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GLOBAL SOLUTIONS FOR ABSTRACT FUNCTIONAL DIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS

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Abstract. In this paper we study the existence of global solutions for a class of abstract functional differential equation with nonlocal conditions. An application is considered.

1. Introduction

In this paper, we discuss the existence of global solutions for an abstract functional differential equation with nonlocal conditions in the form

$$\left\{ \begin{array}{ll} u'(t) & = Au(t) + f(t, u_t, u(\rho(t))), \quad t \in I = [0, \infty), \\ u_0 & = g(u, \varphi) \in \mathcal{B}, \end{array} \right. \tag{1.1}$$

where A is the infinitesimal generator of a C_0-semigroup of bounded linear operators $(T(t))_{t\geq 0}$ defined on a Banach space $(X, \|\cdot\|)$; the history $u_t : (-\infty, 0] \to X$, defined by $u_t(\theta) := u(t+\theta)$, belongs to some abstract phase space $\mathcal{B}$ defined axiomatically, $\varphi \in \mathcal{B}$ and ρ, f, g are appropriated functions.

The study of initial value problems with nonlocal conditions arises to deal specially with some situations in physics. For the importance of nonlocal conditions in different fields, we refer the reader to [1, 2, 3, 4, 5] and the references contained therein.

Differential systems with nonlocal conditions are considered in different works. In Byszewski [1] is studied the existence of mild, strong and classical solutions for the abstract Cauchy problem

$$\begin{aligned} x'(t) &= Ax(t) + f(t, x(t)), \quad t \in [0, a], \\ x(0) &= x_0 + q(t_1, t_2, t_3, ..., t_n, x(\cdot)) \in X, \end{aligned}$$

where A is the generator of a C_0-semigroup of linear operators on a Banach space X; f, q are appropriated functions; $0 < t_1 < .... < t_n \leq a$ are prefixed numbers and the symbol $q(t_1, t_2, t_3, ..., t_n, x(\cdot))$ is used in the sense that the function $x(\cdot)$ can be evaluated only in the points t_i, for instance $q(t_1, t_2, t_3, ..., t_n, x(\cdot)) = \sum_{i=1}^{n} \alpha_i x(t_i)$.

The literature concerning differential equations with nonlocal conditions also include ordinary differential equations, partial differential equations, abstract partial functional differential, integro-differential equations. Related to this matter, we cite

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among others works, [1, 2, 3, 4, 5, 6, 7, 8, 9]. To the best of our knowledge, the study of the existence of " global " solutions for equations described in the abstract form (1.1), is a untreated topic, and this fact, is the main motivation of this work.

In this paper, A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $(T(t))_{t\geq 0}$ defined on a Banach space $(X, \|\cdot\|)$ and M, δ are positive constants such that $\| T(t) \| \leq Me^{-\delta t}$, for every $t \geq 0$. In this work, the phase space $\mathcal{B}$ is a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$ and verifying the following axioms.

(**A**) If $x : (-\infty, a) \to X$, $a > 0$, is continuous on $[0, a)$ and $x_0 \in \mathcal{B}$, then for every $t \in [0, a)$ the following conditions hold:
  (i) x_t is in $\mathcal{B}$.
  (ii) $\| x(t) \| \leq H \| x_t \|_{\mathcal{B}}$.
  (iii) $\| x_t \|_{\mathcal{B}} \leq K(t) \sup\{\| x(s) \| : 0 \leq s \leq t\} + M(t) \| x_0 \|_{\mathcal{B}}$,
  where $H > 0$ is a constant; $K, M : [0, \infty) \to [1, \infty)$, K is continuous, M is locally bounded and H, K, M are independent of $x(\cdot)$.

(**A1**) For the function $x(\cdot)$ in (**A**), the function $t \to x_t$ is continuous from $[0, a)$ into $\mathcal{B}$.

(**B**) The space $\mathcal{B}$ is complete.

Remark 1. In this paper, $\mathfrak{K}$ is a positive constant such that $\sup_{t\geq 0}\{K(t), M(t)\} \leq \mathfrak{K}$. We know from that the functions $M(\cdot), K(\cdot)$ are bounded if, for instance, $\mathcal{B}$ is a fading memory space, see [10, p. 190] for additional details.

Example 1.1. The phase space $\mathbf{C_r} \times \mathbf{L^p}(\varrho, \mathbf{X})$

Let $r \geq 0$, $1 \leq p < \infty$, and let $\varrho : (-\infty, -r] \mapsto \mathbb{R}$ be a nonnegative measurable function which satisfies the conditions (g-5)-(g-6) of [10]. Briefly, this means that ϱ is locally integrable and there exists a non-negative locally bounded function γ on $(-\infty, 0]$ such that $\varrho(\xi + \theta) \leq \gamma(\xi)\varrho(\theta)$, for all $\xi \leq 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set whose Lebesgue measure zero.

The space $\mathcal{B} = C_r \times L^p(\varrho, X)$ consists of all classes of functions $\psi : (-\infty, 0] \mapsto X$ such that ψ is continuous on $[-r, 0]$, Lebesgue-measurable, and $\varrho\|\psi\|^p$ is Lebesgue integrable on $(-\infty, -r)$. The seminorm in $C_r \times L^p(\varrho, X)$ is defined as follows:

$$\| \psi \|_{\mathcal{B}} := \sup\{\|\psi(\theta)\| : -r \leq \theta \leq 0\} + \left(\int_{-\infty}^{-r} \varrho(\theta) \|\psi(\theta)\|^p d\theta \right)^{1/p}.$$

The space $\mathcal{B} = C_r \times L^p(\varrho, X)$ satisfies axioms **(A), (A1),** and **(B).** Moreover, when $r = 0$ and $p = 2$, one can then take $H = 1$, $M(t) = \gamma(-t)^{1/2}$ and $K(t) = 1 + \left(\int_{-t}^{0} \varrho(\theta)\, d\theta \right)^{1/2}$ for $t \geq 0$ (see [10, Theorem 1.3.8]). We also note that if the conditions (g-5)-(g-7) of [10] hold, then $\mathcal{B}$ is a uniform fading memory.

For additional details concerning phase space, we cite [10].

In this paper, $h : [0, \infty) \to \mathbb{R}$ is a positive, continuous and non-decreasing function such that $h(0) = 1$ and $\lim_{t\to\infty} h(t) = \infty$. In the sequel, $C_0(X)$ and $C_h^0(X)$ are the spaces defined by

$$\begin{aligned} C_0(X) &= \{x \in C([0,\infty) : X) : \lim_{t\to\infty} \| x(t) \| = 0\}, \\ C_h^0(X) &= \{x \in C([0,\infty) : X) : \lim_{t\to\infty} \frac{\| x(t) \|}{h(t)} = 0\}, \end{aligned}$$

endowed with the norms $\| x \|_0 = \sup_{t\geq 0} \| x(t) \|$ and $\| x \|_h = \sup_{t\geq 0} \frac{\|x(t)\|}{h(t)}$ respectively. Additionally, we define the space

$$\mathcal{B}C_h^0(X) = \{u \in C(\mathbb{R}, X) ; u_0 \in \mathcal{B}, u|_{[0,\infty)} \in C_h^0(X)\}$$

endowed with the norm $\| u \|_{\mathcal{B}C_h^0} = \| u_0 \|_{\mathcal{B}} + \| u_{|[0,\infty)} \|_h$.

We recall here the following compactness criterion.

Lemma 1.1. *A set $B \subset C_h^0(X)$ is relatively compact in $C_h^0(X)$ if, and only if, B is equicontinuous, $B(t) = \{x(t) : x \in B\}$ is relatively compact in X for all $t \geq 0$ and $\lim_{t\to\infty} \frac{\|x(t)\|}{h(t)} = 0$ uniformly for $x \in B$.*

2. Existence of mild solutions

In this section, we study the existence of mild solutions for the nonlocal abstract Cauchy problem (1.1). To treat this system, we introduce the following conditions.

- $\mathbf{H_1}$ The function $\rho : [0,\infty) \to [0,\infty)$ is continuous and $\rho(t) \leq t$ for every $t \geq 0$.
- $\mathbf{H_2}$ The function $f : I \times \mathcal{B} \times X \to X$ is continuous and there exist an integrable function $m : [0,\infty) \to [0,\infty)$ and a nondecreasing continuous function $W : [0,\infty) \to (0,\infty)$ such that $\| f(t,\psi,x) \| \leq m(t)W(\| \psi \|_{\mathcal{B}} + \| x \|)$, for every $(t,\psi,x) \in [0,\infty) \times \mathcal{B} \times X$.
- $\mathbf{H_3}$ The function $g : \mathcal{B}C_h^0(X) \times \mathcal{B} \to \mathcal{B}$ is continuous and there exists $L_g \geq 0$ such that

$$\| g(u,\varphi) - g(v,\varphi) \|_{\mathcal{B}} \leq L_g \| u - v \|_{\mathcal{B}C_h^0}, \quad u, v \in \mathcal{B}C_h^0(X).$$

- $\mathbf{H_4}$ The function $g(\cdot,\varphi) : \mathcal{B}C_h^0(X) \to \mathcal{B}$ is completely continuous and uniformly bounded. In the sequel, N is a positive constant such that $\| g(\psi,\varphi) \|_{\mathcal{B}} \leq N$ for every $\psi \in \mathcal{B}C_h^0(X)$.

In this work, we adopt the following concept of mild solutions for (1.1).

Definition 2.1. *A function $u \in C(\mathbb{R}, X)$ is called a mild solution of system (1.1) if $u_0 = g(u,\varphi)$ and*

$$u(t) = T(t)g(u,\varphi)(0) + \int_0^t T(t-s)f(s,u_s,u(\rho(s)))ds, \quad t \in I = [0,\infty).$$

Now, we can establish the principal results of this section.

Theorem 2.1. *Assume $\mathbf{H_1}, \mathbf{H_2}$ and $\mathbf{H_4}$ be hold and that the followings properties are verified.*

(a) *The set* $\{T(t)f(s,\psi,x) : (s,\psi,x) \in [0,t] \times B_r(0,\mathcal{B}) \times B_r(0,X)\}$ *is relatively compact in* X *for every* $t \geq 0$ *and all* $r > 0$.

(b) *For every* $K > 0$, $\lim_{t\to\infty} \frac{1}{h(t)} \int_0^t m(s)W(Kh(s))ds = 0$.

If $M\mathfrak{K}(1+H)\int_0^\infty m(s)ds < \int_c^\infty \frac{ds}{W(s)}$, *where* $c = \mathfrak{K}N(1+H)(1+MH)$, *then there exists a mild solution for system (1.1).*

Proof. Let $\Gamma : \mathcal{B}C_h^0(I : X) \to \mathcal{B}C_h^0(X)$ be the operator defined by $(\Gamma u)_0 = g(u,\varphi)$ and

$$\Gamma u(t) \quad = \quad T(t)g(u,\varphi)(0) + \int_0^t T(t-s)f(s,u_s,u(\rho(s)))ds, \qquad t \geq 0.$$

Next we prove that Γ verifies the conditions in [11, Theorem 6.5.4]. To begin, we note that for $u \in \mathcal{B}C_h^0(X)$ and $t \geq 0$, $\| u(t) \| \leq \| u_{|_I} \|_h \, h(t)$. If $\mathfrak{K}$ is the constant in Remark 1, then

$$\frac{\| \Gamma u(t) \|}{h(t)} \quad \leq \quad \frac{MNH}{h(t)} + \frac{M}{h(t)} \int_0^t m(s)W(2\mathfrak{K} \| u \|_{\mathcal{B}C_h^0} h(s))ds,$$

which from **(b)** permits us to conclude that Γ is a function from $\mathcal{B}C_h^0(X)$ into $\mathcal{B}C_h^0(X)$.

Let $(x^n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{B}C_h^0(X)$ and $x \in \mathcal{B}C_h^0(X)$ such that $x^n \to x$ in $\mathcal{B}C_h^0(X)$. Let $\varepsilon > 0$ and $\mathfrak{N} = \sup_{n\in\mathbb{N}} \| x^n \|_{\mathcal{B}C_h^0}$. From condition (**b**), we can select $L > 0$ such that

$$\frac{2MNH}{h(t)} + \frac{M}{h(t)} \int_0^t m(s)W(2\mathfrak{K}\mathfrak{N}h(s))ds \leq \frac{\varepsilon}{2}, \qquad t \geq L. \tag{2.1}$$

From the properties of the functions f, g, we can choose $N_\varepsilon \in \mathbb{N}$ such that

$$\| g(x^n,\varphi) - g(x,\varphi) \|_{\mathcal{B}} \quad \leq \quad \frac{\varepsilon}{4(MH+1)}, \tag{2.2}$$

$$\int_0^L \| f(s,x_s^n,x^n(\rho(s))) - f(s,x_s,x(\rho(s))) \| \, ds \quad \leq \quad \frac{\varepsilon}{4M}, \tag{2.3}$$

for every $n \geq N_\varepsilon$. Under these conditions, we see that

$$\sup\{\frac{\| \Gamma x^n(t) - \Gamma x(t) \|}{h(t)} : t \in [0,L], \; n \geq N_\varepsilon\} \leq \frac{\varepsilon}{2}. \tag{2.4}$$

Moreover, for $t \geq L$ and $n \geq N_\varepsilon$, we find that

$$\begin{aligned}\frac{\| \Gamma x^n(t) - \Gamma x(t) \|}{h(t)} \quad &\leq \quad M\frac{\| g(x^n,\varphi)(0) - g(x,\varphi)(0) \|}{h(t)} \\ &\quad + \frac{M}{h(t)} \int_0^t \| f(s,x_s^n,x^n(\rho(s))) - f(s,x_s,x(\rho(s))) \| \, ds \\ &\leq \quad \frac{2MNH}{h(t)} + \frac{M}{h(t)} \int_0^t m(s)W(2\mathfrak{K}\mathfrak{N}h(s))ds,\end{aligned}$$

and hence,

$$\sup\{\frac{\| \Gamma x^n(t) - \Gamma x(t) \|}{h(t)} : t \geq L,\ n \geq N_\varepsilon\} \leq \frac{\varepsilon}{2}. \tag{2.5}$$

From the inequalities (2.2), (2.4) and (2.5), it follows that $\| \Gamma x^n - \Gamma x \|_{\mathcal{B}C_h^0} \leq \varepsilon$, for every $n \geq N_\varepsilon$, which proves that Γ is continuous.

To prove that Γ is completely continuous, we introduce the decomposition $\Gamma = \Gamma_1 + \Gamma_2$, where $(\Gamma_1 u)_0 = g(u, \varphi)$, $(\Gamma_2 u)_0 = 0$ and

$$\Gamma_1 u(t) = T(t)g(u, \varphi)(0), \qquad t \geq 0, \tag{2.6}$$

$$\Gamma_2 u(t) = \int_0^t T(t-s)f(s, u_s, u(\rho(s)))ds, \qquad t \geq 0. \tag{2.7}$$

From Lemma 1.1 and the properties of the semigroup $(T(t))_{t\geq 0}$, it is easy to see that Γ_1 is completely continuous. Next, we prove that Γ_2 is also completely continuous. From [12, Lemma 3.1], we infer that $\Gamma_2 B_r(0, \mathcal{B}C_h^0) \mid_{[0,a]} = \{\Gamma_2 x_{|_{[0,a]}};\ x \in B_r(0, \mathcal{B}C_h^0)\}$ is relatively compact in $C([0, a], X)$ for every $a > 0$. Considering this property, Lemma 1.1 and the fact that

$$\frac{\| \Gamma_2 x(t) \|}{h(t)} \leq \frac{M}{h(t)} \int_0^t m(s)W(2\mathfrak{K}rh(s))ds \to 0, \text{ as } t \to \infty,$$

uniformly for $x \in B_r(0, \mathcal{B}C_h^0)$, we can conclude that the set $\Gamma_2(B_r(0, \mathcal{B}C_h^0))$ is relatively compact in $\mathcal{B}C_h^0(X)$ for every $r > 0$. Thus, Γ is completely continuous.

To finish the proof, we obtain a priori estimates for the solutions of the integral equation $u = \lambda\Gamma u,\ \lambda \in (0, 1)\}$. Let $\lambda \in (0, 1)$ and $u^\lambda \in \mathcal{B}C_h^0(X)$ be a solution of $\lambda\Gamma z = z$. By using the notation $\alpha^\lambda(t) = \mathfrak{K}(\sup_{s\in[0,t]} \| u^\lambda(s) \| + \| u_0^\lambda \|_{\mathcal{B}})$, for $t \in I$ we see that

$$\begin{aligned} \| u^\lambda(t) \| &\leq MNH + M \int_0^t m(s)W(\| u_s^\lambda \|_{\mathcal{B}} + \| u^\lambda(\rho(s)) \|)ds, \\ &\leq MNH + M \int_0^t m(s)W((1 + H)\alpha^\lambda(s))ds, \end{aligned}$$

and hence,

$$\begin{aligned} \alpha^\lambda(t) &\leq \mathfrak{K} \| g(u, \varphi) \|_{\mathcal{B}} + \mathfrak{K}MNH + \mathfrak{K}M \int_0^t m(s)W(1 + H)\alpha^\lambda(s))ds, \\ &\leq \mathfrak{K}N(1 + MH) + \mathfrak{K}M \int_0^t m(s)W((1 + H)\alpha^\lambda(s))ds. \end{aligned}$$

Defining by β_λ the right hand side of the last inequality, we see that $\beta'_\lambda(t) \leq \mathfrak{K}Mm(t)W((1 + H)\beta_\lambda(t))$, which implies that

$$\int_{(1+H)\beta_\lambda(0)=c}^{(1+H)\beta_\lambda(t)} \frac{ds}{W(s)} \leq M\mathfrak{K}(1 + H) \int_0^\infty m(s)ds < \int_c^\infty \frac{ds}{W(s)}.$$

This inequality permits us to conclude that the set of functions $\{\beta_\lambda : \lambda \in (0,1)\}$ is bounded in $C(\mathbb{R})$ and as consequence that the set $\{u^\lambda : \lambda \in (0,1)\}$ is bounded in $\mathcal{B}C_h^0(X)$.

Now the assertion is a consequence of [11, Theorem 6.5.4]. The proof is complete. ■

Theorem 2.2. *Assume assumptions* $\mathbf{H_1}$, $\mathbf{H_2}$ *and* $\mathbf{H_3}$ *be hold. If the conditions* (**a**) *and* (**b**) *of Theorem 2.1 are valid and*

$$L_g(1+MH) + M \liminf_{r\to\infty} \int_0^{+\infty} \frac{m(s)W((1+2\mathfrak{K})rh(s))}{rh(s)} ds < 1, \tag{2.8}$$

then there exists a mild solution of (1.1).

Proof. Let $\Gamma, \Gamma_1, \Gamma_2$ be the operators introduced in the proof of Theorem 2.1. We claim that there exists $r > 0$ such that $\Gamma(B_r) \subset B_r$, where $B_r = B_r(0, \mathcal{B}C_h^0(X))$. In fact, if this property is false, then for every $r > 0$, there exist $x^r \in B_r$ and $t^r \geq 0$ such that $r <\| (\Gamma x^r)_0 \|_{\mathcal{B}} + \| \frac{\Gamma x^r(t^r)}{h(t^r)} \|$. Consequently,

$$\begin{aligned}
r &\leq \| g(x^r,\varphi) \|_{\mathcal{B}} + M\frac{\| g(x^r,\varphi)(0) \|}{h(t^r)} \\
&\quad + \frac{M}{h(t^r)} \int_0^t m(s) W(\mathfrak{K} \| x_0^r \|_{\mathcal{B}} + \mathfrak{K} \sup_{\theta\in[0,s]} \| x^r(\theta) \| + \| x^r(\rho(s)) \|) ds \\
&\leq L_g \| x^r \|_{\mathcal{B}C_h^0} + \| g(0,\varphi) \|_{\mathcal{B}} + \frac{MH}{h(t^r)} L_g \| x^r \|_{\mathcal{B}C_h^0} \\
&\quad + \frac{M}{h(t^r)} \| g(0,\varphi) \|_{\mathcal{B}} + \frac{M}{h(t^r)} \int_0^{t^r} m(s) W(\mathfrak{K}r + \mathfrak{K}rh(s) + rh(s)) ds \\
&\leq L_g(1+MH)r + (1+M) \| g(0,\varphi) \|_{\mathcal{B}} + M \int_0^{t^r} \frac{m(s)((2\mathfrak{K}+1)rh(s))}{h(s)} ds
\end{aligned}$$

and then,

$$1 \leq L_g(1+MH) + M \liminf_{r\to\infty} \int_0^{+\infty} \frac{m(s)W((2\mathfrak{K}+1)rh(s))}{rh(s)} ds,$$

which is contrary to (2.8).

Let $r > 0$ such that $\Gamma(B_r) \subset B_r$. From the proof of Theorem 2.1, we know that Γ_2 is a completely continuous on B_r. Moreover, from the estimate,

$$\begin{aligned}
&\| \Gamma_1 u - \Gamma_1 v \|_{\mathcal{B}C_h^0} \\
&\quad \leq L_g \| u - v \|_{\mathcal{B}C_h^0} + H \| g(u,\varphi) - g(v,\varphi) \|_{\mathcal{B}} \leq L_g(1+H) \| u - v \|_{\mathcal{B}C_h^0},
\end{aligned}$$

we infer that Γ_1 is a contraction on B_r from which we conclude that Γ is a condensing operator on B_r.

The assertion is now a consequence of [13, Theorem 4.3.2]. ■

3. An application

To complete this work, we study the existence of global solutions for a concrete partial differential equation with nonlocal conditions. In the sequel, $X = L^2[0,\pi]$ and $A : D(A) \subset X \to X$ is the operator $Ax = x''$ with domain $D(A) := \{x \in X : x'' \in X,\ x(0) = x(\pi) = 0\}$. It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on X. Furthermore, A has discrete spectrum with eigenvalues of the form $-n^2, n \in \mathbb{N}$, and corresponding normalized eigenfunctions given by $z_n(x) = (\frac{2}{\pi})^{\frac{1}{2}} \sin(nx)$; the set of functions $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis for X and $T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t}\langle x, z_n\rangle z_n$ for every $x \in X$ and all $t \geq 0$. It follows from this last expression that $(T(t))_{t\geq 0}$ is a compact semigroup on X and that $\| T(t) \| \leq e^{-t}$ for every $t \geq 0$.

As phase space, we choose the space $\mathcal{B} = C_0 \times L^2(\varrho, X)$, see Example 1.1, and we assume that the conditions $(g5)$ and $(g7)$ in [10] are verified. Under these conditions, $\mathcal{B}$ is a fading memory space, and as consequence, there exists $\mathfrak{K} > 0$ such that $\sup_{t\geq 0}\{M(t), K(t)\} \leq \mathfrak{K}$.

Consider the delayed partial differential equation with nonlocal conditions

$$
\begin{aligned}
(3.1)\quad \frac{\partial}{\partial t} w(t,\xi) &= \frac{\partial^2}{\partial \xi^2} w(t,\xi) + \int_{-\infty}^{t} a(t, s-t)\, w(s,\xi)\, ds + b(t, w(\rho(t), \xi)),\\
(3.2)\quad w(t,0) &= w(t,\pi) = 0, \qquad t \geq 0,\\
(3.3)\quad w(s,\xi) &= \sum_{i=1}^{\infty} L_i w(t_i + s, \xi) + \varphi(s,\xi), \qquad s \leq 0,\ \xi \in [0,\pi],
\end{aligned}
$$

for $t \geq 0$ and $\xi \in [0,\pi]$, and where $(L_i)_{i\in\mathbb{N}}, (t_i)_{i\in\mathbb{N}}$ are sequences of real numbers and $\varphi \in \mathcal{B}$.

To treat this system in the abstract form (1.1), we assume $\sum_{i=1}^{\infty} | L_i | h(t_i) < \infty$, the functions $\rho : [0,\infty) \to [0,\infty)$, $a : \mathbb{R}^2 \to \mathbb{R}$ are continuous, $\rho(t) \leq t$ and $L_1(t) = (\int_{-\infty}^{0} \frac{a^2(t,s)}{\varrho(s)} ds)^{1/2} < \infty$, for every $t \geq 0$. We also assume that $b : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous and that there exists a continuous function $L_2 : [0,\infty) \to [0,\infty)$ such that $|b(t,x)| \leq L_2(t)|x|$, for every $t \geq 0$ and $x \in \mathbb{R}$.

By defining the functions $g : \mathcal{B}C_h^0(X) \to \mathcal{B}$, $f : [0,\infty) \times \mathcal{B} \times X \to X$ by $g(u,\varphi) = \varphi + \sum_{i=1}^{\infty} L_i u_{t_i}$ and

$$
f(t,\psi,x)(\xi) = \int_{-\infty}^{0} a(t,s)\, \psi(s,\xi) ds + b(t, x(\rho(t),\xi)),
$$

we can transform system (3.1)-(3.3) into the abstract system (1.1). Moreover, it is easy to see that f, g are continuous, $\| f(t,\psi,x) \| \leq L_1(t) \| \psi \|_{\mathcal{B}} + L_2(t) \| x \|$, for all $(t,\psi,x) \in [0,\infty)$ and g verifies conditions $\mathbf{H_3}$ with $L_g = \mathfrak{K}\sum_{i=1}^{\infty} L_i h(t_i)$.

The next result is a direct consequence of Theorem 2.2. We omit the proof.

Proposition 3.1. *If $\mathfrak{K}\sum_{i=1}^{\infty} L_i h(t_i) + (1+2\mathfrak{K})\int_0^{\infty}(L_1(s) + L_2(s))ds < 1$, then there exists a mild solution of (3.1)-(3.3).*

References

[1] L. Byszewski, Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem, *J. Math. Anal. Appl.* **162** (2) 494-505 (1991).

[2] L. Byszewski, Existence, uniqueness and asymptotic stability of solutions of abstract nonlocal Cauchy problems, *Dynam. Systems Appl.* **5** (4) 595-605 (1996).

[3] L. Byszewski and H. Akca, Existence of solutions of a semilinear functional differential evolution nonlocal problem, *Nonlinear Anal.* **34** (1) 65-72 (1998).

[4] L. Byszewski and H. Akca, On a mild solution of a semilinear functional-differential evolution nonlocal problem, *J. Appl. Math. Stochastic Anal.* **10** (3) 265-271 (1997).

[5] S. K. Ntouyas and P. Tsamatos, Global existence for semilinear evolution equations with nonlocal conditions, *J.Math. Anal. Appl.* **210** 679-687 (1997).

[6] G. L. Karakostas and P. Ch. Tsamatos, Sufficient conditions for the existence of nonnegative solutions of a nonlocal boundary value problem, *Appl. Math. Lett.* **15** (4) 401-407 (2002).

[7] W. E. Olmstead and C. A. Roberts, The one-dimensional heat equation with a nonlocal initial condition, *Appl. Math. Lett.* **10** (3) 89-94 (1997).

[8] E. Hernï¿$\frac{1}{2}$ndez, Existence results for partial neutral functional differential equations with nonlocal conditions, *Dynam. Systems Appl.* **11** (2) 241-252 (2002).

[9] K. Balachandran, J. Y. Park and M. Chandrasekaran, Nonlocal Cauchy problem for delay integrodifferential equations of Sobolev type in Banach spaces, *Appl. Math. Lett.* **15** (7) 845-854 (2002).

[10] Y. Hino, S. Murakami and T. Naito, In Functional-differential equations with infinite delay, *Lecture Notes in Mathematics*, 1473, Springer-Verlag, Berlin (1991).

[11] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York (2003).

[12] E. Hernández and M. McKibben, Some comments on: " Existence of solutions of abstract nonlinear second-order neutral functional integrodifferential equations " *Comput. Math. Appl.* **50** (5-6) 655-669 (2005).

[13] R. H. Martin, Nonlinear Operators and Differential Equations in Banach Spaces, Robert E. Krieger Publ. Co., Florida (1987).

[14] S. Aizicovici and H. Lee, Nonlinear nonlocal Cauchy problems in Banach spaces, *Appl. Math. Lett.* **18** (4) 401-407 (2005).

[15] D. Bahuguna, Existence, uniqueness and regularity of solutions to semilinear nonlocal functional differential problems, *Nonlinear Anal.* **57** (7-8) 1021-1028 (2004).

[16] L. Byszewski and V. Lakshmikantham, Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space, *Appl. Anal.* **40** (1) 11-19 (1991).

[17] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, *Applied Mathematical Sciences*, 44, Springer-Verlag, New York-Berlin (1983).

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