

Some stability and boundedness conditions for non-autonomous differential equations with deviating arguments

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Abstract

In this article, the author studies the stability and boundedness of solutions for the non-autonomous third order differential equation with a deviating argument, r :

$$\begin{aligned}x'''(t) + a(t)x''(t) + b(t)g_1(x'(t-r)) + g_2(x'(t)) + h(x(t-r)) \\ = p(t, x(t), x(t-r), x'(t), x'(t-r), x''(t)),\end{aligned}$$

where $r > 0$ is a constant. Sufficient conditions are obtained; a stability result in the literature is improved and extended to the preceding equation for the case $p(t, x(t), x(t-r), x'(t), x'(t-r), x''(t)) = 0$, and a new boundedness result is also established for the case $p(t, x(t), x(t-r), x'(t), x'(t-r), x''(t)) \neq 0$.

1 Introduction

In 1968, Ponzo [10] considered the following nonlinear third order differential equation without a deviating argument:

$$x'''(t) + a(t)x''(t) + b(t)x'(t) + cx(t) = 0.$$

For the preceding equation, he constructed a positive definite Liapunov function with negative semi-definite time derivative. This established the stability of the null solution.

In this paper, instead of the preceding equation, we consider the following non-autonomous

third order differential equation with a deviating argument, r :

$$\begin{aligned} x'''(t) + a(t)x''(t) + b(t)g_1(x'(t-r)) + g_2(x'(t)) + h(x(t-r)) \\ = p(t, x(t), x(t-r), x'(t), x'(t-r), x''(t)), \end{aligned} \quad (1)$$

which is equivalent to the system:

$$\begin{aligned} x'(t) &= y(t), \\ y'(t) &= z(t), \\ z'(t) &= -a(t)z(t) - b(t)g_1(y(t)) - h(x(t)) + b(t) \int_{t-r}^t g_1'(y(s))z(s)ds \\ &\quad - g_2(y(t)) + \int_{t-r}^t h'(x(s))y(s)ds + p(t, x(t), x(t-r), y(t), y(t-r), z(t)), \end{aligned} \quad (2)$$

where r is a positive constant; the functions a , b , g_1 , g_2 , h and p depend only on the arguments displayed explicitly and the primes in Eq. (1) denote differentiation with respect to $t \in \mathfrak{R}^+ = [0, \infty)$. The functions a , b , g_1 , g_2 , h and p are assumed to be continuous for their all respective arguments on \mathfrak{R}^+ , \mathfrak{R}^+ , \mathfrak{R} , \mathfrak{R} , \mathfrak{R} and $\mathfrak{R}^+ \times \mathfrak{R}^5$, respectively. Assume also that the derivatives $a'(t) \equiv \frac{d}{dt}a(t)$, $b'(t) \equiv \frac{d}{dt}b(t)$, $h'(x) \equiv \frac{d}{dx}h(x)$ and $g_1'(y) \equiv \frac{d}{dy}g_1(y)$ exist and are continuous; throughout the paper $x(t)$, $y(t)$ and $z(t)$ are abbreviated as x , y and z , respectively. Finally, the existence and uniqueness of solutions of Eq. (1) are assumed and all solutions considered are supposed to be real valued.

The motivation of this paper has come by the result of Ponzo [10, Theorem 2]. Our purpose here is to extend and improve the result established by Ponzo [10, Theorem 2] to the preceding non-autonomous differential equation with the deviating argument r for the asymptotic stability of null solution and the boundedness of all solutions, whenever $p \equiv 0$ and $p \neq 0$ in Eq.(1), respectively.

At the same time, it is worth mentioning that one can recognize that by now many significant theoretical results dealt with the stability and boundedness of solutions of nonlinear differential equations of third order without delay:

$$x'''(t) + a_1x''(t) + a_2x'(t) + a_3x(t) = p(t, x(t), x'(t), x''(t)),$$

in which a_1 , a_2 and a_3 are not necessarily constants. In particular, one can refer to the

book of Reissig et al. [11] as a survey and the papers of Ezeilo [4,5], Ezeilo and Tejumola [6], Ponzo [10], Swick [14], Tunç [16, 17, 18, 21], Tunç and Ateş [27] and the references cited in these works for some publications performed on the topic. Besides, with respect our observation from the literature, it can be seen some papers on the stability and boundedness of solutions of nonlinear differential equations of third order with delay (see, for example, the papers of Afuwape and Omeike [2], Omeike [9], Sadek [12], Sinha [13], Tejumola and Tchegnani [15], Tunç ([19, 20], [22-26]), Zhu [28]) and the references thereof).

It should be noted that, to the best of our knowledge, we did not find any work based on the result of Ponzo [10, Theorem 2] in the literature. That is to say that, this work is the first attempt carrying the result of Ponzo [10, Theorem 2] to certain non-autonomous differential equations with deviating arguments. The assumptions will be established here are different from that in the papers mentioned above.

2 Main Results

Let $p(t, x, x(t-r), y, y(t-r), z) = 0$. We establish the following theorem

Theorem 1. In addition to the basic assumptions imposed on the functions $a(t)$, $b(t)$, g_1 , g_2 and h appearing in Eq. (1), we assume that there are positive constants a , α , β , b_1 , b_2 , B , c , c_1 and L such that the following conditions hold:

- (i) $a(t) \geq 2\alpha + a$, $B \geq b(t) \geq \beta$,
 $g_1(0) = g_2(0) = h(0) = 0$,
 $0 < c_1 \leq h'(x) \leq c$, $\alpha\beta - c > 0$,
 $\frac{g_1(y)}{y} \geq b_1 \geq 1$, $\frac{g_2(y)}{y} \geq b_2$, ($y \neq 0$) and $|g_1'(y)| \leq L$.
- (ii) $[\alpha b(t) - c]y^2 \geq 2^{-1}\alpha\alpha'(t)y^2 + b'(t) \int_0^y g_1(\eta)d\eta$.

Then the null solution of Eq. (1) is stable, provided

$$r < \min \left\{ \frac{\alpha b_2}{\alpha(BL + 2c) + c}, \frac{2\alpha}{BL(2 + \alpha) + c} \right\}.$$

Proof. To prove Theorem 1, we define a Lyapunov functional $V(t, x_t, y_t, z_t)$:

$$\begin{aligned}
 2V(t, x_t, y_t, z_t) &= z^2 + 2\alpha yz + 2b(t) \int_0^y g_1(\eta) d\eta + 2 \int_0^y g_2(\eta) d\eta + \alpha a(t)y^2 + 2h(x)y \\
 &\quad + 2\alpha \int_0^x h(\xi) d\xi + \lambda_1 \int_{-r}^0 \int_{t+s}^t y^2(\theta) d\theta ds + \lambda_2 \int_{-r}^0 \int_{t+s}^t z^2(\theta) d\theta ds,
 \end{aligned} \tag{3}$$

where λ_1 and λ_2 are some positive constants which will be specified later in the proof.

Now, from the assumptions $\frac{g_1(y)}{y} \geq b_1 \geq 1$, $\frac{g_2(y)}{y} \geq b_2$, ($y \neq 0$), and $0 < c_1 \leq h'(x) \leq c$, it follows that

$$\begin{aligned}
 2b(t) \int_0^y g_1(\eta) d\eta &= 2b(t) \int_0^y \frac{g_1(\eta)}{\eta} \eta d\eta \geq \beta b_1 y^2 \geq \beta y^2, \\
 2 \int_0^y g_2(\eta) d\eta &= 2 \int_0^y \frac{g_2(\eta)}{\eta} \eta d\eta \geq b_2 y^2, \\
 h^2(x) &= 2 \int_0^x h(\xi) h'(\xi) d\xi \leq 2c \int_0^x h(\xi) d\xi.
 \end{aligned}$$

The preceding inequalities lead to the following:

$$\begin{aligned}
 2V(t, x_t, y_t, z_t) &\geq (z + \alpha y)^2 + \beta[y + \beta^{-1}h(x)]^2 + 2\alpha \int_0^x h(\xi) d\xi - \frac{1}{\beta} h^2(x) \\
 &\quad + b_2 y^2 + \lambda_1 \int_{-r}^0 \int_{t+s}^t y^2(\theta) d\theta ds + \lambda_2 \int_{-r}^0 \int_{t+s}^t z^2(\theta) d\theta ds \\
 &\geq (z + \alpha y)^2 + \beta[y + \beta^{-1}h(x)]^2 + 2\alpha \int_0^x h(\xi) d\xi - \frac{2c}{\beta} \int_0^x h(\xi) d\xi \\
 &\quad + b_2 y^2 + \lambda_1 \int_{-r}^0 \int_{t+s}^t y^2(\theta) d\theta ds + \lambda_2 \int_{-r}^0 \int_{t+s}^t z^2(\theta) d\theta ds.
 \end{aligned}$$

Now, it is clear

$$\begin{aligned}
 2\alpha \int_0^x h(\xi) d\xi - \frac{2c}{\beta} \int_0^x h(\xi) d\xi &= 2\beta^{-1}(\alpha\beta - c) \int_0^x h(\xi) d\xi \\
 &\geq c_1 \beta^{-1}(\alpha\beta - c)x^2.
 \end{aligned}$$

Hence

$$\begin{aligned}
 2V(t, x_t, y_t, z_t) &\geq (z + \alpha y)^2 + \beta[y + \beta^{-1}h(x)]^2 + 2^{-1}c_1\beta^{-1}(\alpha\beta - c)x^2 + b_2 y^2 \\
 &\quad + \lambda_1 \int_{-r}^0 \int_{t+s}^t y^2(\theta) d\theta ds + \lambda_2 \int_{-r}^0 \int_{t+s}^t z^2(\theta) d\theta ds.
 \end{aligned}$$

The preceding inequality allows the existence of some positive constants D_i , ($i = 1, 2, 3$), such that

$$V(t, x_t, y_t, z_t) \geq D_1 x^2 + D_2 y^2 + D_3 z^2 \geq D_4 (x^2 + y^2 + z^2), \tag{4}$$

where $D_4 = \min\{D_1, D_2, D_3\}$, since $\int_{-r}^0 \int_{t+s}^t y^2(\theta) d\theta ds \geq 0$ and $\int_{-r}^0 \int_{t+s}^t z^2(\theta) d\theta ds \geq 0$.

Now, along a trajectory of (2) we find

$$\begin{aligned}
\frac{d}{dt}V(t, x_t, y_t, z_t) &= - [\alpha b(t)g_1(y)y^{-1} + \alpha g_2(y)y^{-1} - h'(x) - 2^{-1}\alpha a'(t)] y^2 + b'(t) \int_0^y g_1(\eta) d\eta \\
&\quad - [a(t) - \alpha] z^2 + zb(t) \int_{t-r}^t g'_1(y(s))z(s) ds + z \int_{t-r}^t h'(x(s))y(s) ds \\
&\quad + \alpha y b(t) \int_{t-r}^t g'_1(y(s))z(s) ds + \alpha y \int_{t-r}^t h'(x(s))y(s) ds \\
&\quad + \lambda_1 y^2 r - \lambda_1 \int_{t-r}^t y^2(s) ds + \lambda_2 z^2 r - \lambda_2 \int_{t-r}^t z^2(s) ds.
\end{aligned} \tag{5}$$

In view of the assumptions of Theorem 1 and the inequality $2|mn| \leq m^2 + n^2$, we find the following inequalities:

$$\begin{aligned}
&[\alpha b(t)g_1(y)y^{-1} + \alpha g_2(y)y^{-1} - h'(x) - 2^{-1}\alpha a'(t)] y^2 - b'(t) \int_0^y g_1(\eta) d\eta \\
&\geq [\alpha b_1 b(t) + \alpha b_2 - c - 2^{-1}\alpha a'(t)] y^2 - b'(t) \int_0^y g_1(\eta) d\eta \\
&\geq [\alpha b(t) - c] y^2 - 2^{-1}\alpha a'(t) y^2 - b'(t) \int_0^y g_1(\eta) d\eta + \alpha b_2 y^2 \\
&\geq \alpha b_2 y^2,
\end{aligned}$$

$$[a(t) - \alpha] z^2 \geq (\alpha + a) z^2,$$

$$zb(t) \int_{t-r}^t g'_1(y(s))z(s) ds \leq \frac{BL}{2} r z^2 + \frac{BL}{2} \int_{t-r}^t z^2(s) ds,$$

$$\alpha y b(t) \int_{t-r}^t g'_1(y(s))z(s) ds \leq \frac{\alpha BL}{2} r y^2 + \frac{\alpha BL}{2} \int_{t-r}^t z^2(s) ds,$$

$$z \int_{t-r}^t h'(x(s))y(s) ds \leq \frac{c}{2} r z^2 + \frac{c}{2} \int_{t-r}^t y^2(s) ds,$$

$$\alpha y \int_{t-r}^t h'(x(s))y(s) ds \leq \frac{\alpha c}{2} r y^2 + \frac{\alpha c}{2} \int_{t-r}^t y^2(s) ds.$$

The substituting of the preceding inequalities into (5) gives

$$\begin{aligned} \frac{d}{dt}V(t, x_t, y_t, z_t) &\leq -\frac{1}{2} [\alpha b_2 - (\alpha BL + \alpha c + 2\lambda_1)r] y^2 - \frac{1}{2} \alpha b_2 y^2 \\ &\quad - a z^2 - \frac{1}{2} [2\alpha - (BL + c + 2\lambda_2)r] z^2 \\ &\quad + [2^{-1}(1 + \alpha)c - \lambda_1] \int_{t-r}^t y^2(s) ds \\ &\quad + [2^{-1}(1 + \alpha)BL - \lambda_2] \int_{t-r}^t z^2(s) ds. \end{aligned}$$

Let $\lambda_1 = \frac{(1+\alpha)c}{2}$ and $\lambda_2 = \frac{(1+\alpha)BL}{2}$. Hence we can write

$$\begin{aligned} \frac{d}{dt}V(t, x_t, y_t, z_t) &\leq -\frac{1}{2} [\alpha b_2 - (\alpha BL + \alpha c + 2\lambda_1)r] y^2 - \frac{1}{2} \alpha b_2 y^2 \\ &\quad - a z^2 - \frac{1}{2} [2\alpha - (BL + c + 2\lambda_2)r] z^2. \end{aligned}$$

Now, the last inequality implies

$$\frac{d}{dt}V(t, x_t, y_t, z_t) \leq -\lambda_3 y^2 - \lambda_4 z^2,$$

for some positive constants λ_3 and λ_4 , provided

$$r < \min \left\{ \frac{\alpha b_2}{\alpha(BL + 2c) + c}, \frac{2\alpha}{BL(2 + \alpha) + c} \right\}.$$

This completes the proof of Theorem 1 (see also Burton [3], Hale [7], Krasovskii [8]).

For the case $p(t, x, x(t-r), y, y(t-r), z) \neq 0$, we establish the following theorem.

Theorem 2. Suppose that assumptions (i)-(ii) of Theorem 1 and the following condition hold:

$$|p(t, x, x(t-r), y, y(t-r), z)| \leq q(t),$$

where $q \in L^1(0, \infty)$. Then, there exists a finite positive constant K such that the solution $x(t)$ of Eq. (1) defined by the initial functions

$$x(t) = \phi(t), x'(t) = \phi'(t), x''(t) = \phi''(t)$$

satisfies

$$|x(t)| \leq K, |x'(t)| \leq K, |x''(t)| \leq K$$

for all $t \geq t_0$, where $\phi \in C^2([t_0 - r, t_0], \mathfrak{R})$, provided

$$r < \min \left\{ \frac{\alpha b_2}{\alpha(L + 2c) + c}, \frac{2\alpha}{BL(2 + \alpha) + c} \right\}.$$

Proof. It is clear that under the assumptions of Theorem 2, the time derivative of functional $V(t, x_t, y_t, z_t)$ satisfies the following:

$$\frac{d}{dt}V(t, x_t, y_t, z_t) \leq -\lambda_3 y^2 - \lambda_4 z^2 + (\alpha y + z)p(t, x, x(t - r), y, y(t - r), z).$$

Hence

$$\frac{d}{dt}V(t, x_t, y_t, z_t) \leq D_5(|y| + |z|)q(t), \tag{6}$$

where $D_5 = \max\{1, \alpha\}$.

In view of the inequality $|m| < 1 + m^2$, it follows from (6) that

$$\frac{d}{dt}V(t, x_t, y_t, z_t) \leq D_5(2 + y^2 + z^2)q(t). \tag{7}$$

By (4) and (7), we get that

$$\begin{aligned} \frac{d}{dt}V(t, x_t, y_t, z_t) &\leq D_5(2 + D_4^{-1}V(t, x_t, y_t, z_t))q(t) \\ &= 2D_5q(t) + D_5D_4^{-1}V(t, x_t, y_t, z_t)q(t). \end{aligned}$$

Integrating the preceding inequality from 0 to t , using the assumption $q \in L^1(0, \infty)$ and the Gronwall-Reid-Bellman inequality, (see Ahmad and Rama Mohana Rao [1]), it follows that

$$\begin{aligned} V(t, x_t, y_t, z_t) &\leq V(0, x_0, y_0, z_0) + 2D_5A + D_5D_4^{-1} \int_0^t V(s, x_s, y_s, z_s)q(s)ds \\ &\leq \{V(0, x_0, y_0, z_0) + 2D_5A\} \exp \left(D_5D_4^{-1} \int_0^t q(s)ds \right) \\ &= \{V(0, x_0, y_0, z_0) + 2D_5A\} \exp(D_5D_4^{-1}A) = K_1 < \infty, \end{aligned} \tag{8}$$

where $K_1 > 0$ is a constant, $K_1 = \{V(0, x_0, y_0, z_0) + 2D_5A\} \exp(D_5D_4^{-1}A)$, and $A = \int_0^\infty q(s)ds$.

Thus, we have from (4) and (8) that

$$x^2 + y^2 + z^2 \leq D_4^{-1}V(t, x_t, y_t, z_t) \leq K,$$

where $K = K_1D_4^{-1}$.

This fact completes the proof of Theorem 2.

Example. Consider nonlinear delay differential equation of third order:

$$\begin{aligned} & x'''(t) + \{11 + (1 + t^2)^{-1}\}x''(t) + 2(1 + e^{-t})x'(t - r) + 4x'(t) + x(t - r) \\ & = \frac{4}{1+t^2+x^2(t)+x^2(t-r)+x'^2(t)+x'^2(t-r)+x''^2(t)}. \end{aligned} \quad (9)$$

Delay differential Eq. (9) may be expressed as the following system:

$$\begin{aligned} x' &= y \\ y' &= z \\ z' &= -\{11 + (1 + t^2)^{-1}\}z - 2(1 + e^{-t})y - 4y - x \\ &+ 2(1 + e^{-t}) \int_{t-r}^t z(s)ds + \int_{t-r}^t y(s)ds \\ &+ \frac{4}{1+t^2+x^2+x^2(t-r)+y^2+y^2(t-r)+z^2}. \end{aligned}$$

Clearly, Eq. (9) is special case of Eq. (1), and we have the following:

$$a(t) = 11 + \frac{1}{1+t^2} \geq 11 = 2 \times 5 + 1,$$

$$\alpha = 5, a = 1,$$

$$b(t) = 1 + \frac{1}{e^t},$$

$$1 \leq 1 + \frac{1}{e^t} \leq 2,$$

$$\beta = 1, B = 2,$$

$$g_1(y) = 2y, g_1(0) = 0,$$

$$\frac{g_1(y)}{y} = 2 = b_1 > 1, (y \neq 0),$$

$$g_1'(y) = 2 = L,$$

$$\int_0^y g_1(\eta)d\eta = \int_0^y 2\eta d\eta = y^2,$$

$$g_2(y) = 4y, g_2(0) = 0,$$

$$\frac{g_2(y)}{y} = 4 = b_2, (y \neq 0),$$

$$h(x) = x, h(0) = 0, h'(x) = 1,$$

$$0 < 2^{-1} < h'(x) \leq 1,$$

$$c_1 = 2^{-1}, c = 1,$$

$$a'(t) = -\frac{2t}{(1+t^2)^2}, (t \geq 0),$$

$$b'(t) = -\frac{1}{e^t}, (t \geq 0),$$

$$\begin{aligned} & p(t, x, x(t-r), y, y(t-r), z) \\ &= \frac{4}{1+t^2+x^2+x^2(t-r)+y^2+y^2(t-r)+z^2} \leq \frac{4}{1+t^2} = q(t). \end{aligned}$$

In view of the above discussion, it follows that

$$\alpha\beta - c = 4 > 0,$$

$$[\alpha b(t) - c]y^2 = [4 + 5e^{-t}]y^2, (t \geq 0),$$

$$\frac{\alpha}{2}a'(t)y^2 + b'(t) \int_0^y g(\eta)d\eta = - \left[\frac{5t}{(1+t^2)^2} \right] y^2 - e^{-t}y^2, (t \geq 0),$$

$$\begin{aligned} [\alpha b(t) - c]y^2 = [4 + 5e^{-t}]y^2 &\geq - \left[\frac{5t}{(1+t^2)^2} \right] y^2 - e^{-t}y^2 \\ &= \frac{\alpha}{2}a'(t)y^2 + b'(t) \int_0^y g(\eta)d\eta, \end{aligned}$$

$$\int_0^\infty q(s)ds = \int_0^\infty \frac{4}{1+s^2}ds = 2\pi < \infty,$$

that is, $q \in L^1(0, \infty)$ and

$$r < \min \left\{ \frac{\alpha b_2}{\alpha(BL + 2c) + c}, \frac{2\alpha}{BL(2 + \alpha) + c} \right\} = \min \left\{ \frac{4}{31}, \frac{10}{29} \right\} = \frac{4}{31}.$$

Thus all the assumptions of Theorems 1 and 2 hold. This shows that the null solution of Eq. (9) is stable and all solutions of the same equation are bounded, when $p(t, x, x(t-r)y, y(t-r), z) = 0$ and $\neq 0$, respectively.

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