -P \cap B(\theta, R_1), -P) = 0. \tag{27}$$

Then, by (19), (24) and (26), we have

$$i(A, P \cap (B(\theta, R_1) \backslash \overline{B(\theta, r)}), P) = 0 - 1 = -1, \tag{28}$$

$$i(A, P \cap (B(\theta, r) \backslash \overline{B(\theta, r_1)}), P) = 1 - 0 = 1. \tag{29}$$

Thus, the operator A has at least two fixed points $x_1 \in P \cap (B(\theta, R_1) \backslash \overline{B(\theta, r)})$ and $x_2 \in P \cap (B(\theta, r) \backslash \overline{B(\theta, r_1)})$, respectively. Obviously, x_1 and x_2 are two distinct positive solutions to the BVP (1).

Similarly, by (20), (25) and (27), we have

$$i(A, -P \cap (B(\theta, R_1) \backslash \overline{B(\theta, r)}), -P) = -1, \tag{30}$$

$$i(A, -P \cap (B(\theta, r) \backslash \overline{B(\theta, r_1)}), -P) = 1 - 0 = 1. \tag{31}$$

Therefore, the operator A has at least two fixed points $x_3 \in (-P) \cap (B(\theta, R_1) \backslash \overline{B(\theta, r)})$ and $x_4 \in (-P) \cap (B(\theta, r) \backslash \overline{B(\theta, r_1)})$, respectively. It is clear that x_3 and x_4 are two distinct negative solutions to the BVP (1).

By Lemmas 5 and 12, (28)-(31), there exist four open subsets O_1, O_2, O_3 and O_4 of E such that

$$O_1 \subset P \cap (B(\theta, R_1) \backslash \overline{B(\theta, r)}), \quad O_2 \subset P \cap (B(\theta, r) \backslash \overline{B(\theta, r_1)}),$$

$$O_3 \subset -P \cap (B(\theta, r) \backslash \overline{B(\theta, r_1)}), \quad O_4 \subset -P \cap (B(\theta, R_1) \backslash \overline{B(\theta, r)}),$$

and

$$\deg(I - A, O_1, 0) = -1, \tag{32}$$

$$\deg(I - A, O_2, 0) = 1, \tag{33}$$

$$\deg(I - A, O_3, 0) = 1, \tag{34}$$

$$\deg(I - A, O_4, 0) = -1. \tag{35}$$

It follows from (21), (22), (33) and (34) that

$$\deg(I - A, B(\theta, r)\backslash(\overline{O}_2 \cup \overline{O}_3) \cup \overline{B(\theta, r_1)}), 0) = 1 - 1 - 1 - 1 = -2, \tag{36}$$

which means that A has at least one fixed point $x_5 \in B(\theta, r)\backslash(\overline{O}_2 \cup \overline{O}_3) \cup \overline{B(\theta, r_1)})$. Similarly, by (21), (23), (32) and (35) we get

$$\deg(I - A, B(\theta, R_1)\backslash(\overline{O}_1 \cup \overline{O}_4 \cup \overline{B(\theta, r)}), 0) = 1 + 1 + 1 - 1 = 2.$$

This implies that A has at least one fixed point $x_6 \in B(\theta, R_1)\backslash(\overline{O}_1 \cup \overline{O}_4 \cup \overline{B(\theta, r)})$. Obviously, x_5 and x_6 are two distinct sign-changing solutions to BVP (1). □

Proof of Theorem 2 According to Theorem 1, the BVP (1) has at least six different nontrivial solutions $x_i \in E$, $i = 1, 2, \cdots, 6$, satisfying

$$x_1,\ x_2 \in \mathring{P}, \quad x_3,\ x_4 \in -\mathring{P}, \quad x_5,\ x_6 \notin P \cup (-P),$$

$$r_1 < \|x_5\| < r < \|x_6\| < R_1.$$

It follows from $f(-x) = -f(x)$ for all $x \in \mathbb{R}$ that $-x_5$ and $-x_6$ are also solution for the BVP (1). Denote $x_7 = -x_5, x_8 = -x_6$. It is clear x_7 and x_8 are two sign-changing solutions, too. So, BVP (1) has eight different nontrivial solutions $x_i,\ i = 1, 2, \cdots, 8$. □

4 An example

Consider the following nonlinear second-order integral boundary-value problem (BVP)

$$\begin{cases} x''(t) + f(x(t)) = 0, \quad 0 \le t \le 1, \\ x(0) = 0, \quad x(1) = \displaystyle\int_0^1 s x(s) \mathrm{d}s, \end{cases} \tag{37}$$

where

$$f(x) = \begin{cases} 40(x - x^{\frac{23}{21}}), \quad |x| \le \dfrac{1}{2}, \\ (n - m)x + \dfrac{3}{2}m - \dfrac{1}{2}n, \quad \dfrac{1}{2} < x < \dfrac{3}{2}, \\ (n - m)x + \dfrac{n}{2} - \dfrac{3}{2}m, \quad -\dfrac{3}{2} < x < -\dfrac{1}{2}, \\ 38(x - x^{\frac{57}{59}}), \quad |x| > \dfrac{3}{2}, \end{cases}$$

in which $m = 20(1 - 2^{-\frac{2}{21}}), n = 57(1 - (\frac{3}{2})^{-\frac{2}{59}})$.

Conclusion: The conclusion of Theorem 2 holds for integral boundary value problem (37).

Let

$$(K_1x)(t) = \int_0^1 \kappa_1(t,s)x(s)\mathrm{d}s, \tag{38}$$

where

$$\kappa_1(t,s) = G(t,s) + \frac{1}{4}ts(1-s^2). \tag{39}$$

Let $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \lambda_{n+1} < \cdots$ be the sequence of positive solutions of the equation

$$(1-x)\sin\sqrt{x} = \sqrt{x}\cos\sqrt{x}. \tag{40}$$

Proof of the conclusion. Let β be a positive number. First we show that the sequence of positive eigenvalues of the operator βK_1 is of the form

$$\frac{\beta}{\lambda_1} > \frac{\beta}{\lambda_2} > \cdots > \frac{\beta}{\lambda_n} > \cdots.$$

Let $\overline{\lambda}$ be a positive eigenvalue of the linear operator βK_1, and $x \in E\backslash\{\theta\}$ be an eigenfunction corresponding to the eigenvalue $\overline{\lambda}$. Then we have

$$\overline{\lambda}x = \beta K_1 x, \text{ i.e., } K_1 x = \frac{\overline{\lambda}}{\beta}x.$$

By Lemma 1, we obtain

$$\Big(\frac{\overline{\lambda}}{\beta}x\Big)'' + x = 0.$$

That is

$$x'' + \frac{\beta}{\overline{\lambda}}x = 0. \tag{41}$$

The differential equation (41) has roots $\pm\sqrt{\frac{\beta}{\overline{\lambda}}}i$. Thus, the general solution of (41) can be expressed by

$$x(t) = C_1\cos t\sqrt{\frac{\beta}{\overline{\lambda}}} + C_2\sin t\sqrt{\frac{\beta}{\overline{\lambda}}}, \quad t\in[0,1].$$

Applying the boundary value condition $x(0)=0$, we obtain that $C_1 = 0$, and so the general solution can be reduced to

$$x(t) = C_2\sin t\sqrt{\frac{\beta}{\overline{\lambda}}}, \quad t\in[0,1],$$

which together with boundary condition $x(1) = \int_0^1 sx(s)\mathrm{d}s$ and integrate by parts shows that

$$\begin{aligned}\sin\sqrt{\tfrac{\beta}{\overline{\lambda}}} &= \int_0^1 s\sin\Big(s\sqrt{\frac{\beta}{\overline{\lambda}}}\Big)\mathrm{d}s\\ &= -\frac{\cos\sqrt{\frac{\beta}{\overline{\lambda}}}}{\sqrt{\frac{\beta}{\overline{\lambda}}}} + \frac{1}{\frac{\beta}{\overline{\lambda}}}\sin\sqrt{\frac{\beta}{\overline{\lambda}}}.\end{aligned} \tag{42}$$

That is

$$\Big(1-\frac{\beta}{\overline{\lambda}}\Big)\sin\sqrt{\frac{\beta}{\overline{\lambda}}} = \sqrt{\frac{\beta}{\overline{\lambda}}}\cos\sqrt{\frac{\beta}{\overline{\lambda}}}.$$

Since the positive solutions of the equation $(1-x)\sin\sqrt{x} = \sqrt{x}\cos\sqrt{x}$ are $0 < \lambda_1 < \lambda_2 < \cdots$, then $\overline{\lambda}$ is one of the values

$$\frac{\beta}{\lambda_1} > \frac{\beta}{\lambda_2} > \cdots \frac{\beta}{\lambda_n} > \cdots,$$

and the eigenfunction corresponding to the eigenvalue $\frac{\beta}{\lambda_n}$ is

$$x_n(t) = C\sin t\sqrt{\lambda_n}, \quad t \in [0,1],$$

where C is a nonzero constant.

Next, we check that all the conditions of Theorem 2 hold. Take $a(s) = s$. It is clear that $\int_0^1 a^2(s)\mathrm{d}s = \frac{1}{3} < 1$. Obviously, f is odd, and condition (H_1) holds. By direct computation, we have $\sigma_1 = \frac{1}{3}, \sigma_1^* = \frac{1}{12}, \gamma = \frac{7}{4}, \frac{6(1-\sigma_1)}{(1-\sigma_1)(3+\gamma)+6\sigma_1^*} = \frac{12}{11}, \lambda_1 \approx 7.5279, \lambda_2 \approx 37.4148, \lambda_3 \approx 86.7993, \alpha_0 = 40, \beta_0 = 38$. By above statement we know that the sequence of positive eigenvalues of the operators $\alpha_0 K_1, \beta_0 K_1$ are of the forms $\frac{\alpha_0}{\lambda_i}, \frac{\beta_0}{\lambda_i}$ $(i = 1, 2, \cdots)$, respectively. Thus, the numbers of the eigenvalues belong to $(1, +\infty)$ of the operators $\alpha_0 K$ and $\beta_0 K$ are both two. Hence, (H_2) holds. Choose $r = 2$. By direct computation, we know

$$|f(x)| \leq \begin{cases} f((\frac{21}{23})^{\frac{21}{2}}) = 40((\frac{21}{23})^{\frac{21}{2}} - (\frac{21}{23})^{\frac{23}{2}}) \approx 1.3382 < 2 \leq \frac{12}{11}r, & |x| \leq 1, \\ f(2) = 38(2 - 2^{\frac{57}{59}}) \approx 1.7649 < 2 \leq \frac{12}{11}r, & 1 < |x| < 2. \end{cases}$$

Thus, we have proved that (H_3) holds for $r = 2$. As a consequence, our conclusion follows from Theorem 2. □

Remark 3. It is clear that $\kappa_1(t,s) \leq \frac{3}{2}s(1-s)$. Thus, the constant $\gamma = \frac{7}{4}$ in this example is not optimal and the optimal constant is $\frac{3}{2}$.

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