Electronic Journal of Qualitative Theory of Differential Equations
2015, No. **100**, 1–16; doi: 10.14232/ejqtde.2015.1.100 http://www.math.u-szeged.hu/ejqtde/

Approximate controllability for second order nonlinear evolution hemivariational inequalities

Xiuwen Li[1], **Zhenhai Liu**[✉2] and **Stanisław Migórski**[3]

[1]College of Mathematics and Statistics, Baise University, Baise 533000, China
[2]Guangxi Key Laboratory of Universities Optimization Control and Engineering Calculation, and College of Sciences, Guangxi University for Nationalities, Nanning 530006, China
[3]Faculty of Mathematics and Computer Science, Jagiellonian University, Chair of Optimization and Control, ul. Łojasiewicza 6, 30-348 Krakow, Poland

Received 18 September 2015, appeared 9 January 2016
Communicated by László Simon

Abstract. The goal of this paper is to study approximate controllability for control systems driven by abstract second order nonlinear evolution hemivariational inequalities in Hilbert spaces. First, the concept of a mild solution of our problem is defined by using the cosine operator theory and the generalized Clarke subdifferential. Next, the existence and the approximate controllability of mild solutions are formulated and proved by means of the fixed points strategy. Finally, an example is provided to illustrate our main results.

Keywords: approximate controllability, evolution inclusion, hemivariational inequality, Clarke subdifferential, cosine operator theory, fixed point.

2010 Mathematics Subject Classification: 35R70, 93B05.

1 Introduction

Let H be a separable Hilbert space and $J = [0, b]$ be a real interval, where $b > 0$. The purpose of this paper is to consider the solvability and approximate controllability of a system governed by the following hemivariational inequality

$$\begin{cases} \langle -x''(t) + Ax(t) + Bu(t),\ v\rangle_H + F^0(t, x(t); v) \geq 0, \quad t \in J, \\ x(0) = x_0,\ x'(0) = y_0, \end{cases} \tag{1.1}$$

where $A\colon D(A) \subset H \to H$ is a closed, linear and densely defined operator which generates a strongly continuous cosine family $\{C(t) \mid t \in J\}$ on H. The notation $F^0(t, \cdot; \cdot)$ stands for the generalized Clarke directional derivative (cf. [5]) of a locally Lipschitz function $F(t, \cdot)\colon H \to \mathbb{R}$, $u \in L^2(J, U)$ is a control function, the admissible control set U is also a Hilbert space, and B is a bounded linear operator from U into H.

✉Corresponding author. Email: 641542785@qq.com (X. Li), zhhliu@hotmail.com (Z. Liu), migorski@ii.uj.edu.pl (S. Migórski).

The variational inequalities in the elliptic and parabolic cases were introduced by Lions and Stampacchia in [20], while the hyperbolic variational inequalities were introduced by Lions in [14]. Many important physical applications were given in the monographs [8] and [16], whose introductions contain many historical details, and a good survey is given in the paper [17]. Concerning the control-theoretical aspects, we would also like to mention the books [15] and [18,19], considered to be the most important references on control of linear partial differential equations. They also contain numerous results closely related to the subject of the present paper.

It is well known that many problems from mechanics (elasticity theory, semipermeability, electrostatics, hydraulics, fluid flow), economics and so on can be modeled by subdifferential inclusions or hemivariational inequalities, and we refer to [32] for more applications of hemivariational inequalities. Recently, the existence of solutions for hemivariational inequalities has been proved by many authors. For instance, the authors of [12,22] considered the problems with elliptic hemivariational inequalities, and in [4,21,27–30], the researchers discussed the problems of parabolic hemivariational inequalities. However, to our knowledge, only a few results on existence of solutions were obtained for hyperbolic hemivariational inequalities, and this is one of our motivation for the study of such hemivariational inequality (1.1) in the present work.

It is well know that the importance of the wave equation is not only because it is the most relevant hyperbolic partial differential equation but also relies on the fact that it models a large class of vibrating phenomena such as vibrations of elastic bodies and the sound propagation. For example, let $\Omega \subset \mathbb{R}^n$ and Δ be the Laplace operator, the wave equation

$$\partial^2 x/\partial t^2 - \Delta x + f = 0 \quad \text{in } \Omega \times (0, b),$$

(here $x = x(t, \theta)$ is the displacement function and f is an arbitrary external forcing function) provides a good application for the amplitude vibrations of an elastic string or a flexible membrane occupying the region Ω under a force acting on the vibrating structure. In applications of mathematical methods to physics, one is often concerned with the study of more complicated nonlinear wave equations such as the Klein–Gordon equation of the form

$$\partial^2 x/\partial t^2 - \Delta x + \mu^2 x + \eta^2 x^3 = 0,$$

where μ, $\eta \in \mathbb{R}$. A large number of work has been devoted to the study of the Cauchy problem for the nonlinear Klein–Gordon equation. One of the nonlinear equations of greatest interest in the development of theoretical physics is the following

$$\partial^2 x/\partial t^2 - \Delta x + f(t, x, x_t) = 0,$$

where $f(t, x, x_t)$ depends nonlinearly on x and x_t and is, in a sense, close to a "monotone" function. Equations of this type arise naturally in many contexts, for instance, in classical mechanics, fluid dynamics, quantum field theory (cf. [37]) and have been extensively studied in the last decade (cf. [10,13,37,38,40]). Motivated by the aforementioned contributions, we are interested in the following model which is met in contact mechanics and can be described by the Clarke subdifferential of a nonconvex function

$$\partial^2 x/\partial t^2 - \Delta x + F = 0 \quad \text{in } \Omega \times (0, b). \tag{1.2}$$

It is supposed that F is a known function of the external force of the form

$$-F(t, \theta) \in \partial j(t, \theta, x(t, \theta)) \quad \text{a.e. } (t, \theta) \in (0, b) \times \Omega. \tag{1.3}$$

Here $\partial j(t,\theta,\eta)$ denotes the generalized gradient of Clarke (cf. [5]) with respect to the third variable of a function $j\colon (0,b)\times\Omega\times\mathbb{R}\to\mathbb{R}$ which is assumed to be locally Lipschitz in η. The multivalued function $\partial j(t,\theta,\cdot)\colon \mathbb{R}\to 2^{\mathbb{R}}$ is generally nonmonotone and it includes the vertical jumps. In a physicist's language, it means that the law is characterized by the generalized gradient of a nonsmooth potential j. The system (1.2), (1.3) can serve as a prototype of a dynamic model of nonmonotone skin friction in plane elasticity. For a description of such problems which include beams in adhesive contact, Kirchhoff plates, and other engineering applications, we refer to [26, 31–33]. We underline that due to the lack of convexity of the function $j(t,\theta,\cdot)$, the above problem cannot be formulated as a variational inequality. Its variational formulation leads to a hyperbolic hemivariational inequality, a simple version of the problem (1.1) with $Ay=\Delta y$.

The first goal of our work is to study the existence of solutions for the system (1.1). Secondly, we are also curious about the fact whether or not the system (1.1) can get a good behavior as desired under the proper action of the law of supply. That is, the main properties of hyperbolic hemivariational inequalities such as time-reversibility and the lack of regularizing effects, have some very interesting and important consequences in control problems, too. At present, optimal control problems for hemivariational inequalities have been examined in a number of publications. In particular, we refer to Haslinger and Panagiotopoulos [11] on the existence of optimal control pairs for a class of coercive hemivariational inequalities, Migórski and Ochal [28] about the optimal control problems with the parabolic hemivariational inequalities, J. Park and S. Park [35] on the optimal control problems for the hyperbolic linear systems, and to Tolstonogov [41, 42] about the optimal control problems for subdifferential type differential inclusions. Very recently, Liu and Li [23] studied the approximate controllability for a class of first order hemivariational inequalities. However, there is still little information available on the approximate controllability of hyperbolic hemivariational inequalities like (1.1). Therefore, it is worth to extend the main results of our previous paper in [23] to the control system (1.1).

The paper is organized as follows. In Section 2 we recall the notation and some basic definitions, and preliminary facts, we use throughout the paper. In Section 3, we are concerned with the existence of mild solutions of the system (1.1). The approximate controllability of our problems is analyzed in Section 4, while Section 5 is devoted to a concrete application of our main results.

2 Preliminaries

For a Banach space E with the norm $\|\cdot\|_E$, E^* denotes its dual and $\langle\cdot,\cdot\rangle_E$ the duality pairing between E^* and E. The symbol $\mathcal{L}(X,Y)$ denotes the space of bounded linear operators from a Banach space X to a Banach space Y. $C(J,E)$ is the Banach space of all continuous functions from $J=[0,b]$ into E equipped with the norm $\|x\|_{C(J,E)}=\sup_{t\in J}\|x(t)\|_E$ and $W^{1,2}(J,E)=\{x\mid x,x'\in L^2(J,E)\}$ denotes the Sobolev space with the norm $\|x\|_{W^{1,2}(J,E)}=\big(\|x\|^2_{L^2(J,E)}+\|x'\|^2_{L^2(J,E)}\big)^{1/2}$, where x' stands for the generalized derivative of x, i.e.,

$$\int_J x'(t)\phi(t)\,dt=-\int_J x(t)\phi'(t)\,dt\quad\text{for all }\phi\in C_0^\infty(J).$$

Let $\mathcal{P}(E)$ be the set of all nonempty subsets of E. We will use the following notation

$$\mathcal{P}_{cv}(E)=\{\Omega\in\mathcal{P}(E)\mid \Omega\text{ is convex}\},\qquad \mathcal{P}_{cp}(E)=\{\Omega\in\mathcal{P}(E)\mid \Omega\text{ is compact}\}.$$

Moreover, $B_r(0)$ and $\overline{B_r(0)}$ denote, respectively, the open and the closed balls in a Banach space E centered at origin and of radius $r > 0$.

The key tool in our main results is the following fixed point theorem stated in [7].

Theorem 2.1. *Let $\mathcal{A}\colon \overline{B_r(0)} \to E$ and $\mathcal{B}\colon \overline{B_r(0)} \to P_{cv,cp}(E)$ be two operators such that*

(a) $\mathcal{A}$ is a single-valued contraction with contraction constant $k < \frac{1}{2}$, and

(b) $\mathcal{B}$ u.s.c. and compact.

Then, either

(i) the operator inclusion $x \in \mathcal{A}x + \mathcal{B}x$ has a solution in $\overline{B_r(0)}$, or

(ii) there exist an element $u \in E$ with $\|u\| = r$ such that $\lambda u \in \mathcal{A}u + \mathcal{B}u$ for some $\lambda > 1$.

Next, we recall some concepts of nonsmooth analysis (see [5] for more details). Let $j\colon E \to \mathbb{R}$ be a locally Lipschitz function on a Banach space E. The Clarke generalized directional derivative $j^0(x; v)$ of j at the point $x \in E$ in the direction $v \in E$ is defined by

$$j^0(x; v) = \limsup_{\lambda \to 0^+,\ \zeta \to x} \frac{j(\zeta + \lambda v) - j(\zeta)}{\lambda}.$$

The generalized gradient of j at $x \in E$ is the subset of E^* given by

$$\partial j(x) = \{x^* \in E^* \mid j^0(x; v) \geq \langle x^*, v\rangle \text{ for all } v \in E\}.$$

In the sequel, we shall study the existence of mild solutions and approximate controllability of the following second order evolution inclusion

$$\begin{cases} x''(t) \in Ax(t) + Bu(t) + \partial F(t, x(t)) & \text{for } t \in J, \\ x(0) = x_0,\ x'(0) = y_0, \end{cases} \tag{2.1}$$

where H is a separable Hilbert space, $A\colon D(A) \subset H \to H$ is a closed, linear and densely defined operator which generates a strongly continuous cosine family $\{C(t) \mid t \in J\}$ on H. The notation ∂F stands for the generalized Clarke subdifferential (cf. [5]) of a locally Lipschitz function $F(t, \cdot)\colon H \to \mathbb{R}$ and $B \in \mathcal{L}(U, H)$. The control function $u \in L^2(J, U)$ where U is a Hilbert space of admissible controls.

We remark that, by the definition of the generalized Clarke subdifferential, problem (2.1) is equivalent to the hemivariational inequality (1.1)

$$\begin{cases} \langle -x''(t) + Ax(t) + Bu(t),\ v\rangle_H + F^0(t, x(t); v) \geq 0 & \text{for a.e. } t \in J, \text{ and all } v \in H, \\ x(0) = x_0,\ x'(0) = y_0. \end{cases}$$

Therefore, if we want to prove the solvability of the hemivariational inequality (1.1), we only need to deal with the inclusion (2.1). Similarly to [23], we say that $x \in W^{1,2}(J, H)$ is a solution of (2.1), if there exists $f \in L^2(J, H)$ such that $f(t) \in \partial F(t, x(t))$ for a.e. $t \in J$ and

$$\begin{cases} x''(t) = Ax(t) + Bu(t) + f(t) & \text{for a.e. } t \in J \\ x(0) = x_0,\ x'(0) = y_0. \end{cases}$$

From now on, in order to obtain the solution to problem (2.1), let us consider the following abstract second order initial value problem

$$\begin{cases} x''(t) = Ax(t) + h(t) & \text{for a.e. } t \in J \\ x(0) = x_0, \ x'(0) = y_0. \end{cases}$$

The following basic results concerning strongly continuous cosine operator can be found in the books [1,9] and in the papers [2,34,36,39].

Definition 2.2. A strongly continuous operator $C\colon \mathbb{R} \to \mathcal{L}(H)$ is called a cosine operator, if $C(0) = I$ (identity operator) and

$$C(t+s) + C(t-s) = 2C(t)C(s) \quad \text{for all } t, s \in \mathbb{R}.$$

The linear operator A defined by

$$D(A) = \{x \in H \mid C(t)x \in C^2(\mathbb{R}; H)\}$$

and

$$Ax = \frac{d^2}{dt^2} C(t)x\Big|_{t=0} \quad \text{for } x \in D(A)$$

is the generator of the strongly continuous cosine operator C, $D(A)$ is the domain of A.

It is known that the generator A is a linear, closed, and densely defined operator on H. Next, let C be a cosine operator on H with generator A. The sine operator $S\colon \mathbb{R} \to \mathcal{L}(H)$ associated with the strongly continuous cosine operator C is defined by

$$S(t)x = \int_0^t C(s)x\, ds \quad \text{for all } t \in \mathbb{R}, \ x \in H.$$

In the sequel, we collect some further properties of a cosine operator C and its relations with both the generator A and the associated sine operator S.

Proposition 2.3. *The following assertions hold.*

(i) There exist $M_A \geq 1$ and $\omega \geq 0$ such that $\|C(t)\| \leq M_A e^{\omega|t|}$ and $\|S(t)\| \leq M_A e^{\omega|t|}$.

(ii) $A \int_s^r S(u)x\, du = (C(r) - C(s))x$ for all $0 \leq s \leq r < \infty$.

(iii) There exists $N \geq 1$ such that $\|S(s) - S(r)\| \leq N|\int_s^r e^{\omega|s|}\, ds|$ for all $0 \leq s \leq r < \infty$.

The uniform boundedness principle, together with Proposition 2.3 (i), implies that $C(t)$ and $S(t)$ are uniformly bounded on J by some positive constants M_C and M_S, respectively.

Now, from the argument above, we may introduce the following concept.

Definition 2.4. For each $u \in L^2(J, U)$, a function $x \in C(J, H)$ is called a mild solution of the system (2.1) if there exists $f \in L^2(J, H)$ such that $f(t) \in \partial F(t, x(t))$ for a.e. $t \in J$ and

$$x(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)(Bu(s) + f(s))\, ds \quad \text{for all } t \in J.$$

Throughout this paper, by a solution of system (2.1), we mean the mild solution.

3 Existence of mild solutions

The purpose of this section is to state the existence of mild solutions for problem (2.1).

We list the following standing hypotheses of this paper.

$\underline{H(F_1)}$: the function $x \mapsto F(t,x)$ is locally Lipschitz for all $t \in J$,

$\underline{H(F_2)}$: the function $t \mapsto F(t,x)$ is measurable for all $x \in H$,

$\underline{H(F_3)}$: there exist a function $k \in L^2(J,\mathbb{R}^+)$ and a constant $l > 0$ such that

$$\|\partial F(t,x)\|_H = \sup\{\|f\|_H \mid f \in \partial F(t,x)\} \le k(t) + l\|x\|_H \ \text{ a.e. } t \in J, \text{ and all } x \in H,$$

$\underline{H(S)}$: the sine operator $S(t)$ associated with the operator A is compact for all $t \ge 0$.

Next, we define a multivalued operator $\mathcal{N}\colon L^2(J,H) \to 2^{L^2(J,H)}$ by

$$\mathcal{N}(x) = \{w \in L^2(J,H) \mid w(t) \in \partial F(t,x(t)) \ \text{ a.e. } t \in J\} \ \text{ for all } x \in L^2(J,H). \tag{3.1}$$

From [31, Lemma 5.3], we know that the multifunction $\mathcal{N}$ has nonempty, convex and weakly compact values for each $x \in L^2(J,H)$. Moreover, we have the following lemma.

Lemma 3.1 ([30, Lemma 11]). *If $H(F_1)$–$H(F_3)$ hold, then the operator $\mathcal{N}$ satisfies: if $x_n \to x$ in $L^2(J,H)$, $w_n \to w$ weakly in $L^2(J,H)$ and $w_n \in \mathcal{N}(x_n)$, then $w \in \mathcal{N}(x)$.*

In the sequel, for any $x \in C(J,H) \subset L^2(J,H)$, we can define a multivalued operator $\mathcal{B}\colon C(J,H) \to 2^{C(J,H)}$ as follows

$$\mathcal{B}(x) = \left\{\varphi \in C(J,H) \mid \varphi(t) = \int_0^t S(t-s)f(s)\,ds, \ \ f \in \mathcal{N}(x)\right\}. \tag{3.2}$$

The following property of the multivalued operator $\mathcal{B}$ is an essential result for proving the existence of mild solutions for system (2.1).

Lemma 3.2. *For each $u \in L^2(J,U)$, under the hypotheses $H(F_1)$–$H(F_3)$ and $H(S)$, the multivalued operator $\mathcal{B}\colon C(J,H) \to 2^{C(J,H)}$ is completely continuous, u.s.c., and has compact and convex values.*

Proof. Firstly, for all $x \in C(J,H)$, the convexity of values of the operator $\mathcal{B}(x)$ is obvious by the convexity of $\mathcal{N}(x)$. Next, for convenience, we divide the proof into two steps.

Step 1: We show that the multivalued operator $\mathcal{B}$ is completely continuous and has compact values.

First, we show that the operator $\mathcal{B}$ is bounded, i.e., for all $x \in B_r(0)$ with $r > 0$, $\mathcal{B}(B_r(0))$ is bounded in $C(J,H)$. In fact, for all $x \in B_r(0)$, $\varphi \in \mathcal{B}(x)$, by using $H(F_3)$ and the Hölder inequality, we obtain, for all $t \in J$

$$\begin{aligned}\|\varphi(t)\|_H &\le \int_0^t \|S(t-s)f(s)\|_H\,ds \le M_S \int_0^t (k(s) + l\|x(s)\|_H)\,ds \\ &\le M_S\left(\|k\|_{L^2(J,\mathbb{R}^+)}\sqrt{b} + lrb\right).\end{aligned}$$

Therefore, $\mathcal{B}(B_r(0))$ is a bounded subset in $C(J,H)$.

Next, we prove that $\{\mathcal{B}(x) \mid x \in B_r(0)\}$ is equicontinuous. To this end, let $0 < \tau_1 < \tau_2 \le b$ and $\delta > 0$ be small enough. Then, we obtain

$$\begin{aligned}\|\varphi(\tau_2) - \varphi(\tau_1)\|_H &\le \int_0^{\tau_1} \|[S(\tau_2 - s) - S(\tau_1 - s)]f(s)\|_H\, ds + \int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)f(s)\|_H\, ds \\ &\le \int_0^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)\|(k(s) + lr)\, ds + M_S \int_{\tau_1}^{\tau_2} (k(s) + lr)\, ds \\ &\le \sup_{s\in[0,\tau_1-\delta]} \|S(\tau_2 - s) - S(\tau_1 - s)\| \left(\|k\|_{L^2(J,\mathbb{R}^+)} \sqrt{b} + lrb \right) \\ &\quad + M_S \left(\|k\|_{L^2(J,\mathbb{R}^+)} (2\sqrt{\delta} + \sqrt{\tau_2 - \tau_1}) + lr(2\delta + \tau_2 - \tau_1) \right).\end{aligned}$$

Since Proposition 2.3 (iii) implies the continuity of $S(t)$ in the uniform operator topology, it can be easily seen that the right-hand side of the above inequality is independent of $x \in B_r(0)$ and tends to zero, as $\tau_2 \to \tau_1$. Hence, we obtain that $\{\mathcal{B}(x) \mid x \in B_r(0)\}$ is an equicontinuous subset of $C(J, H)$.

Finally, from the assumption $H(S)$ and by the definition of a relatively compact set, it is not difficult to check that $\{\varphi(t) \mid \varphi \in \mathcal{B}(B_r(0))\}$ is relatively compact in H. Thus, by the generalized Ascoli–Arzelà Theorem, we get that $\mathcal{B}$ is a multivalued compact map.

Step 2: The operator $\mathcal{B}$ has a closed graph.

Let $x_n \in C(J, H)$, $y_n \in \mathcal{B}(x_n)$ be such that $x_n \to \overline{x}$ and $y_n \to \overline{y}$. We will prove that $\overline{y} \in \mathcal{B}(\overline{x})$. Thus $y_n \in \mathcal{B}(x_n)$ implies that there exists $f_n \in \mathcal{N}(x_n)$ such that for all $t \in J$, we have

$$y_n(t) = \int_0^t S(t - s) f_n(s)\, ds.$$

Define the linear continuous operator $G\colon L^2(J, H) \to C(J, H)$ by

$$(Gf)(\cdot) = \int_0^{\cdot} S(\cdot - s) f(s)\, ds \quad \text{for } f \in L^2(J, H).$$

Since $x_n \to \overline{x}$, it follows from Lemma 3.1 and the compactness of the operator $(Gf)(\cdot) = \int_0^{\cdot} S(\cdot - s) f(s)\, ds$ that

$$\overline{y}(t) = \int_0^t S(t - s)\overline{f}(s)\, ds$$

for some $\overline{f} \in \mathcal{N}(\overline{x})$, i.e., $\mathcal{B}$ has a closed graph. Therefore, since $\mathcal{B}$ takes compact values, by Proposition 3.3.12(2) of [31], we know that $\mathcal{B}$ is u.s.c. The proof of the lemma is complete. □

We are now in a position to obtain the main result of this section.

Theorem 3.3. *For each $u \in L^2(J, U)$, if the hypotheses $H(F_1)$–$H(F_3)$ and $H(S)$ are satisfied, then the system (2.1) has a mild solution on J.*

Proof. It is clear that the multivalued map $\mathcal{F}: C(J, H) \to 2^{C(J,H)}$ defined by

$$\mathcal{F}(x) = \left\{ h \in C(J, H) \mid h(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t - s)(f(s) + Bu(s))\, ds,\ f \in \mathcal{N}(x) \right\}$$

has nonempty values, since $\mathcal{N}(x) \neq \emptyset$. In view of the definition of $\mathcal{F}$, the problem of finding mild solutions of (2.1) is equivalent to obtaining fixed points of $\mathcal{F}$. To prove this, we set $\mathcal{F} = \mathcal{A} + \mathcal{B}$, where $\mathcal{B}$ is defined by (3.2) and $\mathcal{A}$ is given by

$$\mathcal{A}(x) = C(t)x_0 + S(t)y_0 + \int_0^t S(t - s)Bu(s)\, ds \quad \text{for } t \in J.$$

According to Theorem 2.1, it is sufficient to show that there exists no element $x \in C(J, H)$ with $\|x\| = r$ such that $\lambda x \in \mathcal{A}x + \mathcal{B}x$ for some $\lambda > 1$.

Indeed, let $\lambda x \in \mathcal{A}(x) + \mathcal{B}(x)$ with $\lambda > 1$, and suppose that there exists $f \in \mathcal{N}(x)$ such that

$$\lambda x(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f(s)\,ds + \int_0^t S(t-s)Bu(s)\,ds.$$

Then, by the assumptions, we obtain

$$\begin{aligned}
\|x(t)\|_H &\le \|C(t)x_0\|_H + \|S(t)y_0\|_H + \left\|\int_0^t S(t-s)f(s)\,ds\right\|_H + \left\|\int_0^t S(t-s)Bu(s)\,ds\right\|_H \\
&\le M_C\|x_0\|_H + M_S\|y_0\|_H + M_S\int_0^t (k(s) + l\|x(s)\|_H)\,ds + M_S\|B\|\int_0^t \|u(s)\|_U\,ds \\
&\le \rho + M_S l\int_0^t \|x(s)\|_H\,ds,
\end{aligned}$$

where

$$\rho = M_C\|x_0\|_H + M_S\big(\|y_0\|_H + \big(\|k\|_{L^2(J,\mathbb{R}^+)} + \|B\|\|u\|_{L^2(J,U)}\big)\sqrt{b}\big).$$

Applying the Gronwall inequality, from the last expression, we obtain

$$\|x(t)\|_H \le \rho e^{M_S l t},$$

which implies

$$\|x\|_C \le \rho e^{M_S l b} =: r.$$

We set

$$K_r = \{x \in C(J, H) \mid \|x\|_C < r + 1\}.$$

Clearly, K_r is an open subset of $C(J, H)$. As an immediate consequence of Lemma 3.2, $\mathcal{B}\colon \overline{K_r} \to \mathcal{P}_{cv,cp}(H)$ is u.s.c. and compact and it is not difficult to get that $\mathcal{A}\colon \overline{K_r} \to H$ is a single-valued contraction with $k < \frac{1}{2}$. Furthermore, from the choice of $\overline{K_r}$, there is no $x \in C(J, H)$ with $\|x\| = r$ such that $\lambda x \in \mathcal{A}x + \mathcal{B}x$ for some $\lambda > 1$.

Thus, by Theorem 2.1, we obtain that the operator inclusion $x \in \mathcal{A}x + \mathcal{B}x$ has a solution in $\overline{K_r}$ which is a mild solution of system (2.1). The proof of the theorem is complete. □

4 Approximate controllability results

Controllability is one of the fundamental concepts in mathematical control theory. This is a qualitative property of dynamical control systems and it is of particular importance in control theory. In recent years, controllability problems for various types of nonlinear dynamical systems in infinite dimensional spaces by using different kinds of approaches have been considered in many publications, see [2,3,23–25,34,36] and the references therein. In this section, we turn our attention to approximate controllability of second order evolution inclusion (2.1). Following [6], we recall the following notions.

Definition 4.1.

(a) Control system (2.1) is said to be exactly controllable on J if, for all x_0, $x_1 \in H$, there exists $u \in L^2(J, U)$ such that the mild solution to system (2.1) satisfies $x(0; u) = x_0$ and $x(b; u) = x_1$.

(b) Control system (2.1) is approximately controllable on J if, for every x_0, $x_1 \in H$, and for every $\varepsilon > 0$, there exists a control $u \in L^2(J, U)$ such that the mild solution x of the problem

(2.1) satisfies $x(0) = x_0$ and $\|x(b) - x_1\| < \varepsilon$. Equivalently, we may say that the attainable set of system (2.1) with the initial value x_0 at the terminal time b, i.e., $K_b(F) = \{x(b) \in H \mid x(\cdot)$ is a mild solution of system (2.1) corresponding to $u \in L^2(J, U)\}$ is dense in H.

In order to analyze the approximate controllability of problem (2.1), we shall consider the linear system which is associated with (2.1), namely

$$\begin{cases} x''(t) = Ax(t) + Bu(t) & \text{for } t \in J, \\ x(0) = x_0, \ x'(0) = y_0. \end{cases} \tag{4.1}$$

The controllability map corresponding to the linear system (4.1) is defined by

$$\Gamma_0^b = \int_0^b S(b-s)BB^*S^*(b-s)\,ds,$$

which is a nonnegative, bounded, and linear operator on H. Here B^* denotes the adjoint of B and $S^*(t-s)$ is the adjoint of $S(t-s)$. Therefore, the inverse of $\varepsilon I + \Gamma_0^b$ exists, for any $\varepsilon > 0$, so the resolvent

$$R(\varepsilon, -\Gamma_0^b) = (\varepsilon I + \Gamma_0^b)^{-1}$$

is well defined. The resolvent is useful in the study of the controllability properties of system (4.1). In this respect, we state a useful characterization of the approximate controllability for (4.1) in terms of the resolvent.

Lemma 4.2 ([3, Theorem 2]). *The linear system (4.1) is approximately controllable on J if and only if $\varepsilon R(\varepsilon, -\Gamma_0^b) \to 0$, as $\varepsilon \to 0^+$ in the strong operator topology.*

At this point, we develop our fixed point approach. For any $x \in C(J, H) \subset L^2(J, H)$, there holds $\mathcal{N}(x) \neq \emptyset$, where $\mathcal{N}$ is defined by (3.1). Hence, for every $\varepsilon > 0$, we can define the multivalued map $\mathcal{F}_\varepsilon \colon C(J, H) \to 2^{C(J,H)}$ as follows

$$\mathcal{F}_\varepsilon(x) = \left\{ h \in C(J,H) \mid h(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)\left(f(s) + Bu_\varepsilon(s)\right) ds, \ f \in \mathcal{N}(x) \right\}$$

for $x \in C(J, H)$, where

$$u_\varepsilon(t) = B^*S^*(b-t)R(\varepsilon, \Gamma_0^b)\left(x_1 - C(b)x_0 - S(b)y_0 - \int_0^b S(b-s)f(s)\,ds\right) \quad \text{for } t \in J.$$

Similarly, as in the proof of Theorem 3.3, we decompose $\mathcal{F}_\varepsilon$ in the following way

$$\mathcal{F}_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon,$$

where

$$\mathcal{A}_\varepsilon(x) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)BB^*S^*(b-s)R(\varepsilon, \Gamma_0^b)\left(x_1 - C(b)x_0 - S(b)y_0\right) ds,$$

and

$$\mathcal{B}_\varepsilon(x) = \left\{ \varphi \in C(J,H) \,\middle|\, \varphi(t) = \int_0^t S(t-s)\Big(f(s) - BB^*S^*(b-s)R(\varepsilon, \Gamma_0^b) \times \int_0^b S(b-\tau)f(\tau)\,d\tau\Big)\,ds, \ f \in \mathcal{N}(x) \right\}$$

for $x \in C(J, H)$. The following property of the multivalued operator $\mathcal{B}_\varepsilon$ is an essential result for obtaining the main results of this section.

Lemma 4.3. *Suppose that hypotheses $H(S)$, $H(F_1)$ and $H(F_2)$ are satisfied. Moreover, we assume that*

$\underline{H(F_4)}$: *there exists a multivalued function $G\colon J \to 2^H$ with weakly compact values which is square integrable such that*

$$\partial F(t,x) \subset G(t) \quad \text{for a.e. } t \in J, \text{ all } x \in H.$$

Then for all $\varepsilon > 0$ the operator $\mathcal{B}_\varepsilon\colon C(J,H) \to 2^{C(J,H)}$ is completely continuous, u.s.c., and has compact and convex values.

Proof. The proof is similar to the one of Lemma 3.2 and for this reason we omit it here. □

Theorem 4.4. *Suppose all of the hypotheses of Lemma 4.3. Then for all $\varepsilon > 0$ the map $\mathcal{F}_\varepsilon$ has a fixed point on J provided the system (4.1) is approximately controllable.*

Proof. From the decomposition of $\mathcal{F}_\varepsilon$, our problem is reduced to find the solutions of the operator inclusion $x \in \mathcal{A}_\varepsilon x + \mathcal{B}_\varepsilon x$. According to Theorem 2.1, it is sufficient to show that there is no element $x \in C(J,H)$ with $\|x\| = R$ such that $\lambda x \in \mathcal{A}_\varepsilon x + \mathcal{B}_\varepsilon x$ for some $\lambda > 1$.

Indeed, let $\lambda x \in \mathcal{A}_\varepsilon x + \mathcal{B}_\varepsilon x$ with $\lambda > 1$, and assume that there exists $f \in \mathcal{N}(x)$ such that

$$\lambda x(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)\left(f(s) + Bu_\varepsilon(s)\right) ds,$$

where

$$u_\varepsilon(t) = B^* U^*(b,t) R(\varepsilon, \Gamma_0^b)\left(x_1 - C(b)x_0 - S(b)y_0 - \int_0^b S(b-s)f(s)\,ds\right) \quad \text{for } t \in J.$$

From the assumption $H(F_4)$, we know that there exists $g \in L^2(J,\mathbb{R}^+)$ such that $\|\partial F(t,x)\|_H = \sup\{\|f\|_H \mid f \in \partial F(t,x)\} \le g(t)$ for a.e. $t \in J$. Then, we obtain

$$\begin{aligned}
\|x(t)\|_H &\le \|C(t)x_0\|_H + \|S(t)y_0\|_H + \left\|\int_0^t S(t-s)f(s)\,ds\right\|_H + \left\|\int_0^t S(t-s)Bu_\varepsilon(s)\,ds\right\|_H \\
&\le M_C\|x_0\|_H + M_S\|y_0\|_H + M_S\int_0^t g(s)\,ds \\
&\quad + \int_0^t \frac{M_S^2\|B\|^2}{\varepsilon}\left(\|x_1\|_H + M_C\|x_0\|_H + M_S\|y_0\|_H + M_S\int_0^b g(\tau)\,d\tau\right) ds \\
&\le \left(1 + \frac{M_S^2\|B\|^2 b}{\varepsilon}\right)\varrho + \frac{M_S^2\|B\|^2 b}{\varepsilon}\|x_1\|_H =: R,
\end{aligned}$$

where $\varrho = M_C\|x_0\|_H + M_S\left(\|y_0\|_H + \|g\|_{L^2(J,\mathbb{R}^+)}\sqrt{b}\right)$. We set

$$K_R = \{x \in C(J,H) \mid \|x\|_C < R+1\}.$$

Clearly, K_R is an open subset of $C(J,H)$. As an immediate consequence of Lemma 4.3, $\mathcal{B}_\varepsilon\colon \overline{K_R} \to \mathcal{P}_{cv,cp}(H)$ is u.s.c. and compact and it is also easy to see that $\mathcal{A}_\varepsilon\colon \overline{K_R} \to H$ is a single-valued contraction with contraction constant $k < \frac{1}{2}$. Moreover, by the choice of $\overline{K_R}$, there is no $x \in C(J,H)$ with $\|x\| = R$ such that $\lambda x \in \mathcal{A}_\varepsilon x + \mathcal{B}_\varepsilon x$ for some $\lambda > 1$.

Thus, by Theorem 2.1, the operator inclusion $x \in \mathcal{A}_\varepsilon x + \mathcal{B}_\varepsilon x$ has a solution in $\overline{K_R}$ which is also a fixed point of $\mathcal{F}_\varepsilon$ on J. The proof is complete. □

From now on, with the aforementioned theorems in mind, we deliver the second main result of this paper.

Theorem 4.5. *Assume the hypotheses of Lemma 4.3. Then system (2.1) is approximately controllable on J if the system (4.1) is approximately controllable on J.*

Proof. From Theorem 4.4, we know that the operator $\mathcal{F}_\varepsilon$ has a fixed point in $C(J,H)$ for all $\varepsilon > 0$. Let x^ε be a fixed point of $\mathcal{F}_\varepsilon$ in $C(J,H)$. It is easy to see that any fixed point of $\mathcal{F}_\varepsilon$ is a mild solution of (2.1) for control u_ε. Therefore, there exists $f^\varepsilon \in \mathcal{N}(x^\varepsilon)$ such that

$$\begin{aligned} x^\varepsilon(t) \in C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)\Big(f^\varepsilon(s) + BB^*S^*(b-s) \\ \times R(\varepsilon,\Gamma_0^b)\Big(x_1 - C(b)x_0 - S(b)y_0 - \int_0^b S(b-\tau)f^\varepsilon(\tau)d\tau\Big)\Big)\, ds. \end{aligned}$$

Now, denote

$$G(f^\varepsilon) = x_1 - C(b)x_0 - S(b)y_0 - \int_0^b S(b-\tau)f^\varepsilon(\tau)d\tau.$$

Then, by the property $I - \Gamma_0^b R(\varepsilon,\Gamma_0^b) = \varepsilon R(\varepsilon,\Gamma_0^b)$, we obtain

$$x^\varepsilon(b) = x_1 - \varepsilon R(\varepsilon,\Gamma_0^b)G(f^\varepsilon).$$

This fact, combined with hypothesis $H(F_4)$ and the Dunford–Pettis Theorem, guarantees that the set $\{f^\varepsilon(\cdot)\}$ is weakly compact in $L^2(J,H)$. Thus, there is a subsequence, still denoted by $\{f^\varepsilon(\cdot)\}$ that converges weakly in $L^2(J,H)$ to $f(\cdot)$. We set

$$Q = x_1 - C(b)x_0 - S(b)y_0 - \int_0^b S(b-\tau)f(\tau)\, d\tau.$$

It is easy to see that

$$\|G(f^\varepsilon) - Q\| \le \left\| \int_0^b S(b-\tau)\,(f^\varepsilon(\tau) - f(\tau))\, d\tau \right\| \le \int_0^b \|S(b-\tau)\,(f^\varepsilon(\tau) - f(\tau))\|\, d\tau \to 0,$$

as $\varepsilon \to 0^+$ due to the compactness of the operator

$$f \mapsto \int_0^{\cdot} S(\cdot - \tau)f(\tau)\, d\tau \colon L^2(J,H) \to C(J,H).$$

Moreover, from the last inequality, we get

$$\begin{aligned} \|x^\varepsilon(b) - x_1\| &= \|\varepsilon R(\varepsilon,\Gamma_0^b)G(f^\varepsilon)\| \\ &\le \|\varepsilon R(\varepsilon,\Gamma_0^b)(Q)\| + \|\varepsilon R(\varepsilon,\Gamma_0^b)\,(G(f^\varepsilon) - Q)\,\| \\ &\le \|\varepsilon R(\varepsilon,\Gamma_0^b)(Q)\| + \|G(f^\varepsilon) - Q\| \to 0, \quad \text{as } \varepsilon \to 0^+, \end{aligned}$$

which implies that system (2.1) is approximately controllable on J. The proof is complete. □

5 An example

In this section we provide an example which illustrates the abstract results of this paper. We consider a controlled system modeled by an evolution partial differential equation. The system is described by the classical wave equation involving a multivalued subdifferential term.

Let Ω be a bounded open subset of $\mathbb{R}^n$ with boundary $\partial\Omega$ of class C^2. The system reads as follows

$$\begin{cases} y_{tt} = \Delta y + u + f, & (x,t) \in \Omega \times (0,b) \\ y(t,x) = 0, & (x,t) \in \partial\Omega \times [0,b] \\ y(0,x) = y_0(x),\ y_t(0,x) = y_1(x), & x \in \Omega, \end{cases} \tag{5.1}$$

where f is a known multivalued function of y of the form

$$-f(x,t) \in \partial F(x,t,y(x,t)) \quad \text{a.e. } (x,t) \in \Omega \times (0,b). \tag{5.2}$$

Here $\partial F(x,t,\xi)$ denotes the Clarke's generalized gradient with respect to the last variable of a function $F\colon \Omega \times (0,b) \times \mathbb{R} \to \mathbb{R}$ which is assumed to be locally Lipschitz in ξ. The multivalued function $\partial F(x,t,\cdot)\colon \mathbb{R} \to 2^{\mathbb{R}}$ is generally nonmonotone and it includes the vertical jumps. Dynamic problems modeled by (5.1) and (5.2) arise in the theory of contact mechanics for elastic bodies in many engineering applications. In such problems, the set Ω represents a plane deformable purely elastic body which remains in contact with another medium introducing frictional effects. In the framework of small deformations, the body is subjected to nonmonotone friction skin effects (skin friction, adhesion, etc.), f is the reaction force of the constraint introducing the skin effect (e.g. due to the gluing material), y is the displacement field, and u is interpreted as the given external loading (a control variable). The condition (5.2) describes a possibly multivalued reaction-displacement law. Since the function $F(x,t,\cdot)$ is not convex in general, the relation (5.2) models the interior nonmonotone contact condition which provides a realistic description of friction and adhesive laws. More details on modeling and applications can be found in [26,31–33] and references therein.

In our example, we denote

$$H = L^2(\Omega), \qquad D(A) = H^2(\Omega) \cap H_0^1(\Omega), \qquad Ay = \Delta y.$$

Let moreover $\mathbb{A} = i(-A)^{1/2}$. It is known that $\mathbb{A}$ generates a C_0-group of operators $e^{\mathbb{A}t}$ on H. The strongly continuous operator-valued function

$$C(t) = \frac{1}{2}\left(e^{\mathbb{A}t} + e^{-\mathbb{A}t}\right) \quad \text{for } t \in \mathbb{R},$$

is called the cosine operator generated by A. It is convenient to introduce the operators

$$S(t) = \frac{\mathbb{A}^{-1}}{2}\left(e^{\mathbb{A}t} - e^{-\mathbb{A}t}\right) \quad \text{for } t \in \mathbb{R}.$$

The operator $S(t)$ is called the sine operator associated with $C(t)$. For more properties of the operators $C(t)$ and $S(t)$, we refer to [34].

Next, we consider the function $F\colon (0,b) \times H \to \mathbb{R}$ be given by

$$F(t,y) = \int_0^b j(x,t,y(x))\,dx \quad \text{for a.e. } t \in (0,b),\ \text{all } y \in H,$$

where

$$j(x,t,z) = \int_0^z \phi(x,t,\theta)\,d\theta \quad \text{for } (x,t) \in \Omega \times (0,b),\ z \in \mathbb{R}.$$

We admit the following assumptions. The function $\phi\colon \Omega \times (0,b) \times \mathbb{R} \to \mathbb{R}$ is such that

(i) for all $x \in \Omega$, $z \in \mathbb{R}$, $\phi(\cdot,x,z)\colon (0,b) \to \mathbb{R}$ is measurable,

(ii) for all $t \in (0,b)$, $z \in \mathbb{R}$, $\phi(t,\cdot,z)\colon \Omega \to \mathbb{R}$ is continuous,

(iii) for all $z \in \mathbb{R}$, there exists a constant $c_1 > 0$ such that $|\phi(\cdot,\cdot,z)| \le c_1(1+|z|)$,

(iv) for every $z \in \mathbb{R}$, $\phi(\cdot,\cdot,z \pm 0)$ exists.

If ϕ satisfies condition (iii), then we have $\partial j(z) \subset [\underline{\phi}(z), \overline{\phi}(z)]$ for $z \in \mathbb{R}$ (we omit (x,t) here), where $\underline{\phi}(z)$ and $\overline{\phi}(z)$ denote the essential supremum and essential infimum of ϕ at z (see [5, p. 34]).

If ϕ satisfies conditions (i)–(iv), then the function j defined above is such that

(i) for all $x \in \Omega$, $z \in \mathbb{R}$, $j(\cdot,x,z)$ is measurable and $j(\cdot,\cdot,0) \in L^2(\Omega \times (0,b))$,

(ii) for all $t \in (0,b)$, $z \in \mathbb{R}$, $j(t,\cdot,z)\colon \Omega \to \mathbb{R}$ is continuous,

(iii) for all $(x,t) \in \Omega \times (0,b)$, $j(x,t,\cdot)\colon \mathbb{R} \to \mathbb{R}$ is locally Lipschitz,

(iv) there exists a constant $c_2 > 0$ such that

$$|\eta| \le c_2(1+|z|) \quad \text{for all } \eta \in \partial j(x,t,z),\ (x,t) \in \Omega \times (0,b),$$

(v) there exists a constant $c_3 > 0$ such that

$$j^0(x,t,z;-z) \le c_3(1+|z|) \quad \text{for all } (x,t) \in \Omega \times (0,b).$$

Thus, combining (5.1) with (5.2), we arrive to problem (1.1). Finally, it is known (see [43], p. 358) that the linear system corresponding to (5.1) is approximately controllable on $J = [0,b]$. Therefore, all the hypotheses of Theorem 4.5 are satisfied, and the system (5.1) is approximately controllable on J.

Acknowledgements

The work was supported by NNSF of China Grants Nos. 11271087, 61263006, NSF of Guangxi Grant No. 2014GXNSFDA118002, Special Funds of Guangxi Distinguished Experts Construction Engineering, the Project of Baise University No. 2014KB01; the Project of Guangxi Education Department grant No. KY2015LX391, the Marie Curie International Research Staff Exchange Scheme Fellowship within the 7th European Community Framework Programme under Grant Agreement No. 295118, and the National Science Center of Poland under the Maestro Advanced Project No. DEC-2012/06/A/ST1/00262. The third author is also partially supported by the International Project co-financed by the Ministry of Science and Higher Education of Republic of Poland under Grant No. W111/7.PR/2012.

The authors are gratefully indebted to Prof. L. Simon and the anonymous referees for their insightful comments that improved the paper.

References

[1] W. Arendt, C. Batty, M. Hieber, F. Neubrander, *Vector-valued Laplace transforms and Cauchy problems*, Monographs in Mathematics, Vol. 96, Birkhäuser/Springer Basel AG, Basel, 2011. MR2798103

[2] P. Balasubramaniam, P. Muthukumar, Approximately controllability of second order stochastic distributed implicit functional differential systems with infinite delay, *J. Optim. Theory Appl.* **143**(2009), 225–244. MR1934390; url

[3] A. E. Bashirov, N. I. Mahmudov, On concepts of controllability for deterministic and stochastic systems, *SIAM J. Control Optim.* **37**(1999), No. 6, 1808–1821. MR1720139; url

[4] S. Carl, D. Motreanu, Extremal solutions of quasilinear parabolic inclusions with generalized Clarke's gradient, *J. Differential Equations* **191**(2003), 206–233. MR1973288; url

[5] F. H. Clarke, *Optimization and nonsmooth analysis*, Wiley, New York, 1983. MR0709590

[6] R. F. Curtain, H. Zwart, *An introduction to infinite dimensional linear systems theory*, Springer-Verlag, New York, 1995. MR1351248; url

[7] B. C. Dhage, Multi-valued mappings and fixed points II, *Tamkang J. Math.* **37**(2006), No. 1, 27–46. MR2223941

[8] G. Duvaut, J.-L. Lions, *Inequalities in mechanics and physics*, Springer, 1976. MR0521262

[9] H. O. Fattorini, *Second order linear differential equations in Banach spaces*, North-Holland Mathemtical Studies, Vol. 108, North-Holland Publishing Co., Amsterdam, 1985. MR0797071

[10] V. Georgiev, G. Todorova, Existence of a solution of the wave equation with nonlinear damping and source terms, *J. Differential Equations* **109**(1994), 295–308. MR1273304; url

[11] J. Haslinger, P. D. Panagiotopoulos, Optimal control of systems governed by hemivariational inequalities. Existence and approximation results, *Nonlinear Anal.* **24**(1995), No. 1, 105–119. MR1308473; url

[12] Y. Huang, Z. H. Liu, S. Migórski, Elliptic hemivariational inequalities with nonhomogeneous Neumann boundary conditions and their applications to static frictional contact problems, *Acta Appl. Math.* **138**(2015), 153–170. MR3365585

[13] H. A. Levine, Some additional remarks on the nonexistence of global solutions to nonlinear wave equations, *SIAM J. Math. Anal.* **5**(1974), 138–146. MR0399682

[14] J.-L. Lions, Sur un nouveau type de problème pour non linéaire opérateurs hyperboliques du 2e ordre (in French), *Sém. J. Leray, Collége de France* **II**(1965–66), 17–33. url

[15] J.-L. Lions, *Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles* (in French), Dunod, Paris; Gauthier-Villars, Paris, 1968. MR0244606

[16] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires* (in French), Dunod; Gauthier-Villars, Paris, 1969. MR0259693

[17] J.-L. Lions, Inéquations variationnelles d'évolution, International Congress of Mathématicians, Nice, 1970.

[18] J.-L. Lions, *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 1* (in French), Masson, Paris, 1988. MR953547

[19] J.-L. Lions, *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 2* (in French), Masson, Paris, 1988. MR963060

[20] J.-L. Lions, G. Stampacchia, Variational inequalities, *Comm. Pure Appl. Math.* **20**(1967), 493–519. MR0216344

[21] Z. H. Liu, Browder–Tikhonov regularization of non-coercive evolution hemivariational inequalities, *Inverse Problems* **21**(2005), 13–20. MR2146161; url

[22] Z. H. Liu, On boundary variational-hemivariational inequalities of elliptic type, *Proc. Roy. Soc. Edinburgh Sect. A* **140**(2010), No. 2, 419–434. MR2608584

[23] Z. H. Liu, X. W. Li, Approximate controllability for a class of hemivariational inequalities, *Nonlinear Anal. Real World Appl.* **22**(2015), 581–591. MR3280853; url

[24] Z. H. Liu, X. W. Li, Approximate controllability of fractional evolution systems with Riemann–Liouville fractional derivatives, *SIAM J. Control Optim.* **53**(2015), No. 4, 1920–1933. MR3369987; url

[25] Z. H. Liu, X. W. Li, D. Motreanu, Approximate controllability for nonlinear evolution hemivariational inequalities in Hilbert spaces, *SIAM J. Control Optim.* **53**(2015), No. 5, 3228–3244. MR3411720; url

[26] Z. H. Liu, S. Migórski, Noncoercive damping in dynamic hemivariational inequality with application to problem of piezoelectricity, *Discrete Contin. Dyn. Syst. Ser. B* **9**(2008), 129–143. MR2358347

[27] S. Migórski, On existence of solutions for parabolic hemivariational inequalities, *J. Comput. Applied Math.* **129**(2001), 77–87. MR1823211

[28] S. Migórski, A. Ochal, Optimal control of parabolic hemivariational inequalities, *J. Global Optim.* **17**(2000), 285–300. MR1807980

[29] S. Migórski, A. Ochal, Existence of solutions for second order evolution inclusions with application to mechanical contact problems, *Optimization* **55**(2006), No. 1–2, 101–120. MR2221727

[30] S. Migórski, A. Ochal, Quasi-static hemivariational inequality via vanishing acceleration approach, *SIAM J. Math. Anal.* **41**(2009), 1415–1435. MR2540272; url

[31] S. Migórski, A. Ochal, M. Sofonea, *Nonlinear inclusions and hemivariational inequalities. Models and analysis of contact problems*, Advances in Mechanics and Mathematics, Vol. 26, Springer, New York, 2013. MR2976197

[32] Z. Naniewicz, P. D. Panagiotopoulos, *Mathematical theory of hemivariational inequalities and applications*, Marcel Dekker, New York, 1995. MR1304257

[33] P. D. Panagiotopoulos, G. Pop, On a type of hyperbolic variational-hemivariational inequalities, *J. Appl. Anal.* **5**(1999), No. 1, 95–112. MR1690058

[34] L. Pandolfi, The controllability of the Gurtin–Pipkin equation: a cosine operator approach, *Appl. Math. Optim.* **64**(2011), No. 3, 467–468. MR2846257; url

[35] J. Y. Park, S. H. Park, Existence of solutions and optimal control problems for hyperbolic hemivariational inequalities, *ANZIAM J.* **47**(2005), 51–63. MR2159851

[36] R. Sakthivel, N. I. Mahmudov, J. H. Kim, On controllability of second order nonlinear impulsive differential systems, *Nonlinear Anal.* **71**(2009), 45–52. MR2518010; url

[37] I. Segal, Non-linear semi-groups, *Ann. of Math. (2)* **78**(1963), 339–364. MR0152908

[38] L. Simon, *Application of monotone type operators to nonlinear PDEs*, Prime Rate Kft., Budapest, 2013.

[39] S. Sivasankaran, M. Mallika Arjunan, V. Vijayakumar, Existence of global solutions for second order impulsive abstract partial differential equations, *Nonlinear Anal.* **74**(2011), No. 17, 6747–6757. MR2834074; url

[40] G. Todorova, E. Vitillaro, Blow-up for nonlinear dissipative wave equations in $\mathbb{R}^n$, *J. Math. Anal. Appl.* **303**(2005), 242–257. MR2113879

[41] A. A. Tolstonogov, Relaxation in nonconvex optimal control problems with subdifferential operators, *J. Math. Sci. (N. Y.)* **140**(2007), No. 6, 850–872. MR2179455

[42] A. A. Tolstonogov, Control systems of subdifferential type depending on a parameter, *Izv. Math.* **72(5)**(2008), 985–1022.

[43] M. Tucsnak, G. Weiss, *Observation and control for operator semigroups*, Birkhäuser Verlag, Basel, 2009. MR2502023