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Bounded and Periodic Solutions of Nonlinear Integro-differential Equations with Infinite Delay

Manuel Pinto *
Departamento de Matemáticas
Facultad de Ciencias - Universidad de Chile
Casilla 653, Santiago - Chile

Abstract: By using the concept of integrable dichotomy, the fixed point theory, functional analysis methods and some new technique of analysis, we obtain new criteria for the existence and uniqueness of bounded and periodic solutions of general and periodic systems of nonlinear integro-differential equations with infinite delay.

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1 Introduction.

Based on the exponential dichotomy of a linear nonautonomous system

$$x' = A(t)x, \tag{1}$$

several qualitative properties of the solutions of ordinary and functional differential equations have been well investigated (see e.g.[11-23,27-35,37-38]). In particular, the existence of bounded and periodic solutions of several families of quasilinear systems has been advantageously studied with the help of

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the Green matrix $G(t,s)$ of system (1) and concluding that for any bounded function f

$$x(t) = \int_{-\infty}^{\infty} G(t,s)f(s)ds \tag{2}$$

is a bounded solution of the non-homogeneous linear system

$$x' = A(t)x + f(t). \tag{3}$$

Nevertheless, notice that similar results can be obtained by the more general condition:

$$\sup_{t\in\mathbf{R}} \int_{-\infty}^{\infty} |G(t,s)|ds < \infty, \tag{4}$$

that is, if system (1) has an integrable dichotomy. For example, condition (4) holds for any integrable (h,k)-dichotomy without need of being exponential [7,8,14].

Under the assumptions that A and f are periodic and (1) has an integrable dichotomy, we will study the periodicity of the solution x given by (2). The existence of periodic solutions of functional differential equations has been discussed extensively in theory and in practice (for example, see [1-3,7-10,14-19] and the references cited therein), but there are few papers considering integrable dichotomies.

In this paper, under condition (4) we consider systems of the type (see e.g.[2,3])

$$y'(t) = A(t)y(t) + \int_{-\infty}^{t} C(t,s,y(s))ds + g(t,y(t),y(t-\tau(t))). \tag{5}$$

The investigation of integro-differential equations with delay had a big impulse when the Volterra integral equations with linear convolutions appeared, see for example, Burton [2-5], Corduneanu [9,10], Hale [17], Hale-Lunel [19], Gopalsamy [20], Lakshmikantham et al [23], Yoshizawa [37,38], etc. The existence of bounded and periodic solutions of nonlinear Volterra equations with infinite delay has been extensively discussed by Burton and others under boundedness conditions (see [2-5]). Several papers treat on this subject, see for example [28-35]. After introducing the space of $BC_{(-\infty,\rho]}$ by combining Lyapunov function (functional) and fixed point theory, sufficient conditions which guarantee the existence of periodic solutions of a variety of infinite delay systems

$$y'(t) = f(t, y_t) \tag{6}$$

have been obtained. Very soon, in several works linear integro-differential equations,

$$y'(t) = A(t)y(t) + \int_{-\infty}^{t} C(t,s)y(s)ds + f(t) \tag{7}$$

were studied under some sufficient conditions which guarantee the existence of periodic solutions of system (7) . Recently, Chen [6] considered a kind of integro-differential equations more general than (7),

$$y'(t) = A(t)y(t) + \int_{-\infty}^{t} C(t,s)y(s)ds + g(t, y(t)) + f(t). \tag{8}$$

By using exponential dichotomy and fixed point theorem, he discusses the existence, uniqueness and stability of periodic solutions of (8). Beside its theoretical interest, the study of integro-differential equations with delay has great importance in applications. For these reasons the theory of integro-differential equations with infinite delay has drawn the attention of several authors (see [2,3,9-13,16-23,29,31,37,38]).

This paper is organized as follows. In next section, some definitions and preliminary results are introduced. Important properties of integrable dichotomies are obtained. In particular, any integrable (h, k)-dichotomy satisfies our requirements. Section 3 is devoted to establish some criteria for the existence and uniqueness of bounded and periodic solutions of system (5), which include systems as (6) and (8). Integrable dichotomies and Banach and Schauder fixed point theorems will be fundamental to obtain the results. Finally, in section 4 we show some examples, where our results can be applied.

2 Bounded and periodic solutions of nonhomogeneous systems.

Let $\mathbf{C}^n$ and $\mathbf{R^n}$ denote the sets of complex and real vectors, and $|x|$ any convenient norm for $x \in \mathbf{C}^n$, also let $\mathbf{C} = \mathbf{C}^1$, $\mathbf{R} = \mathbf{R^1}$ and $\mathbf{R}_+ = (0, \infty)$.

Now, we recall some of the definitions (see[1,14,15,24-26]), concerning integrable dichotomy and the notion of $(h,k)-$dichotomy for linear nonautonomous ordinary differential equations . A solution-matrix $\Phi(t)$ of system (1) is said to be a fundamental-matrix, if $\Phi(0) = I$. For a projection matrix P, we define $G = G_P$ a Green matrix as:

$$G(t,s) = \begin{cases} \Phi(t)P\Phi^{-1}(s), \text{ for } t \geq s, \\ -\Phi(t)(I-P)\Phi^{-1}(s), \text{ for } s > t. \end{cases}$$

Definition 1 . *System (1) is said to have an integrable dichotomy , if there exist a projection P and $\mu \in \mathbf{R}_+$ such that its Green matrix G satisfies:*

$$\sup_{t\in\mathbf{R}} \int_{-\infty}^{\infty} |G(t,s)|ds = \mu. \tag{9}$$

Definition 2 *Let h, k : $\mathbf{R} \to \mathbf{R}_+$ be two positive continuous functions. The linear system (1) is said to possess an $(h,k)-$dichotomy, if there are a projection matrix P and a positive constant K such that for all $t, s \in \mathbf{R}$ the following inequality holds:*

$$|G(t,s)| \leq K g_{h,k}(t,s),$$

where

$$g_{h,k}(t,s) = \begin{cases} h(t)h(s)^{-1}, t \geq s \\ k(s)k(t)^{-1}, s > t \end{cases}$$

and $h(t)^{-1}$ *denotes* $1/h(t)$.

Definition 3 *We say that system (1) has an integrable (h,k)-dichotomy if system (1) has an (h,k)-dichotomy for which there exists $\mu_{h,k} > 0$ such that*

$$\sup_{t\in\mathbf{R}} \int_{-\infty}^{\infty} g_{h,k}(t,s)ds = \mu_{h,k}.$$

Let us state our main hypothesis on the linear system (1).

(I) System (1) has an integrable dichotomy .

(D) System (1) satisfies (I) with a projection P such that $\Phi(t)P\Phi^{-1}(t)$ is bounded.

If the system (1) has an (h,k)-dichotomy integrable then necessarily the dichotomy satisfies condition (D).

Remark 4 *Obviously, the case $h(t) = e^{-\beta t}, k(t) = e^{-\alpha t}, \beta, \alpha > 0$, yields an exponential dichotomy, but $(h,k)-$ dichotomy systems are more general than these ones(see, section 4 and for example [7,p.73, 14]).*

However, h and k have an exponential domination:

Lemma 1 . Let $\varphi : \mathbf{R} \to (0,\infty)$ and $\psi : \mathbf{R} \to (0,\infty)$ be two locally integrable functions, satisfying for $\mu > 0$ constant

$$\varphi(t)\int_{-\infty}^{t} \varphi(s)^{-1}ds \leq \mu \quad , \tag{10}$$

$$\psi(t)\int_{t}^{\infty} \psi(s)^{-1}ds \leq \mu \quad . \tag{11}$$

Then for any $t_0 \in \mathbf{R}, \varphi(t) \leq ce^{-\mu^{-1}t},\ t \geq t_0,$ and $\psi(t) \leq ce^{\mu^{-1}t}$ for $t \leq t_0$, where $c > 0$.

Proof. If $u(t) = \int_{-\infty}^{t} \varphi(s)^{-1}ds$ then $u' = \varphi^{-1} \geq \mu^{-1}u$ by (10). So, $u(t) \geq u(t_0)e^{\mu^{-1}(t-t_0)}$ for $t \geq t_0$. Therefore $\varphi(t) \leq \mu u(t)^{-1} \leq \mu u(t_0)e^{-\mu^{-1}(t-t_0)}$. To solve (10), let $v(t) = \int_{t}^{\infty} \psi(s)^{-1}ds$. We have $v \leq -\mu v'$, *i.e.* $(ve^{\mu^{-1}t})' \geq 0$ or $v(t_0) - v(t)e^{\mu^{-1}(t-t_0)} \leq 0$. By (11), $\psi(t) \leq \mu v(t_0)^{-1}e^{\mu^{-1}(t-t_0)}$ for $t \leq t_0$. ∎

Corollary 1. For every integrable (h,k)- dichotomy, there exist constants $\alpha, M > 0$ such that

$$h(t) \leq Me^{-\alpha t} \text{ for all } t \geq 0; k(t)^{-1} \leq Me^{\alpha t} \text{ for all } t \leq 0.$$

Proposition 1. If system (1) has an integrable dichotomy, then $x(t) = 0$ is the unique bounded solution of system (1).

Proof. Define $B_0 \subset \mathbf{C}^n$ to be the set of initial conditions $\xi \in \mathbf{C}^n$ pertaining to bounded solutions of Eq (1). Take any vector $\xi \in \mathbf{C}^n$ and assume first that $(I-P)\xi \neq 0$. Define $\phi(t)^{-1} = |\Phi(t)(I-P)\xi|$, we may write

$$\int_t^\infty \Phi(t)(I-P)\xi\phi(s)ds = \int_t^\infty \Phi(t)(I-P)\Phi^{-1}(s)\Phi(s)(I-P)\xi\phi(s)ds.$$

So using the integrability of the dichotomy, we have

$$\int_t^\infty \phi(s)ds \leq \mu\phi(t), \qquad \text{uniformly in } t.$$

For this, $\liminf\limits_{s\in[t,\infty)} \phi(s) = 0$, and then $|\Phi(t)(I-P)\xi|$ must be unbounded.

If we assume now that $P\xi \neq 0$, then defining $\phi(t)^{-1} = |\Phi(t)P\xi|$, we perform the same procedure, with the integral over the interval $(-\infty, t]$ we conclude that $\liminf\limits_{s\in(-\infty,t]} \phi(s) = 0$, which means $|\Phi(t)P\xi|$ must be unbounded. Thus $B_0 = \{0\}$ and the only bounded solution of system (1) is $x(t) = 0$. ■

Proposition 2. If system (1) satisfies condition (I) then the system (3) has exactly one bounded solution x, which can be represented by (2).

Proof. It is not difficult to check that $x(t)$, given by (2), is a bounded solution of (3).If there exists another bounded solution $z(t)$, then $x(t) - z(t)$ is a bounded solution of the homogeneous linear system (1). By Proposition 1, $x(t) \equiv z(t)$. The uniqueness of the bounded solution of (3) is proved. ■

From now on, the boundedness of $\Phi(t)P\Phi^{-1}(t)$ is fundamental.

Proposition 3. If the linear system (1) satisfies hypothesis (D), then the projector P is unique, i.e., P is decided uniquely by the integrable dichotomy.

Proof. Firstly, prove that for an integrable dichotomy we have that $|\Phi(t)P|$ is bounded for $t \geq t_0 (t_0 \in \mathbf{R})$ and $|\Phi(t)(I-P)|$ is bounded for $t \leq t_0$. Let $\varphi(t) = |\Phi(t)P|$ and $\psi(t) = |\Phi(t)(I-P)|$. We have

$$\int_{-\infty}^t \Phi(t)P\varphi(s)^{-1}ds = \int_{-\infty}^t \Phi(t)P\Phi^{-1}(s)\Phi(s)P\varphi(s)^{-1}ds.$$

If follows from (9) and the last inequality that φ satisfies (10):

$$\int_{-\infty}^{t} \varphi(t)\varphi(s)^{-1}ds \leq \mu.$$

By (9), $\psi(t) = |\Phi(t)(I-P)|$ similarly satisfies (11):

$$\int_{t}^{\infty} \psi(t)\psi(s)^{-1}ds \leq \mu.$$

So, Lemma 1 implies that for every $t_0 \in \mathbf{R}$, there exists $M > 0$ constant such that

$$|\Phi(t)P| \leq M \quad \text{for } \; t \geq t_0 \; \text{ and } \; |\Phi(t)(I-P)| \leq M \quad \text{for } \; t \leq t_0. \tag{12}$$

Assume now that there exists another projector $\tilde{P}$ satisfying the integrability condition (9) , i.e.,

$$\int_{-\infty}^{t} |\Phi(t)\tilde{P}\Phi^{-1}(s)|ds + \int_{t}^{\infty} |\Phi(t)(I-\tilde{P})\Phi^{-1}(s)|ds \leq \tilde{\mu}.$$

Similar to the above discussion , there exists $\tilde{M} > 0$ such that

$$|\Phi(t)\tilde{P}| \leq \tilde{M} \text{ for all } t \geq 0, |\Phi(t)(I-\tilde{P})| \leq \tilde{M} \text{ for all } \; t \leq 0. \tag{13}$$

Take any $\xi \in \mathbf{C}^n$, for $t \geq 0$, it follows from (12) and (13) that

$$\begin{aligned} |\Phi(t)P(I-\tilde{P})\xi| =& |\Phi(t)P\Phi^{-1}(0)\Phi(0)(I-\tilde{P})\xi| \\ \leq& |\Phi(t)P\Phi^{-1}(0)||\Phi(0)(I-\tilde{P})\xi| \leq M|(I-\tilde{P})\xi|, \; (t \geq 0), \end{aligned}$$

where M is constant. On the other hand, for $t \leq 0$, it follows from (12), (13), and the boundedness of $\Phi(t)P\Phi^{-1}(t)$ that

$$\begin{aligned} |\Phi(t)P(I-\tilde{P})\xi| =& |\Phi(t)P\Phi^{-1}(t)\Phi(t)(I-\tilde{P})\Phi^{-1}(0)\Phi(0)(I-\tilde{P})\xi| \\ \leq& |\Phi(t)P\Phi^{-1}(t)||\Phi(t)(I-\tilde{P})\Phi^{-1}(0)||\Phi(0)(I-\tilde{P})\xi| \qquad (14) \\ \leq& M|(I-\tilde{P})\xi|, \; (t \leq 0), \end{aligned}$$

where M is constant.

It follows from (14) that for any $\xi \in \mathbf{C}^n$, $x(t) = \Phi(t)P(I - \tilde{P})\xi$ is the bounded solution of system (1). By Proposition 1, we have $P(I - \tilde{P})\xi = 0$, which implies $P = P\tilde{P}$. Since $\Phi(t)(I - P)\Phi^{-1}(t)$ is also bounded, similar to the above discussion, we also have $(I - P)\tilde{P} = 0$, i.e., $\tilde{P} = P\tilde{P}$.Therefore, $P = P\tilde{P} = \tilde{P}$. This shows that the projection P is unique. ■

The bounded matrix $\Phi(t)P\Phi^{-1}(t)$ is periodic if A is so.

Proposition 4. Let the linear system (1) satisfy condition (D), where A is a T periodic matrix , then $\Phi(t)P\Phi^{-1}(t)$ is also a $T-$periodic function.
Proof. By the periodicity, we note that $\Phi(t + T)$ is also a solution matrix of (1). Obviously, we have $\Phi(t + T) = \Phi(t)C = \Phi(t)\Phi(T)$, using $\Phi(0) = I$. Note that $\tilde{P} = \Phi(T)P\Phi^{-1}(T)$ is also a projection.Since $\Phi(t)\tilde{P}\Phi^{-1}(s) = \Phi(t+T)P\Phi^{-1}(s + T)$, the dichotomy is also integrable with $\tilde{P}$. By Proposition 3, the projection P is unique. Thus, $\Phi(T)P\Phi^{-1}(T) = P$. Therefore, $\Phi(t + T)P\Phi^{-1}(t + T) = \Phi(t)P\Phi^{-1}(t)$, i.e., $\Phi(t)P\Phi^{-1}(t)$ is $T-$periodic function. ■

Finally, we obtain two important consequences to the non homogeneous linear system (3).

Proposition 5. Let all conditions in Proposition 4 hold and $f(t)$ is a T-periodic function. Then system (3) has exactly one $T-$periodic solution, which can be represented as (2).
Proof. By Proposition 4, it is not difficult to check that $x(t)$, given by (2), is a $T-$ periodic solution and so it is a bounded solution. Then, by Proposition 2, the result follows. ■

3 Existence of bounded and periodic solutions

In this section, we will prove some results about the existence and uniqueness of bounded and periodic solutions of system (5). Let the corresponding space of the initial conditions φ:

$$BC(-\infty, t_0] = \{\varphi : (-\infty, t_0] \to \mathbf{R}^n / \varphi(t) \text{ is a bounded continuous function}\}$$

and for any $\varphi \in BC(-\infty, t_0]$, define the norm $|\varphi| = \sup\{|\varphi(t)|/t \in (-\infty, t_0]\}$. Let $x(t, t_0, \varphi)$ (or $x(t, \varphi)$, $x(t)$ for convenience) denote the solution of the system (5) with bounded continuous initial function $\varphi \in BC(-\infty, t_0]$.

Consider the following nonlinear integro-differential equation with both continuous delay and discrete delay of the form

$$y'(t) = A(t)y(t) + F_1(t, y(t), y(t-\tau(t))) + F_2(t, y_t), \tag{15}$$

where F_1 involves bounded delay and F_2 unbounded delay. Typically system (15) has the form:

$$y'(t) = A(t)y(t) + g(t, y(t), y(t-\tau(t))) + \int_{-\infty}^{t} C(t, s, y(s))ds \tag{16}$$

and its linear system (1) has an integrable dichotomy , where $y \in \mathbf{C}^n$, $A(t) = (a_{ij}(t))_{n\times n}$, $A(t)$ is continuous on $\mathbf{R}$, $g : \mathbf{R}\times\mathbf{C}^n\times\mathbf{C}^n \to \mathbf{C}^n$ is continuous, $C : \mathbf{R}\times\mathbf{R}\times\mathbf{C^n} \to \mathbf{C^n}$ is continuous. Now we introduce the following conditions:

Dichotomy conditions:

- (I) The linear system (1) possesses an integrable dichotomy with projection P and constant μ.
- (D) The linear system (1) satisfies condition (I) and $\Phi(t)P\Phi^{-1}(t)$ is bounded.

  Periodic conditions:

- (P) $A(t+T) = A(t)$, $g(t+T, y, z) = g(t, y, z)$, $\tau(t+T) = \tau(t)$, $C(t+T, s+T, y) = C(t, s, y)$.

  Lipschitz conditions:

- (L_1) There exists a nonnegative constant L_1 such that $2L_1 < \mu^{-1}$ and

  $$|g(t, x_1, y_1) - g(t, x_2, y)| \le L_1(|x_1 - x_2| + |y_1 - y_2|),\ x_i, y_i \in \mathbf{C}^n, t \in \mathbf{R}.$$

- (L_2) There exists a continuous function $\lambda : \mathbf{R^2} \to [0, \infty)$ such that: for $t, s \in \mathbf{R}$ and any $y_1, y_2 \in \mathbf{C^n}$ we have

  $$|C(t, s, y_1) - C(t, s, y_2)| \le \lambda(t, s)|y_1 - y_2|$$

  and

  $$\int_{-\infty}^{t} \lambda(t, s)ds \le L_2, L_2 < \mu^{-1}, \int_{-\infty}^{t} |C(t, s, 0)|ds \le \rho_2.$$

  Estimation conditions :

- (E_1) For every real $r > 0$ there exist c_1, ρ_1 nonnegative constants with $2c_1 < \mu^{-1}$ such that

$$|g(t,x,y)| \le c_1(|x| + |y|) + \rho_1 \text{ for} |x|, |y| \le r \text{ uniformly for } t \in \mathbf{R}.$$

- (E_2) For every real $r > 0$ there exist two continuous function $\lambda, \gamma : \mathbf{R^2} \to [0, \infty)$ such that

$$|C(t,s,y)| \le \lambda(t,s)|y| + \gamma(t,s); t, s \in \mathbf{R}, |y| \le r$$

and nonnegative constants c_2, ρ_2 such that:

$$\int_{-\infty}^{t} \lambda(t,s)ds \le c_2 \ , \ c_2 < \mu^{-1}; \int_{-\infty}^{t} \gamma(t,s)ds \le \rho_2.$$

Continuity conditions.

- (C_1) The function g: $\mathbf{R} \times \mathbf{C^n} \times \mathbf{C^n} \to \mathbf{C^n}$ is continuous .

- (C_2) F_2 is a continuous functional in the following sense. Let $r > 0; t, s \in \mathbf{R}$ and $y_1, y_2 \in \mathbf{C^n}, |y_i| \le r, i = 1, 2$. For any $\epsilon > 0$ there exist $\delta > 0$ and $\gamma : \mathbf{R}^2 \to [0, \infty)$ a function such that $|y_1 - y_2| \le \delta$ implies

$$|C(t,s,y_1) - C(t,s,y_2)| \le \epsilon\gamma(t,s), t, s \in \mathbf{R},$$

where $\rho = \sup\limits_{t\in\mathbf{R}} \int\limits_{-\infty}^{t} \gamma(t,s)ds < \infty$.

Now we are ready to state our main results.

For bounded solutions, we obtain the following results.

Theorem 1: The conditions $(I), (L_1), (L_2)$ and $2L_1 + L_2 < \mu^{-1}$ imply that the system (16) has a unique bounded solution.

Theorem 2: The conditions $(I), (E_1), (E_2), (C_1), (C_2)$ and $2c_1 + c_2 < \mu^{-1}$ imply that system (16) has at least one bounded solution.

Analogously, for periodic solutions we have.

Theorem 3: If $2L_1 + L_2 < \mu^{-1}$ and the asumptions (D), (P) , (L_1) and (L_2) hold, then system (16) has exactly one T- periodic solution.

Theorem 4: If $2c_1 + c_2 < \mu^{-1}$ and the asumptions $(D), (P), (E_1), (E_2)$, (C_1) and (C_2) hold, then system (16) has at least one T-periodic solution.

Remark 5 *Several known results to integro-differential equations using exponential dichotomy theory are special cases of our Theorems. In particular, they have been extended to integrable (h, k)- dichotomy. Obviously, the generalization requires only a dichotomy satisfying condition (D).*

Remark 6 *Considering system (5)*

$$x' = A(t)x + g(t, x(t), x(t - \tau(t))) \tag{17}$$

Krasnoselskii (see[36]) proved that if A is a stable constant matrix, without delay and $\lim\limits_{|x|+|y|\to+\infty} \frac{|g(t,x,y)|}{|x|+|y|} = 0$, then system (17) has at least one periodic solution. In our case, applying Theorem 4, this result is also valid for the retarded system (17), requiring the hypothesis: (D), (P), (C_1) and $|g(t, x, y)| \leq c_1(|x| + |y|) + \rho_1$, with $2c_1 < \mu^{-1}$ and ρ_1 constants for $|x| + |y| \leq r, r > 0$.

Remark 7 *As a special case, when $\lambda(t, s) = \lambda_1(t-s)$, the smallness condition in (L_2) can be reduced to $\int_{-\infty}^{t} \lambda(t, s)ds = \int_{-\infty}^{t} \lambda_1(t - s)ds = \int_0^{+\infty} \lambda_1(s)ds < \mu^{-1}$.*

We will prove only Theorems 3 and 4 about periodic solutions because, with the obvious differences, the proofs of Theorems 1 and 2 are respectively similar. In Theorems 1 and 2, we will use Proposition 2, while in Theorems 3 and 4, we will use Proposition 5. In all of them, we need the same operator defined on the Banach space

$$B = \{u : \mathbf{R} \to \mathbf{C^n} | u \text{ is continuous and bounded}\}$$

provided with the supremum-norm.

Proof of Theorem 3. Consider the Banach space

$$P = \{ \ u : \mathbf{R} \to \mathbf{C^n} | u(t) \text{ is continuous } T - \text{periodic function}\}$$

provided with the norm $\|u\| = \sup \{|u(t)| : 0 \leq t \leq T\}$. For any $u \in P$, consider the integro-differential correspondence:

$$y'(t) = A(t)y(t) + g(t, u(t), u(t-\tau(t))) + \int_{-\infty}^{t} C(t, s, u(s))ds = A(t)y(t) + F(t, u), \tag{18}$$

where F is the functional:

$$F(r,u) = \int\limits_{-\infty}^{r} C(r,s,u(s))ds + g(r,u(r),u(r-\tau(r))). \tag{19}$$

By the conditions (D) and (P) and Proposition 5, $F(t,u)$ is T-periodic in t and system (18) has exactly one $T-$periodic solution which can be written as

$$y_u(t) = \int_{-\infty}^{\infty} G(t,r)F(r,u)dr. \tag{20}$$

So, the operator $\Gamma : P \to P$ given by

$$\Gamma u(t) = y_u(t), \quad u \in P \tag{21}$$

is well defined and any fixed point of Γ is a T-periodic solution of system (18). By (L_1) and (L_2), we shall prove that Γ is a contraction mapping in P. In fact, for any $u_1, u_2 \in P$, it follows from (19), (20) and the conditions in Theorem 3 that

$$\begin{aligned} |F(r,u_1) - F(r,u_2)| &\leq \int\limits_{-\infty}^{r} \lambda(r,s)|u_1(s) - u_2(s)|ds + 2L_1\|u_1 - u_2\| \\ &\leq (2L_1 + L_2)\|u_1 - u_2\|, \text{and} \\ |\Gamma u_1(t) - \Gamma u_2(t)| &\leq \int\limits_{-\infty}^{\infty} |G(t,r)||F(r,u_1) - F(r,u_2)|dr. \\ &\leq \mu(2L_1 + L_2)\|u_1 - u_2\|. \end{aligned}$$

It follows from $\mu(2L_1 + L_2) < 1$, that Γ is a contraction mapping. Therefore Γ has exactly one fixed point u in P. It is easy to check that u is the unique T-periodic solution of (18).

Proof of Theorem 4. Take the Banach space P and the operator Γ defined in the proof of Theorem 3. Now by using Schauder's fixed point theorem, we shall prove that Γ has at least one fixed point under the assumption of Theorem 4.

In order to prove this,we set $B_r = \{u \in P/\|u\| \leq r\}$ and $\mathbf{C}_r = \{(x,y) \in \mathbf{C}^{2n}/|x|, |y| \leq r\}$.

Lemma 1: There exists $N \in \mathbb{N}$ such that $\Gamma : B_N \to B_N$.

Proof. If not, for any $n \in \mathbb{N}$, there exists $u_n \in B_n$ such that $\|\Gamma u_n\| > n$. For any sufficiently small ϵ, it follows from (E_1) and (E_2) that there exists sufficiently large $N \in \mathbb{N}$ such that if $n > N$ then

$$\frac{|F(r,u_n)|}{n} \leq 2c_1 + \frac{\rho_1}{n} + \int\limits_{-\infty}^{r} \lambda(r,s)\frac{|u_n(s)|ds}{n} + \int\limits_{-\infty}^{r} \frac{\gamma(r,s)ds}{n} \tag{22}$$

$$\leq 2c_1 + c_2 + \varepsilon.$$

Therefore, it follows from the assumption in Theorem 4 and (22) that

$$\frac{|\Gamma u_n(t)|}{n} \leq \frac{1}{n}\left\{\int_{-\infty}^{\infty} |G(t,r)\|F(r,u_n)|dr\right\} \leq \mu(2c_1 + c_2 + \epsilon).$$

As $\mu(2c_1+c_2) < 1$, taking sufficiently small ϵ, we have $\mu(2c_1+c_2)+\mu\epsilon < 1$. Therefore, it follows that $\lim\limits_{n\to\infty} \sup \frac{\|\Gamma u_n\|}{n} < 1$, which implies that for sufficiently large n, $\frac{\|\Gamma u_n\|}{n} < 1$. This is a contradiction to $\|\Gamma u_n\| > n$. Thus there exists $N \in \mathbb{N}$ such that $\Gamma : B_N \to B_N$.

Lemma 2: ΓB_N is a relatively compact set of P.

Proof. In fact,since $\Gamma B_N \subset B_N$, $\{\Gamma u(t)/u \in B_N\}$ is uniformly bounded. Moreover, by (E_1) and (E_2), proceeding as in (22), we have for $(t,u) \in [0,T] \times B_N : |F(t,u)| \leq (2c_1 + c_2)N + \rho_1 + \rho_2$. For any $u \in B_N$, $\left|\frac{d\Gamma u(t)}{dt}\right|$ is bounded for $(t,u) \in [0,T] \times B_N$ because

$$\frac{d\Gamma u(t)}{dt} = \frac{dy_u(t)}{dt} = A(t)y_u(t) + F(t,u).$$

Therefore, $\{\Gamma u(t)/u \in B_N\}$ is equicontinuous. It follows from Ascoli-Arzela theorem that ΓB_N is a relatively compact subset of B.

Lemma 3: Γ is continuous on B_N.

Proof. Since $g(t,x,y)$ is uniformly continuous on $[0,T] \times \mathbf{C}_N$ and $g(t+T,x,y) = g(t,x,y)$, $g(t,x,y)$ is uniformly continuous on $\mathbf{R} \times \mathbf{C}_N$. Therefore, for any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that if $|x_1 - x_2| + |y_1 - y_2| < \delta$ $((x_i, y_i) \in \mathbf{C}_N,\ i = 1, 2)$, then

$$|g(t, x_1, y_2) - g(t, x_2, y_1)| \leq \frac{\epsilon}{2\mu}(t \in \mathbf{R}).$$

Moreover for $|y_1 - y_2| < \delta$ we have, by (C_2), $|C(t, s, y_1) - C(t, s, y_2)| \leq \epsilon_1\gamma(t, s)$ and then $|F_2(t, y_1) - F_2(t, y_2)| \leq \epsilon_1 \int_{-\infty}^{t} \gamma(t, s)ds \leq \frac{\epsilon\mu^{-1}}{2}$. So

$$|F_2(t, y_1) - F_2(t_1, y_2)| \leq \frac{\epsilon}{2\mu}(t \in \mathbf{R}).$$

Therefore for any $u_1, u_2 \in B_N$, $r \in \mathbf{R}$, there exists $\delta = \delta(\epsilon)$ such that if $\|u_1 - u_2\| < \delta$, then we have

$$\begin{array}{rcl} |F(r, u_1 - F(r, u_2)| & \leq & \varepsilon\mu^{-1} \text{ and} \\ |\Gamma u_1(t) - \Gamma u_2(t)| & \leq & \int\limits_{-\infty}^{\infty} |G(t, s)|\epsilon\mu^{-1}ds \leq \epsilon. \end{array}$$

Therefore, Γ is continuous on B_N.

From the above three Lemmas, $\Gamma : B_N \to B_N$ is completely continuous. Therefore by Schauder's fixed point theorem, there exists at least one fixed point in ΓB_N. It follows from (18), (20) and (21) that the fixed point is just the $T-$periodic solution of system (18). The proof of Theorem 4 is complete.

4 Examples

4.1) First, we show a large class of integrable dichotomy (see, for example, Coppel[8,p.73], [1,14]). Let $\{a_k\}_{k\in\mathbb{Z}}$ be a positive sequence such that $\sum_{k\in\mathbb{Z}} a_k$ converges and $\inf\limits_{k\in\mathbb{Z}} a_k^{-1} = c > 0$. Define for $k \in \mathbb{Z}$, $I_k = [k - a_k^2, k + a_k^2]$.

Let $\xi : R \to (0, \infty)$ be a continuously differentiable function given by $\xi(t) \equiv c$ except on I_k, where $\xi(k) = a_k^{-1}$ and ξ on I_k lies between c and a_k^{-1}. We have

$$\sum\nolimits_{k\in\mathbb{Z}} \int\limits_{I_k} \xi(s)ds \leqslant v < \infty.$$

Consider the scalar differential equation

$$x' = a(t)x,\ a(t) = -\alpha + \xi'(t)\xi(t)^{-1},\ \alpha > 0 \tag{23}$$

with solutions

$$x(t) = x_0 e^{-\alpha t}\xi(t) := x_0\phi(t).$$

We have

$$\phi(k + a_k^2)\phi(k)^{-1} \leqslant c a_k^{-1} e^{-\alpha a_k^2} \to \infty \text{ as } k \to \infty$$

and equation (23) is not exponentially stable. However, equation (23) has an integrable dichotomy

$$\int_{-\infty}^{t} \phi(t)\phi(s)^{-1}ds \leqslant \int_{-\infty}^{t} e^{-\alpha(t-s)}ds + \sum_{k=-\infty}^{[t+1]} \int_{I_k} \xi(s)ds \leqslant \alpha^{-1} + v < \infty$$

i.e. there exists μ such that for $t \in R$, we have:

$$\int_{-\infty}^{t} \phi(t)\phi(s)^{-1}ds \leqslant \mu. \tag{24}$$

So, in this way a big class of linear differential equations of type (23), satisfying (24), can be built.

In a similar way the above construction may be modified to obtain equation (23) satisfying

$$\int_{t}^{\infty} \phi(t)\phi(s)^{-1}ds \leqslant \mu < \infty \tag{25}$$

but not "exponentially stable" at $-\infty$.

Furthermore, if we construct the diagonal matrix $A(t) = diag\{a_1(t), a_2(t), ..., a_n(t)\}$ with a_i of different types satisfying (24) or (25), then the linear system

$$x' = A(t)x \tag{26}$$

has an integrable dichotomy, which clearly satisfies also condition (D).

4.2) Let $\lambda : \mathbf{R}^2 \to [0, \infty)$ and $\gamma : \mathbf{R}^2 \to [0, \infty)$ be two functions satisfying

$$\sup_{t\in R} \int_{-\infty}^{t} \lambda(t,s)ds \leqslant \vartheta, \ \sup_{t\in R} \int_{-\infty}^{t} \gamma(t,s)ds \leqslant \vartheta.$$

Consider the integro-differential equation:

$$\begin{aligned} y' &= a(t)y + (\sin t)y^5(t) + (1+\cos^2 t)\ y^8(t-2) \\ &\quad + \int_{-\infty}^{t} \ln\left[1 + |y(s)|^3 \lambda(t,s) + \gamma(t,s)\right] ds, \end{aligned} \tag{27}$$

where the solutions of the linear equation (23) have an integrable dichotomy, that is, satisfying (24) or (25).

The conditions of Theorem 2 are fulfilled. Indeed,
i) $g(t,x,y) = (\sin t)x^5 + (1+\cos^2 t)\ y^8$ satisfies (E1): $|g(t,x,y)| \leqslant c_1(r)\,(|x| + |y|) + \vartheta_1$ for every x, y such that $|x|\,, |y| \leqslant r$, uniformly in $t \in \mathbf{R}$. Moreover, there exists r^* such that $2c_1(r^*) \leqslant \mu^{-1}$, where μ satisfies (24) or (25).
ii) $C(t,s,y) = \ln\left[1 + |y^3|\,\lambda(t,s) + \gamma(t,s)\right]$ satisfies (E2): $|C(t,s,y)| \leqslant c_2(r)\lambda(t,s)\,|y| + \gamma(t,s)$ for $|y| \leqslant r$. Moreover, there exists r^* such that $c_2(r^*)\vartheta \leqslant \mu^{-1}$.
iii) g satisfies (C1) and C satisfies (C2).
Indeed, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|y_1 - y_2| \leqslant \delta$ implies

$$|C(t,s,y_1) - C(t,s,y_2)| \leqslant \varepsilon c_2(r)\lambda(t,s) \text{ for } t, s \in \mathbf{R}.$$

Furthermore, there exists r^* such that $2c_1(r^*) + c_2(r^*)\vartheta \leqslant \mu^{-1}$, where μ satisfies (24) or (25).
Then, by Theorem 2, equation (27) has at least one bounded solution.

4.3) Thus many examples can be constructed where our results can be applied.

Consider the integro-differential system

$$y' = A(t)y + B(t)g(y(t), y(t - r(t))) + \int_{-\infty}^{t} \left[\Lambda(t-s)f(y(s)) + R(t-s)\right] ds \tag{28}$$

where
i) $x' = A(t)x$ has an integrable dichotomy (e.g. as in (26)).
ii) B is a bounded matrix: $|B(t)| \leqslant b$; $g : \mathbf{C}^n \times \mathbf{C}^n \to \mathbf{C}^n$ is a continuous function and $|g(x,y)| \leqslant c_1(r)(|x| + |y|) + \vartheta_1$ for $|x|\,, |y| \leqslant r$.
iii) $\int_0^\infty |\Lambda(s)| ds = \vartheta < \infty$, $\int_0^\infty |R(s)| ds = \vartheta < \infty$, $f : \mathbf{C}^n \to \mathbf{C}^n$ is a continuous function such that $|f(y)| \leqslant c_2(r)|y| + \vartheta_2$ for $|y| \leqslant r$.
The hypothesis of Theorem 2 are fulfilled. Then if there exists r^* such that $2c_1(r^*)b + c_2(r^*)\vartheta \leqslant \mu^{-1}$ (μ given by (9)), Theorem 2 implies that there exists at least a bounded solution of system (28).

4.4) Similar results can be obtained under global Lipschitz conditions (L1) and (L2). On the other hand, the periodic situation can be treated in the same way.

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