Multiple positive solutions for (n-1, 1)-type semipositone conjugate boundary value problems of nonlinear fractional differential equations*

Chengjun Yuan†

School of Mathematics and Computer, Harbin University, Harbin 150086, Heilongjiang, P.R.China.

Abstract. In this paper, we consider (n-1, 1)-type conjugate boundary value problem for the nonlinear fractional differential equation

$$\begin{aligned}
&\mathbf{D}_{0+}^{\alpha}u(t)+\lambda f(t,u(t))=0, \quad 0<t<1, \lambda>0,\\
&u^{(j)}(0)=0, \quad 0\le j\le n-2,\\
&u(1)=0,
\end{aligned}$$

where λ is a parameter, $\alpha\in(n-1,n]$ is a real number and $n\ge 3$, and $\mathbf{D}_{0+}^{\alpha}$ is the Riemann-Liouville's fractional derivative, and f is continuous and semipositone. We give properties of Green's function of the boundary value problems, and derive an interval of λ such that any λ lying in this interval, the semipositone boundary value problem has multiple positive solutions.

Key words. Riemann-Liouville's fractional derivative; fractional differential equation; boundary value problem; positive solution; fractional Green's function; fixed-point theorem.

MR(2000) Subject Classifications: 34B15

1 Introduction

We consider the (n-1, 1)-type conjugate boundary value problem of nonlinear fractional differential equation involving Riemann-Liouville's derivative

$$\begin{aligned}
&\mathbf{D}_{0+}^{\alpha}u(t)+\lambda f(t,u(t))=0, \quad 0<t<1, \lambda>0,\\
&u^{(j)}(0)=0, \quad 0\le j\le n-2,\\
&u(1)=0,
\end{aligned} \tag{1.1}$$

where λ is a parameter, $\alpha\in(n-1,n]$ is a real number and $n\ge 3$, $\mathbf{D}_{0+}^{\alpha}$ is the Riemann-Liouville's fractional derivative, $f:(0,1)\times[0,+\infty)\to(-\infty,+\infty)$ is a sign-changing continuous function. As far as we know, there are few papers which deal with the boundary value problem for nonlinear fractional differential equation.

Because of fractional differential equation's modeling capabilities in engineering, science, economy, and other fields, the last few decades has resulted in a rapid development of the theory of fractional differential equation; see [1]-[7] for a good overview. Within this development, a fair amount of the theory has been devoted to initial and boundary value problems problems (see [9]-[20]). In most papers, the definition of fractional derivative is the Riemann-Liouville's fractional derivative or the Caputo's fractional derivative. For details see references.

In this paper, we give sufficient conditions for the existence of positive solution of the semipositone boundary value problems (1.1) for a sufficiently small $\lambda>0$ where f may change sign. Our analysis relies on nonlinear alternative of Leray-Schauder type and Krasnosel'skii's fixed-point theorems.

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†Corresponding author: C.J. Yuan, *E-mail address:* ycj7102@163.com

2 Preliminaries

For completeness, in this section, we will demonstrate and study the definitions and some fundamental facts of Riemann-Liouville's derivatives of fractional order which can be found in [3].

Definition 1.1 [3] The integral

$$I_{0+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad x > 0$$

where $\alpha > 0$, is called Riemann-Liouville fractional integral of order α.

Definition 1.2 [3] For a function $f(x)$ given in the interval $[0, \infty)$, the expression

$$D_{0+}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dx})^n \int_0^x \frac{f(t)}{(x-t)^{\alpha-n+1}} dt$$

where $n = [\alpha] + 1, [\alpha]$ denotes the integer part of number α, is called the Riemann-Liouville fractional derivative of order s.

From the definition of Riemann-Liouville's derivative, we can obtain the statement.

As examples, for $\mu > -1$, we have

$$\mathbf{D}_{0+}^{\alpha} x^{\mu} = \frac{\Gamma(1+\mu)}{\Gamma(1+\mu-\alpha)} x^{\mu-\alpha}$$

giving in particular $\mathbf{D}_{0+}^{\alpha} x^{\alpha-m},\ m = i, 2, 3, \cdots, N$, where N is the smallest integer greater than or equal to α.

Lemma 2.1 *Let $\alpha > 0$, then the differential equation*

$$\mathbf{D}_{0+}^{\alpha} u(t) = 0$$

has solutions $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},\ c_i \in \mathbb{R},\ i = 1,,2\ldots,n$, as unique solutions, where n is the smallest integer greater than or equal to α.

As $\mathbf{D}_{0+}^{\alpha} I_{0+}^{\alpha} u = u$ From the lemma 2.1, we deduce the following statement.

Lemma 2.2 *Let $\alpha > 0$, then*

$$I_{0+}^{\alpha} \mathbf{D}_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

for some $c_i \in \mathbb{R},\ i = 1, 2, \ldots, n,\ n$ is the smallest integer greater than or equal to α.

The following a nonlinear alternative of Leray-Schauder type and Krasnosel'skii's fixed-point theorems, will play major role in our next analysis.

Theorem 2.3 ([12]) *Let X be a Banach space with $\Omega \subset X$ be closed and convex. Assume U is a relatively open subsets of Ω with $0 \in U$, and let $S : \overline{U} \to \Omega$ be a compact, continuous map. Then either*

1. *S has a fixed point in $\overline{U}$, or*
2. *there exists $u \in \partial U$ and $\nu \in (0,1)$, with $u = \nu S u$.*

Theorem 2.4 ([8]) *Let X be a Banach space, and let $P \subset X$ be a cone in X. Assume Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $S : P \to P$ be a completely continuous operator such that, either*

1. *$\|Sw\| \le \|w\|,\ w \in P \cap \partial\Omega_1,\ \|Sw\| \ge \|w\|,\ w \in P \cap \partial\Omega_2$, or*
2. *$\|Sw\| \ge \|w\|,\ w \in P \cap \partial\Omega_1,\ \|Sw\| \le \|w\|\ w \in P \cap \partial\Omega_2$.*

Then S has a fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3 Green's Function and Its Properties

In this section, we derive the corresponding Green's function for boundary value problem (1.1), and obtained some properties of the Green's function. First of all, we find the Green's function for boundary-value problem (1.1).

Lemma 3.1 *Let $h(t) \in C[0,1]$ be a given function, then the boundary-value problem*

$$\begin{aligned} &\mathbf{D}_{0+}^{\alpha} u(t) + h(t) = 0, \quad 0 < t < 1, 2 \le n-1 < \alpha \le n, \\ &u^{(j)}(0) = 0, \quad 0 \le j \le n-2, \\ &u(1) = 0 \end{aligned} \tag{3.1}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)h(s)ds, \tag{3.2}$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-1}, & 0 \le t \le s \le 1. \end{cases} \tag{3.3}$$

Here $G(t,s)$ is called the Green's function of boundary value problem (3.1).

Proof By means of the Lemma2.2, we can reduce (3.1) to an equivalent integral equation

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n} - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds.$$

From $u^{(j)}(0) = 0, 0 \le j \le n-2$, we have $c_j = 0, 2 \le j \le n$. Then

$$u(t) = c_1 t^{\alpha-1} - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds.$$

From $u(1) = 0$, we have $c_1 = \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds$. Then, the unique solution of (3.1) is

$$u(t) = \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds = \int_0^1 G(t,s)h(s)ds.$$

Lemma 3.2 *The Green's function $G(t,s)$ defined by* (3.3) *has the following properties:*

(R1) $G(t,s) = G(1-s, 1-t)$, for $t,s \in [0,1]$,

(R2) $\Gamma(\alpha)k(t)q(s) \le G(t,s) \le (\alpha-1)q(s)$, for $t,s \in [0,1]$,

(R3) $\Gamma(\alpha)k(t)q(s) \le G(t,s) \le (\alpha-1)k(t)$, for $t,s \in [0,1]$,

where

$$k(t) = \frac{t^{\alpha-1}(1-t)}{\Gamma(\alpha)}, \quad q(s) = \frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha)}. \tag{3.4}$$

Proof (R1) From the definition of $G(t,s)$, it is obviously that $G(t,s) = G(1-s, 1-t)$, for $t,s \in (0,1)$.

(R2) For $s \le t$, we have $1-s \ge 1-t$, then

$$\begin{aligned} \Gamma(\alpha)G(t,s) &= t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1} = (\alpha-1)\int_{t-s}^{t-ts} x^{\alpha-2}dx \\ &\le (\alpha-1)(t-ts)^{\alpha-2}((t-ts)-(t-s)) \\ &= (\alpha-1)t^{\alpha-2}(1-s)^{\alpha-2}(1-t)s \\ &\le (\alpha-1)s(1-s)^{\alpha-1} \\ &= \Gamma(\alpha)(\alpha-1)q(s) \end{aligned}$$

and

$$\begin{aligned}\Gamma(\alpha)G(t,s) &= t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1} = (t-ts)^{\alpha-2}(t-ts) - (t-s)^{\alpha-2}(t-s) \\ &\geq (t-ts)^{\alpha-2}(t-ts) - (t-ts)^{\alpha-2}(t-s) \\ &= t^{\alpha-2}(1-s)^{\alpha-2}(1-t)s \\ &\geq t^{\alpha-1}(1-t)s(1-s)^{\alpha-1} \\ &= \Gamma^2(\alpha)k(t)q(s).\end{aligned}$$

For $t \leq s$, since $\alpha > 2$, we have

$$\begin{aligned}\Gamma(\alpha)G(t,s) &= t^{\alpha-1}(1-s)^{\alpha-1} \leq (\alpha-1)t^{\alpha-2}t(1-s)^{\alpha-1} \\ &\leq (\alpha-1)t^{\alpha-2}s(1-s)^{\alpha-1} \leq (\alpha-1)s(1-s)^{\alpha-1} \\ &= \Gamma(\alpha)(\alpha-1)q(s)\end{aligned}$$

and

$$\Gamma(\alpha)G(t,s) = t^{\alpha-1}(1-s)^{\alpha-1} \geq t^{\alpha-1}(1-t)s(1-s)^{\alpha-1} = \Gamma^2(\alpha)k(t)q(s).$$

Thus $\Gamma(\alpha)k(t)q(s) \leq G(t,s) \leq \Gamma(\alpha)(\alpha-1)q(s)$, for $t, s \in (0,1)$.

(R3) From (R1) and (R2), we have

$$\Gamma(\alpha)k(t)q(s) \leq G(t,s) = G(1-s,1-t) \leq (\alpha-1)q(1-t) = (\alpha-1)k(t).$$

This is completes the proof.

4 Main Results

We make the following assumptions:

(H_1) $f(t,u) \in C([0,1] \times [0,+\infty), (-\infty,+\infty))$, moreover there exists a function $g(t) \in L^1([0,1],(0,+\infty))$ such that $f(t,u) \geq -g(t)$, for any $t \in (0,1)$, $u \in [0,+\infty)$.

(H_1^*) $f(t,u) \in C((0,1) \times [0,+\infty), (-\infty,+\infty))$ may be singular at $t = 0, 1$, moreover there exists a function $g(t) \in L^1((0,1),(0,+\infty))$ such that $f(t,u) \geq -g(t)$, for any $t \in (0,1)$, $u \in [0,+\infty)$.

(H_2) $f(t,0) > 0$, for any $t \in [0,1]$.

(H_3) There exists $[\theta_1,\theta_2] \in (0,1)$ such that $\lim\limits_{u\uparrow+\infty} \min\limits_{t\in[\theta_1,\theta_2]} \frac{f(t,u)}{u} = +\infty$.

(H_4) $0 < \int_0^1 q(s)g(s)ds < +\infty$ and $\int_0^1 q(s)f(s,y)ds < +\infty$ for any $y \in [0,R]$, $R > 0$ is any constant.

In fact, we only consider the boundary value problem

$$\begin{aligned}&\mathbf{D}_{0+}^{\alpha}x(t) + \lambda[f(t,[x(t)-v(t)]^*) + g(t)] = 0, \quad 0<t<1, n-1<\alpha\leq n, \lambda>0, \\ &x^{(j)}(0) = 0, \quad 0 \leq j \leq n-2, \\ &x(1) = 0\end{aligned} \tag{4.1}$$

where

$$y(t)^* = \begin{cases} y(t), & y(t) \geq 0; \\ 0, & y(t) < 0. \end{cases}$$

and $v(t) = \lambda \int_0^1 G(t,s)g(s)ds$, which is the solution of the boundary value problem

$$\begin{aligned}&-\mathbf{D}_{0+}^{\alpha}v(t) = \lambda g(t), \quad 0<t<1, n-1<\alpha\leq n, \lambda>0, \\ &v^{(j)}(0) = 0, \quad 0 \leq j \leq n-2, \\ &v(1) = 0.\end{aligned}$$

We will show there exists a solution x for the boundary value problem (4.1) with $x(t) \geq v(t), t \in [0,1]$. If this is true, then $u(t) = x(t) - v(t)$ is a nonnegative solution (positive on $(0,1)$) of the boundary value problem (1.1). Since for any $t \in (0,1)$,

$$-\mathbf{D}_{0+}^{\alpha} u - \mathbf{D}_{0+}^{\alpha} v = \lambda[f(t,u) + g(t)],$$

we have

$$-\mathbf{D}_{0+}^{\alpha} u = \lambda f(t,u).$$

As a result, we will concentrate our study on the boundary value problem (4.1).

We note that $x(t)$ is a solution of (4.1) if and only if

$$x(t) = \lambda \int_0^1 G(t,s)(f(s,[x(s)-v(s)]^*) + g(s))ds, \quad 0 \leq t \leq 1. \tag{4.2}$$

For our constructions, we shall consider the Banach space $E = C[0,1]$ equipped with standard norm $\|x\| = \max_{0\leq t\leq 1} |x(t)|, x \in X$. We define a cone P by

$$P = \{x \in X | x(t) \geq \frac{t^{\alpha-1}(1-t)}{\alpha-1}\|x\|, \quad t \in [0,1], \alpha \in (n-1,n], n \geq 3\}.$$

Lemma 4.1 *Assume (H_1) (or (H_1^*)) is satisfied and define the integral operator $T: P \to E$ by*

$$Tx(t) = \lambda \int_0^1 G(t,s)(f(s,[x(s)-v(s)]^*) + g(s))ds, \quad 0 \leq t \leq 1, x \in P. \tag{4.3}$$

Then $T: P \to P$ is completely continuous.

Proof First, we prove that $T: P \to P$.

Notice from (4.3) and Lemma 3.2 that, for $x \in P$, $Tx(t) \geq 0$ on $[0,1]$ and

$$\begin{aligned} Tx(t) &= \lambda \int_0^1 G(t,s)(f(s,[x(s)-v(s)]^*) + g(s))ds \\ &\leq \lambda \int_0^1 (\alpha-1)q(s)(f(s,[x(s)-v(s)]^*) + g(s))ds, \end{aligned}$$

then $\|Tx\| \leq \int_0^1 (\alpha-1)q(s)(f(s,[x(s)-v(s)]^*) + g(s))ds$.

On the other hand, we have

$$\begin{aligned} Tx(t) &= \lambda \int_0^1 G(t,s)(f(s,[x(s)-v(s)]^*) + g(s))ds \\ &\geq \lambda \int_0^1 t^{\alpha-2}(1-t)q(s)(f(s,[x(s)-v(s)]^*) + g(s))ds \\ &\geq \frac{t^{\alpha-2}(1-t)}{\alpha-1}\lambda \int_0^1 (\alpha-1)q(s)(f(s,[x(s)-v(s)]^*) + g(s))ds \\ &\geq \frac{t^{\alpha-2}(1-t)}{\alpha-1}\|Tx\|. \end{aligned}$$

Thus, $T(P) \subset P$. In addition, from f is continuous it follows that T is continuous.

Next, we show T is uniformly bounded.

Let $D \subset P$ be bounded, i.e. there exists a positive constant $L > 0$ such that $\|y\| \leq L$, for all $y \in D$. Let $M = \max_{0\leq t\leq 1, 0\leq y\leq L} |f(t,y) + e(t)| + 1$, then for $x \in D$, from the Lemma 3.1, one has

$$\begin{aligned} |Ty(t)| &\leq \int_0^1 |G(t,s)(f(s,[x(s)-v(s)]^*) + g(s))|ds \\ &\leq \int_0^1 |(\alpha-1)q(s)(f(s,[x(s)-v(s)]^*) + g(s))|ds \\ &\leq (\alpha-1)M \int_0^1 q(s)ds \end{aligned}$$

Hence, $T(D)$ is bounded.

Finally, we show T is equicontinuous.

For all $\varepsilon > 0$, each $u \in P$, $t_1, t_2 \in [0,1]$, $t_1 < t_2$, let

$$\eta = \min\{\frac{1}{2}, \frac{\Gamma(\alpha)\varepsilon}{M\alpha}\},$$

we will prove that $|Tu(t_2) - Tu(t_1)| < \varepsilon$, when $t_2 - t_1 < \eta$. One has

$$\begin{aligned}
&|Tu(t_2) - Tu(t_1)| \\
&= |\int_0^1 G(t_2,s)(f(s,[x(s)-v(s)]^*)+g(s))ds - \int_0^1 G(t_1,s)(f(s,[x(s)-v(s)]^*)+g(s))ds| \\
&\le \int_0^1 |G(t_2,s)-G(t_1,s))(f(s,[x(s)-v(s)]^*)+g(s))|ds \\
&\le M\int_0^1 |G(t_2,s)-G(t_1,s))|ds \\
&\le M(\int_0^{t_1} |(G(t_2,s)-G(t_1,s))|ds + \int_{t_2}^1 |(G(t_2,s)-G(t_1,s))|ds + \int_{t_1}^{t_2} |(G(t_2,s)-G(t_1,s))|ds) \\
&\le M(\int_0^{t_1} \frac{(t_2^{\alpha-1}-t_1^{\alpha-1})(1-s)^{\alpha-1} + ((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1})}{\Gamma(\alpha)}ds \\
&\quad + \int_{t_2}^1 \frac{(t_2^{\alpha-1}-t_1^{\alpha-1})(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds + \int_{t_1}^{t_2} \frac{(t_2^{\alpha-1}-t_1^{\alpha-1})(1-s)^{\alpha-1}+(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}ds) \\
&\le M(\int_0^1 \frac{(t_2^{\alpha-1}-t_1^{\alpha-1})(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds + \int_0^{t_1} \frac{-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}ds + \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}ds) \\
&\le \frac{M}{\Gamma(\alpha)}((\alpha-1)t_2^{\alpha-2}\eta \int_0^1 (1-s)^{\alpha-1}ds + \frac{1}{\alpha}(t_2^\alpha - t_1^\alpha)) \\
&\le \frac{M}{\Gamma(\alpha)}((\alpha-1)t_2^{\alpha-2}\eta \int_0^1 (1-s)^{\alpha-1}ds + t_2^{\alpha-1}\eta) \\
&\le \frac{M}{\Gamma(\alpha)}((\alpha-1)\int_0^1 (1-s)^{\alpha-1}ds + 1)\eta \\
&\le \frac{M\alpha\eta}{\Gamma(\alpha)} < \varepsilon.
\end{aligned}$$

Thus, we obtain

$$|Tu(t_2) - Tu(t_1)| < \frac{M\alpha\eta}{\Gamma(\alpha)} < \varepsilon.$$

By means of the Arzela-Ascoli theorem, $T : P \to P$ is completely continuous.

If condition (H_1) is replaced by (H_1^*), let

$$T_n x(t) = \lambda \int_0^1 G(t,s)(f_n^*(s,[x(s)-v(s)]^*)+g(s))ds, \quad n \ge 2,$$

where

$$f_n^*(t,y) = \begin{cases} \inf\{f(t,y), f(\frac{1}{n},y)\}, & 0 < t \le \frac{1}{n}; \\ f(t,y), & \frac{1}{n} \le t \le \frac{n-1}{n}; \\ \inf\{f(t,y), f(\frac{n-1}{n},y)\}, & \frac{n-1}{n} \le t < 1. \end{cases}$$

It is easy to see that $f_n^*(t,y) \in C([0,1]\times[0,\infty))$ is bounded and $0 \le f_n^*(t,y) \le f(t,y), t \in (0,1)$. By repeating the similar proof above, we get that T_n is a completely continuous operator on P for each $n \ge 2$. Furthermore, for any $R > 0$, set $\Omega_R = \{u \in P : \|u\| \le R\}$, then T_n converges uniformly to T on Ω_n as $n \to \infty$. In fact, for $R > 0$ and

$u \in \Omega_R$, we have

$$
\begin{aligned}
&|T_n u(t) - Tu(t)| \\
&= |\int_0^1 G(t,s)(f_n^*(s,[x(s)-v(s)]^*) - f(s,[x(s)-v(s)]^*))ds| \\
&\le \int_0^1 (\alpha-1)q(s)|f_n^*(s,[x(s)-v(s)]^*) - f(s,[x(s)-v(s)]^*)|ds \\
&= \int_0^{\frac{1}{n}} (\alpha-1)q(s)|f_n^*(s,[x(s)-v(s)]^*) - f(s,[x(s)-v(s)]^*)|ds \\
&+ \int_{\frac{n-1}{n}}^1 (\alpha-1)q(s)|f_n^*(s,[x(s)-v(s)]^*) - f(s,[x(s)-v(s)]^*)|ds \\
&\le \int_0^{\frac{1}{n}} (\alpha-1)q(s) \max_{0\le y\le R} |f_n^*(s,y) - f(s,y)|ds \\
&+ \int_{\frac{n-1}{n}}^1 (\alpha-1)q(s) \max_{0\le y\le R} |f_n^*(s,y) - f(s,y)|ds \to 0 \quad (n\to\infty).
\end{aligned}
$$

So we conclude that T_n converges uniformly to T on Ω_n as $n \to \infty$. Thus, T is completely continuous.

The proof is completed.

Theorem 4.2 *Suppose that (H_1)-(H_2) hold. Then there exists a constant* $\overline{\lambda} > 0$ *such that, for any* $0 < \lambda \le \overline{\lambda}$, *the boundary value problem* (1.1) *has at least one positive solutions.*

Proof Fixed $\delta \in (0,1)$, from (H_2), Let $0 < \varepsilon < 1$ be such that

$$f(t,z) \ge \delta f(t,0), \quad \text{for} \quad 0 \le t \le 1, \quad 0 \le z \le \varepsilon. \tag{4.4}$$

Suppose

$$0 < \lambda < \frac{\varepsilon}{2c\overline{f}(\varepsilon)} := \overline{\lambda}$$

where $\overline{f}(\varepsilon) = \max\limits_{0\le t\le 1, 0\le z\le\varepsilon} \{f(t,z) + g(t)\}$ and $c = \int_0^1 (\alpha-1)q(s)ds$. Since

$$\lim_{z\downarrow 0} \frac{\overline{f}(z)}{z} = +\infty$$

and

$$\frac{\overline{f}(\varepsilon)}{\varepsilon} < \frac{1}{2c\lambda},$$

then exist a $R_0 \in (0,\varepsilon)$ such that

$$\frac{\overline{f}(R_0)}{R_0} = \frac{1}{2c\lambda}.$$

Let $x \in P$ and $\nu \in (0,1)$ be such that $x = \nu T(x)$, we claim that $\|x\| \ne R_0$. In fact, if $\|x\| = R_0$, then

$$
\begin{aligned}
\|x\| = \nu\|Tx\| &\le \nu\lambda \int_0^1 (\alpha-1)q(s)[f(s,[x(s)-v(s)]^*) + g(s)]ds \\
&\le \lambda \int_0^1 (\alpha-1)q(s)[f(s,[x(s)-v(s)]^*) + g(s)]ds \\
&\le \lambda \int_0^1 (\alpha-1)q(s) \max_{0\le s\le 1; 0\le z\le R_0} [f(s,z) + g(s)]ds \\
&\le \lambda \int_0^1 (\alpha-1)q(s)\overline{f}(R_0)ds \\
&\le \lambda c\overline{f}(R_0),
\end{aligned}
$$

that is

$$\frac{\overline{f}(R_0)}{R_0} \geq \frac{1}{c\lambda} > \frac{1}{2c\lambda} = \frac{\overline{f}(R_0)}{R_0}$$

which implies that $\|x\| \neq R_0$. Let $U = \{x \in P : \|x\| < R_0\}$. By the nonlinear alternative theorem of Leray-Schauder type, T has a fixed point $x \in \overline{U}$. Moreover, if we combine (4.3), (4.4) and $R_0 < \varepsilon$, we obtain

$$\begin{aligned}
x(t) &= \lambda \int_0^1 G(t,s)[f(s,[x(s)-v(s)]^*) + g(s)]ds \\
&\geq \lambda \int_0^1 G(t,s)[\delta f(s,0) + g(s)]ds \\
&\geq \lambda[\delta \int_0^1 G(t,s)f(s,0)ds + \int_0^1 G(t,s)g(s)ds] \\
&> \lambda \int_0^1 G(t,s)g(s)ds \\
&= v(t) \quad \text{for} \quad t \in (0,1).
\end{aligned}$$

Let $u(t) = x(t) - v(t) > 0$. Then (1.1) has a positive solution u and $\|u\| \leq \|x\| \leq R_0 < 1$.

The proof of the theorem is completed.

Theorem 4.3 *Suppose that (H_1^*) and (H_3)-(H_4) hold. Then there exists a constant $\lambda^* > 0$ such that, for any $0 < \lambda \leq \lambda^*$, the boundary value problem* (1.1) *has at least one positive solution.*

Proof Let $\Omega_1 = \{x \in C[0,1] : \|x\| < R_1\}$, where $R_1 = \max\{1, r\}$ and $r = \int_0^1 \frac{(\alpha-1)^2}{\Gamma(\alpha)} g(s)ds$. Choose

$$\lambda^* = \min\{1, R_1[\int_0^1 (\alpha-1)q(s)[\max_{0\leq z\leq R_1} f(s,z) + g(s)]ds]^{-1}\}.$$

Then for any $x \in P \cap \partial\Omega_1$, then $\|x\| = R_1$ and $x(s) - v(s) \leq x(s) \leq \|x\|$, we have

$$\begin{aligned}
\|Tx(t)\| &\leq \lambda \int_0^1 (\alpha-1)q(s)[f(s,[x(s)-v(s)]^*) + g(s)]ds \\
&\leq \lambda \int_0^1 (\alpha-1)q(s)[f(s,[x(s)-v(s)]^*) + g(s)]ds \\
&\leq \lambda \int_0^1 (\alpha-1)q(s)[\max_{0\leq z\leq R_1} f(s,z) + g(s)]ds \\
&\leq R_1 = \|x\|.
\end{aligned}$$

This implies

$$\|Tx\| \leq \|x\|, x \in P \cap \partial\Omega_1.$$

On the other hand, choose a constant $N > 0$ such that

$$\frac{\lambda N}{2(\alpha-1)} \min_{\theta_1\leq t\leq\theta_2} k(t) \int_{\theta_1}^{\theta_2} s^\alpha (1-s)^\alpha ds \geq 1.$$

By the assumptions (H_3), for any $t \in [\theta_1, \theta_2]$, there exists a constant $B > 0$ such that

$$\frac{f(t,z)}{z} > N, \quad \text{namely} \quad f(t,z) > Nz, \quad \text{for} \quad z > B.$$

Choose $R_2 = \max\{R_1+1, 2\lambda r, \frac{2(\alpha-1)(B+1)}{\min\limits_{\theta_1\le t\le\theta_2}\{t^{\alpha-1}(1-t)\}}\}$, and let $\Omega_2 = \{x \in C[0,1] : \|x\| < R_2\}$, then for any $x \in P\cap\partial\Omega_2$, we have

$$\begin{aligned} x(t) - v(t) &= x(t) - \lambda\int_0^1 G(t,s)g(s)ds \\ &\geq x(t) - \frac{t^{\alpha-1}(t-1)}{\alpha-1}\lambda\int_0^1 \frac{(\alpha-1)^2}{\Gamma(\alpha)}g(s)ds \\ &\geq x(t) - \frac{x(t)}{\|x\|}\lambda r \\ &\geq (1-\frac{\lambda r}{R_2})x(t) \\ &\geq \frac{1}{2}x(t) \geq 0,\ \ t\in[0,1]. \end{aligned}$$

And then

$$\begin{aligned} \min_{\theta_1\le t\le\theta_2}\{x(t)-v(t)\} \geq \min_{\theta_1\le t\le\theta_2}\{\frac{1}{2}x(t)\} &\geq \min_{\theta_1\le t\le\theta_2}\{\frac{t^{\alpha-1}(1-t)}{2(\alpha-1)}\|x\|\} \\ &= \frac{R_2\min\limits_{\theta_1\le t\le\theta_2} t^{\alpha-1}(1-t)}{2(\alpha-1)} \geq B+1 > B. \end{aligned}$$

$$\begin{aligned} \|Tx(t)\| &\geq \max_{0\le t\le 1}\lambda\int_0^1 \Gamma(\alpha)k(t)q(s)[f(s,[x(s)-v(s)]^*)+g(s)]ds \\ &\geq \max_{0\le t\le 1}\lambda\Gamma(\alpha)k(t)\int_0^1 q(s)f(s,[x(s)-v(s)]^*)ds \\ &\geq \lambda\min_{\theta_1\le t\le\theta_2}\Gamma(\alpha)k(t)\int_{\theta_1}^{\theta_2} q(s)f(s,x(s)-v(s))ds \\ &\geq \lambda\min_{\theta_1\le t\le\theta_2}\Gamma(\alpha)k(t)\int_{\theta_1}^{\theta_2} q(s)N(x(s)-v(s))ds \\ &\geq \lambda\min_{\theta_1\le t\le\theta_2}\Gamma(\alpha)k(t)\int_{\theta_1}^{\theta_2} q(s)\frac{N}{2}x(s)ds \\ &\geq \frac{\lambda N}{2(\alpha-1)}\min_{\theta_1\le t\le\theta_2}k(t)\int_{\theta_1}^{\theta_2}\Gamma(\alpha)q(s)s^{\alpha-1}(1-s)\|x\|ds \\ &\geq \frac{\lambda N}{2(\alpha-1)}\min_{\theta_1\le t\le\theta_2}k(t)\int_{\theta_1}^{\theta_2} s^{\alpha}(1-s)^{\alpha}ds\|x\| \\ &\geq \|x\|. \end{aligned}$$

$$\|Tx\| \geq \|x\|, x\in P\cap\partial\Omega_2.$$

Condition (2) of Krasnoesel'skii's fixed-point theorem is satisfied. So T has a fixed point x with $r < \|x\| < R_2$ such that

$$\begin{aligned} &-\mathbf{D}_{0+}^{\alpha}x(t) = \lambda(f(s,[x(s)-v(s)]^*)+g(s)), \quad 0<t<1, n-1<\alpha\le n, \\ &x^{(j)}(0) = 0, \quad 0\le j\le n-2, \\ &x(1) = 0. \end{aligned}$$

Since $r < \|x\|$,

$$
\begin{aligned}
x(t)-v(t) &\geq \frac{t^{\alpha-1}(1-t)}{\alpha-1}\|x\| - \lambda\int_0^1 G(t,s)g(s)ds \\
&\geq \frac{t^{\alpha-1}(1-t)}{\alpha-1}\|x\| - \frac{t^{\alpha-1}(1-t)}{\alpha-1}\lambda\int_0^1 \frac{(\alpha-1)^2}{\Gamma(\alpha)}g(s)ds \\
&\geq \frac{t^{\alpha-1}(1-t)}{\alpha-1}\|x\| - \frac{t^{\alpha-1}(1-t)}{\alpha-1}\lambda r \\
&\geq \frac{t^{\alpha-1}(1-t)}{\alpha-1}r - \frac{t^{\alpha-1}(1-t)}{\alpha-1}\lambda r \\
&\geq \frac{t^{\alpha-1}(1-t)}{\alpha-1}(1-\lambda)r \\
&> 0,\ t\in(0,1).
\end{aligned}
$$

Let $u(t) = x(t) - v(t)$, then $u(t)$ is a positive solution of the boundary value problem (1.1).

The proof of the theorem is completed.

Remark. In Theorem 4.3, f may be singular at $t = 0$ and 1.

Since condition (H_1) implies condition (H_1^*) and (H_4), and from the proof of Theorem 4.2 and 4.3, we have immediately the following theorem:

Theorem 4.4 *Suppose that (H_1)-(H_3) hold. Then the boundary value problem* (1.1) *has at least two positive solutions for $\lambda > 0$ sufficiently small.*

In fact, let $0 < \lambda < \min\{\overline{\lambda}, \lambda^*\}$, then the boundary value problem (1.1) has at least two positive solutions u_1 and u_2.

5 Examples

To illustrate the usefulness of the results, we give some examples.

Example 1. Consider the boundary value problem

$$
\begin{aligned}
&D_{0+}^{\alpha}u(t) + \lambda(u^a(t) + \tfrac{1}{(t-t^2)^{\frac{1}{2}}}\cos(2\pi u(t))) = 0, \quad 0<t<1, \lambda>0, \\
&u^{(j)}(0) = 0, \quad 0 \leq j \leq n-2, \\
&u(1) = 0,
\end{aligned} \tag{5.1}
$$

where $a > 1$. Then, if $\lambda > 0$ is sufficiently small, then (5.1) has a positive solutions u with $u(t) > 0$ for $t \in (0, 1)$.

To see this we will apply Theorem 4.3 with

$$
f(t,u) = u^a(t) + \frac{1}{(t-t^2)^{\frac{1}{2}}}\cos(2\pi u(t)), \quad g(t) = \frac{1}{(t-t^2)^{\frac{1}{2}}}.
$$

Clearly

$$
f(t,0) = \tfrac{1}{(t-t^2)^{\frac{1}{2}}} > 0, \quad f(t,u) + g(t) \geq u^a(t) > 0, \lim_{u\uparrow+\infty}\tfrac{f(t,u)}{u} = +\infty, \quad for \quad t\in(0,1), u>0.
$$

Namely (H_1^*) and (H_2)-(H_4) hold. From $\int_0^1 \frac{1}{(s-s^2)^{\frac{1}{2}}}ds = \frac{\pi}{2}$, set $R_1 = \frac{2(\alpha-1)\pi}{\Gamma(\alpha)}$, then $R_1 > \int_0^1 G(t,s)g(s)ds$, we have

$$
\begin{aligned}
\textstyle\int_0^1 (\alpha-1)q(s)[\max\limits_{0\leq z\leq R_1} f(s,z) + g(s)]ds &\leq \textstyle\int_0^1 (\alpha-1)q(s)[(\frac{2(\alpha-1)\pi}{\Gamma(\alpha)})^a + \frac{2}{(s-s^2)^{\frac{1}{2}}}]ds \\
&\leq \tfrac{\alpha-1}{\Gamma(\alpha)}((\tfrac{2(\alpha-1)\pi}{\Gamma(\alpha)})^a + 2\pi)
\end{aligned}
$$

and $\lambda^* = \min\{1, \frac{\Gamma^a(\alpha)}{(\alpha-1)^a(2\pi)^{a-1}+\Gamma^a(\alpha)}\}$. Now, if $\lambda < \lambda^*$, Theorem 4.3 guarantees that (5.1) has a positive solutions u with $\|u\| \geq 2$.

Example 2. Consider the boundary value problem

$$\begin{aligned}&D_{0+}^{\alpha}u(t)+\lambda(u^2(t)-\tfrac{9}{2}u(t)+2)=0, \quad 0<t<1, \lambda>0,\\&u^{(j)}(0)=0, \quad 0\le j\le n-2,\\&u(1)=0,\end{aligned} \tag{5.2}$$

Then, if $\lambda>0$ is sufficiently small, then (5.2) has two solutions u_i with $u_i(t)>0$ for $t\in(0,1), i=1,2$.

To see this we will apply Theorem 4.4 with (here $0<R_1<1<R_2$ will be chosen below)

$$f(t,u)=u^2(t)-\frac{9}{2}u(t)+2, \quad g(t)=4.$$

Clearly

$$f(t,0)=2>0, \quad f(t,u)+g(t)\ge \tfrac{15}{16}>0, \lim_{u\uparrow+\infty}\tfrac{f(t,u)}{u}=+\infty, \quad for \quad t\in(0,1),$$

Namely (H_1)-(H_3) hold. Let $\delta=\frac{1}{4}$ and $\varepsilon=\frac{1}{4}$, we may have

$$\overline{\lambda}=\frac{\varepsilon}{2c(\max\limits_{0\le x\le\varepsilon}f(t,x)+4)}=\frac{1}{48c}, \tag{5.3}$$

where $c=\int_0^1(\alpha-1)q(s)ds$. Now, if $\lambda<\overline{\lambda}$, Theorem 4.2 guarantees that (5.2) has a positive solutions u_1 with $\|u_1\|\le\frac{1}{4}$.

Next, let $R_1=5$, we have

$$\int_0^1(\alpha-1)q(s)[\max_{0\le z\le R_1}f(s,z)+g(s)]ds=\int_0^1(\alpha-1)q(s)[\frac{9}{2}+4]ds=\frac{17c}{2}$$

and $\lambda^*=\min\{1,\frac{10}{17c}\}$. Now, if $\lambda<\lambda^*$, Theorem 4.3 guarantees that (5.2) has a positive solutions u_2 with $\|u_2\|\ge 5$.

So, since all the conditions of Theorem 4.4 are satisfied , if $\lambda<\min\{\overline{\lambda},\lambda^*\}$, Theorem 4.4 guarantees that (5.2) has two solutions u_i with $u_i(t)>0$ for $t\in(0,1), i=1,2$.

Example 3. Consider the boundary value problem

$$\begin{aligned}&D_{0+}^{\alpha}u(t)+\lambda(u^a(t)+\cos(2\pi u(t)))=0, \quad 0<t<1, \lambda>0,\\&u^{(j)}(0)=0, \quad 0\le j\le n-2,\\&u(1)=0,\end{aligned} \tag{5.4}$$

where $a>1$. Then, if $\lambda>0$ is sufficiently small, then (5.4) has two solutions u_i with $u_i(t)>0$ for $t\in(0,1), i=1,2$.

To see this we will apply Theorem 4.4 with

$$f(t,u)=u^a(t)+\cos(2\pi u(t)), \quad g(t)=2.$$

Clearly

$$f(t,0)=1>0, \quad f(t,u)+g(t)\ge u^a(t)+1>0, \lim_{u\uparrow+\infty}\tfrac{f(t,u)}{u}=+\infty, \quad for \quad t\in(0,1),$$

Namely (H_1)-(H_3) hold. By a similar way of example 2, if $\lambda>0$ is sufficiently small, Theorem 4.4 guarantees that (5.4) has two solutions u_i with $u_i(t)>0$ for $t\in(0,1), i=1,2$.

References

[1] A. Ghorbani, *Toward a new analytical method for solving nonlinear fractional differential equations*, Computer Methods in Applied Mechanics and Engineering, **197**(2008), 4173-4179

[2] I. Podlubny, *Fractional Differential Equations, Mathematics in Science and Engineering*, vol, 198, Academic Press, New Tork/Londin/Toronto, 1999.

[3] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integral And Derivatives (Theory and Applications)*. Gordon and Breach, Switzerland, 1993.

[4] A. Ashyralyev, *A note on fractional derivatives and fractional powers of operators*, J. Math. Anal. Appl. **357**(2009), 232-236.

[5] S. P. Mirevski, L. Boyadjiev, R. Scherer, *On the Riemann-Liouville fractional calculus, g-Jacobi functions and F-Gauss functions*, Appl. Math. Comp., **187**(2007), 315-325

[6] A. Mahmood, S. Parveen, A. Ara, N. A. Khan, *Exact analytic solutions for the unsteady flow of a non-Newtonian fluid between two cylinders with fractional derivative model*, Communications in Nonlinear Science and Numerical Simulation, **14**(2009), 3309-3319.

[7] G. Jumarie, *Table of some basic fractional calculus formulae derived from a modified Riemann-Liouville derivative for non-differentiable functions*, Appl. Math. Lett. **22**(2009), 378-385.

[8] M. A. Krasnosel'skii, *Positive Solutions of Operator Equations*, Noordhoff Gronigen, Netherland, 1964.

[9] V. Lakshmikantham, S. Leela, *A Krasnoselskii-Krein-type uniqueness result for fractional differential equations*, Nonlinear Anal. TMA, **71**(2009), 3421-3424.

[10] Yong Zhou, Feng Jiao, Jing Li, *Existence and uniqueness for fractional neutral differential equations with infinite delay*, Nonlinear Anal. TMA, **71**(2009), 3249-3256.

[11] N. Kosmatov, *Integral equations and initial value problems for nonlinear differential equations of fractional order*, Nonlinear Analysis: Theory, Methods & Applications, Volume 70, Issue 7, 1 April 2009, Pages 2521-2529

[12] Chuanzhi Bai, *Positive solutions for nonlinear fractional differential equations with coefficient that changes sign*, Nonlinear Anal., **64**(2006), 677-685

[13] Shuqin Zhang, *Existence of positive solution for some class of nonlinear fractional differential equations*, J. Math. Anal. Appl. **278**(2003), 136-148.

[14] M. Benchohra, S. Hamani, S. K. Ntouyas, *Boundary value problems for differential equations with fractional order and nonlocal conditions*, Nonlinear Anal. TMA, **71**(2009), 2391-2396.

[15] Zhanbing Bai, Haishen Lü, *Positive solutions for boundary-value problem of nonlinear fractional differential equation*, J. Math. Anal. Appl. **311**(2005), 495-505.

[16] Shuqin Zhang, *The existence of a positive solution for a nonlinear fractional differential equation*, J. Math. Anal. Appl. **252**(2000), 804-812.

[17] Chengjun Yuan, Daqing Jiang, and Xiaojie Xu, *Singular positone and semipositone boundary value problems of nonlinear fractional differential equations*, Mathematical Problems in Engineering, Volume 2009, Article ID 535209, 17 pages.

[18] Xinwei Su, *Boundary value problem for a coupled system of nonlinear fractional differential equations*, Appl. Math. Lett. **22**(2009), 64-69.

[19] B. Ahmad, S. Sivasundaram, *Existence results for nonlinear impulsive hybrid boundary value problems involving fractional differential equations*, Nonlinear Anal. Hybrid Systems, **3**(2009), 251-258.

[20] M. Benchohra, S. Hamani, *The method of upper and lower solutions and impulsive fractional differential inclusions* Nonlinear Anal.: Hybrid Systems, **3** (2009), 433-440.

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