



On solvability of nonlinear boundary value problems with integral condition for the system of hyperbolic equations

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Abstract. For a system of hyperbolic equations of second order a nonlinear boundary value problem with integral condition is considered. By introducing new unknown functions, the investigated problem is reduced to an equivalent problem involving a one-parametered family of boundary value problems with integral condition and integral relations. Conditions for the existence of classical solutions to the nonlinear boundary value problem with an integral condition for a system of hyperbolic equations are obtained. Algorithms for finding solutions are constructed, and their convergence is established.

Keywords: nonlinear boundary value problem, integral condition, hyperbolic equation, solvability.

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1 Introduction


The aim of this paper is to investigate a nonlinear boundary value problem with integral condition for the system of hyperbolic equations with mixed derivatives

$$\frac{\partial^2 u}{\partial t \partial x} = A(t, x) \frac{\partial u}{\partial x} + f\left(t, x, u, \frac{\partial u}{\partial t}\right), \quad u \in R^n, \quad (1.1)$$

$$P(x) \frac{\partial u(0, x)}{\partial x} + S(x) \frac{\partial u(T, x)}{\partial x} + g_1\left(x, u(0, x), u(T, x), \frac{\partial u(0, x)}{\partial t}, \frac{\partial u(T, x)}{\partial t}\right) + \int_0^T L(\tau, x) \frac{\partial u(\tau, x)}{\partial x} d\tau + \int_0^T g_2\left(\tau, x, u(\tau, x), \frac{\partial u(\tau, x)}{\partial \tau}\right) d\tau = 0, \quad x \in [0, \omega], \quad (1.2)$$

$$u(t, 0) = \psi(t), \quad t \in [0, T], \quad (1.3)$$

where $u(t, x) = \text{col}(u_1(t, x), u_2(t, x), \dots, u_n(t, x))$ is the desired function, $\bar{\Omega} = [0, T] \times [0, \omega]$, the $(n \times n)$ matrices $A(t, x)$, $L(t, x)$, $P(x)$, $S(x)$ and the n -vector-function $f(t, x)$ are continuous

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on $\bar{\Omega}$, the n -vector-function $\psi(t)$ is continuously differentiable on $[0, T]$. Assumptions about the functions $f: \bar{\Omega} \times R^n \times R^n \rightarrow R^n$, $g_1: [0, \omega] \times R^n \times R^n \times R^n \times R^n \rightarrow R^n$, $g_2: \bar{\Omega} \times R^n \times R^n \rightarrow R^n$ will be given below.

Intensive study of boundary value problems with data on characteristics for hyperbolic equations with mixed derivative started in 1960 with works of L. Cesari [9]. Periodic and nonlocal boundary value problems, which belong to this class of problems, have been studied by many authors. For the review and bibliography we refer the reader to [12, 13, 21, 22]. The most general formulation of linear boundary value problems with data on two characteristics was studied in [1, 2]. In these works the sufficient conditions for the unique solvability were obtained and the ways of finding solutions to the boundary value problem with data on characteristics of system of hyperbolic equations were proposed. Subsequently, the well-posedness criteria for this problem were established in terms of initial data [3–5]. To do this, there have been introduced some new unknown functions, indicating first-order derivatives of the desired function. Thus, the problem was reduced to an equivalent problem involving a family of two-point boundary value problems for ordinary differential equations and integral relations. Equivalence of well-posedness of both considered problem and family of two-point boundary value problems is proved.

The results obtained for the linear boundary value problems were extended to the quasi-linear systems of hyperbolic equations [6, 7]. Sufficient conditions for existence of a unique classical solution were identified for the linear boundary problem with the data on characteristics for the system of quasi-linear hyperbolic equations. In recent years, the boundary value problems with integral conditions are of great interest to specialists [11, 16, 19, 23]. Mathematical modelling of various processes in physics, chemistry and biology leads to the boundary value problems with integral conditions for partial differential equations. For example, some problems arising in the dynamics of groundwater [20, 24] can be reduced to a nonlocal problem with integral condition for hyperbolic equations with mixed derivative. In [8] the linear boundary value problem with integral condition for a system of hyperbolic equations, corresponding to (1.1)–(1.3), was investigated. With the new approach proposed for boundary value problems with data on characteristics without integral terms, we established the necessary and sufficient conditions for the well-posedness of linear boundary value problems with integral condition for a system of hyperbolic equations with mixed derivative. In papers [17, 18] a model of an oscillator, which is described by hyperbolic equations, was considered. There, some linear and nonlinear boundary value problems for hyperbolic equations were studied. Application of asymptotic methods for solving boundary value problems for partial differential equations allowed one to investigate the buffer phenomenon. The role of nonlinear boundary conditions in the models of an oscillator with distributed parameters is demonstrated. This leads to the study of nonlinear boundary value problems for systems of hyperbolic equations with a mixed derivative. We also note that mathematical modelling of fluid mechanical processes leads to boundary value problems for nonlinear hyperbolic equations of higher order [14, 15]. Introducing new functions, we may reduce the nonlinear hyperbolic equations of higher order to the system of quasi-linear hyperbolic equations with a mixed derivative.

In this paper we study the existence problems for classical solutions to nonlinear boundary value problem for hyperbolic equations (1.1)–(1.3) and methods of constructing their approximate solutions. The results and methods of [8] are extended to the new class of problems – nonlinear boundary value problems with integral condition for a system of hyperbolic equations. We establish sufficient conditions for the unique solvability of nonlinear boundary value

problem (1.1)–(1.3) in terms of the right-hand side of the system, the boundary functions and kernels of integral terms. Algorithms for finding the solution of the considered problem are constructed and their convergence are shown. The results can be used in the numerical solving of application problems.

Function $u(t, x) \in C(\bar{\Omega}, R^n)$, which has the partial derivatives $\frac{\partial u(t, x)}{\partial x} \in C(\bar{\Omega}, R^n)$, $\frac{\partial u(t, x)}{\partial t} \in C(\bar{\Omega}, R^n)$, $\frac{\partial^2 u(t, x)}{\partial t \partial x} \in C(\bar{\Omega}, R^n)$ is called the classical solution to problem (1.1)–(1.3), if for $(t, x) \in \bar{\Omega}$ it satisfies system (1) and boundary conditions (1.2), (1.3).

Here $C(\bar{\Omega}, R^n)$ is a space of functions $u: \bar{\Omega} \rightarrow R^n$, continuous on $\bar{\Omega}$, with the norm

$$\|u\|_0 = \max_{(t, x) \in \bar{\Omega}} \|u(t, x)\|.$$

Introduce $C^{1,1}(\bar{\Omega}, R^n)$ as a space of functions $u: \bar{\Omega} \rightarrow R^n$, continuous on $\bar{\Omega}$ and continuously differentiable with respect to t and x , with the norm $\|u\|_1 = \max(\|u\|_0, \|\frac{\partial u}{\partial x}\|_0, \|\frac{\partial u}{\partial t}\|_0)$.

2 Reduction of problem (1.1)–(1.3) to an equivalent problem and the main result

Let us introduce new unknown functions $v(t, x) = \frac{\partial u(t, x)}{\partial x}$, $w(t, x) = \frac{\partial u(t, x)}{\partial t}$. This leads to reduction of nonlinear boundary value problem (1.1)–(1.3) to the following problem:

$$\frac{\partial v}{\partial t} = A(t, x)v + f(t, x, u, w), \quad (t, x) \in \bar{\Omega}, \quad (2.1)$$

$$\begin{aligned} P(x)v(0, x) + S(x)v(T, x) + \int_0^T L_2(\tau, x)v(\tau, x) d\tau \\ = -g_1(x, u(0, x), u(T, x), w(0, x), w(T, x)) \\ - \int_0^T g_2(\tau, x, u(\tau, x), w(\tau, x)) d\tau, \quad x \in [0, \omega], \end{aligned} \quad (2.2)$$

$$u(t, x) = \psi(t) + \int_0^x v(t, \xi) d\xi, \quad w(t, x) = \dot{\psi}(t) + \int_0^x \frac{\partial v(t, \xi)}{\partial t} d\xi, \quad (2.3)$$

Condition (1.3) contains in integral relations (2.3).

Triple of functions $\{v(t, x), u(t, x), w(t, x)\}$, continuous on $\bar{\Omega}$, is called the solution to problem (2.1)–(2.3), if the function $v(t, x) \in C(\bar{\Omega}, R^n)$ has a continuous derivative in respect to t on $\bar{\Omega}$ and satisfies boundary value problem with integral condition for the system of ordinary differential equations (2.1), (2.2), where the functions $u(t, x)$ and $w(t, x)$ are determining from equalities (2.3) by $v(t, x)$ and $\frac{\partial v(t, x)}{\partial t}$. The constant $x \in [0, \omega]$ plays a role of parameter for problem (2.1)–(2.3).

Problems (1.1)–(1.3) and (2.1)–(2.3) are equivalent. Let $u^*(t, x)$ be the classical solution to problem (1.1)–(1.3). Then the triple of functions $\{v^*(t, x), u^*(t, x), w^*(t, x)\}$, where $v^*(t, x) = \frac{\partial u^*(t, x)}{\partial x}$, $w^*(t, x) = \frac{\partial u^*(t, x)}{\partial t}$ becomes the solution to problem (2.1)–(2.3). The converse is also true. If the triple of functions $\{\tilde{v}(t, x), \tilde{u}(t, x), \tilde{w}(t, x)\}$ is the solution to problem (2.1)–(2.3), which we may assume, then the function $\tilde{u}(t, x)$ is a classical solution to problem (1.1)–(1.3).

Under fixed $u(t, x)$, $w(t, x)$ we may consider the system of equations (2.1) with condition (2.2) as a one-parametered family of boundary value problems with integral condition for system of ordinary differential equations. Integral conditions (2.3) allow us to determine the functions $u(t, x)$, $w(t, x)$ via the solution to the family of boundary value problems with integral condition for system of ordinary differential equations.

Thus, the solution to the nonlinear boundary value problem with integral condition for the system of hyperbolic equations (1.1)–(1.3) depends on the solutions to the family of boundary value problems with integral condition for a system of ordinary differential equations.

Consider the following one-parametered family of boundary value problems with integral condition for a system of ordinary differential equations

$$\frac{\partial v}{\partial t} = A(t, x)v + \tilde{f}(t, x), \quad t \in [0, T], \quad x \in [0, \omega], \quad v \in R^n, \quad (2.4)$$

$$P_2(x)v(0, x) + S_2(x)v(T, x) + \int_0^T L_2(\tau, x)v(\tau, x)d\tau = \tilde{g}(x), \quad x \in [0, \omega], \quad (2.5)$$

where $\tilde{f}(t, x) \in C(\bar{\Omega}, R^n)$ and $\tilde{g}(x) \in C([0, \omega], R^n)$.

Function $v: \bar{\Omega} \rightarrow R^n$, continuous on $\bar{\Omega}$ and continuously differentiable with respect to t on Ω , is called the solution to the one-parametered family of boundary value problems with integral condition (2.4), (2.5), if given any $(t, x) \in \bar{\Omega}$ it satisfies the system (2.4) and given any $x \in [0, \omega]$ it satisfies the conditions (2.5).

Definition 2.1. A one-parametered family of boundary value problems with integral condition (2.4), (2.5) is called well-posed if for arbitrary $\tilde{f}(t, x) \in C(\bar{\Omega}, R^n)$ and $\tilde{g}(x) \in C([0, \omega], R^n)$ it has the unique solution $v(t, x) \in C(\bar{\Omega}, R^n)$, and the following estimate is satisfied:

$$\max_{t \in [0, T]} \|v(t, x)\| \leq K \max\left(\max_{t \in [0, T]} \|\tilde{f}(t, x)\|, \|\tilde{g}(x)\|\right),$$

where the constant K does not depend on $\tilde{f}(t, x)$, $\tilde{g}(x)$, and $x \in [0, \omega]$.

Note that the family of boundary value problems with integral condition for systems of ordinary differential equations (2.4), (2.5) belongs to a non-Fredholm problems, i.e. the existence of only trivial solution to the corresponding homogeneous family of boundary value problems does not imply the existence of a unique solution to the family of nonhomogeneous boundary value problems. Let us illustrate this on the following example. Consider a family of boundary value problems on $[0, 1] \times [0, 1]$

$$\frac{\partial v}{\partial t} = \left(x - \frac{1}{2}\right)v + 1, \quad (2.4')$$

$$v(0, x) = v(1, x). \quad (2.5')$$

The homogeneous problem corresponding to (2.4'), (2.5') is

$$\frac{\partial v}{\partial t} = \left(x - \frac{1}{2}\right)v, \quad (2.4_0)$$

$$v(0, x) = v(1, x). \quad (2.5_0)$$

The general solution to equation (2.4₀), (2.5₀) has the form: $v(t, x) = C(x)e^{(x-\frac{1}{2})t}$. Substituting it into (2.5₀), we get

$$C(x) = e^{x-\frac{1}{2}}C(x), \quad (2.6_0)$$

where $C(x)$ is an arbitrary function continuous on $[0, 1]$. Equality (2.6₀) is fulfilled for all $x \in [0, 1]$, if $C(x) = 0$. Thus, the problem (2.4₀), (2.5₀) has only the trivial solution $v(t, x) = 0$ for all $x \in [0, 1]$. Despite this, for all $x \in [0, 1]$, the family of nonhomogeneous boundary value problems (2.4'), (2.5') does not have any solutions.

We introduce the following sets:

$$\begin{aligned} G_0(\psi, \dot{\psi}, \rho) &= \{(t, x, u, w) : (t, x) \in \bar{\Omega}, \|u - \psi(t)\| < \rho, \|w - \dot{\psi}(t)\| < \rho\}, \\ G_1(\psi, \dot{\psi}, \rho) &= \left\{ (x, u_1, u_2, w_1, w_2) : x \in [0, \omega], \|u_1 - \psi(0)\| < \rho, \|u_2 - \psi(T)\| < \rho, \right. \\ &\quad \left. \|w_1 - \dot{\psi}(0)\| < \rho, \|w_2 - \dot{\psi}(T)\| < \rho \right\}, \\ G_2(\psi, \dot{\psi}, \rho) &= \{(t, x, u, w) : (t, x) \in \bar{\Omega}, \|u - \psi(t)\| < \rho, \|w - \dot{\psi}(t)\| < \rho\}, \\ S(\psi(t), \rho) &= \{u \in C^{1,1}(\bar{\Omega}, R^n) : \|u - \psi\|_1 < \rho\}. \end{aligned}$$

Let the functions f, g_1, g_2 fulfill the following assumptions.

a) Under fixed u, w the function $f(t, x, u, w)$ is continuous by $(t, x) \in \bar{\Omega}$ and it satisfies a Lipschitz condition with respect to u and w on the set $G_0(\psi, \dot{\psi}, \rho)$, i.e.

$$\|f(t, x, u, w) - f(t, x, \bar{u}, \bar{w})\| \leq l_1(t, x)\|u - \bar{u}\| + l_2(t, x)\|w - \bar{w}\|,$$

where $l_i(t, x) \geq 0$ are functions continuous on $\bar{\Omega}$, $i = 1, 2$.

b) Under fixed u, w the function $g_1(x, u_1, u_2, w_1, w_2)$ is continuous by $x \in [0, \omega]$ and it satisfies a Lipschitz condition with respect to u and w on the set $G_1(\psi, \dot{\psi}, \rho)$, i.e.

$$\begin{aligned} &\|g_1(x, u_1, u_2, w_1, w_2) - g_1(x, \bar{u}_1, \bar{u}_2, \bar{w}_1, \bar{w}_2)\| \\ &\leq d_1(x)\|u_1 - \bar{u}_1\| + \tilde{d}_1(x)\|u_2 - \bar{u}_2\| + d_2(x)\|w_1 - \bar{w}_1\| + \tilde{d}_2(x)\|w_2 - \bar{w}_2\|, \end{aligned}$$

where $d_i(x) > 0, \tilde{d}_i(x) > 0$ are functions, continuous on $[0, \omega]$, $i = 1, 2$.

c) Under fixed u, w , the function $g_2(t, x, u, w)$ is continuous by $(t, x) \in \bar{\Omega}$ and it satisfies a Lipschitz condition with respect to u and w on the set $G_2(\psi, \dot{\psi}, \rho)$, i.e.

$$\|g_2(t, x, u, w) - g_2(t, x, \bar{u}, \bar{w})\| \leq h_1(t, x)\|u - \bar{u}\| + h_2(t, x)\|w - \bar{w}\|,$$

where $h_i(t, x) > 0$ are functions, continuous on $\bar{\Omega}$, $i = 1, 2$.

Suppose, that

$$\begin{aligned} \tilde{f}^{(0)}(t, x) &= f(t, x, \psi(t), \dot{\psi}(t)), \\ \tilde{g}^{(0)}(x) &= -g_1(x, \psi(0), \psi(T), \dot{\psi}(0), \dot{\psi}(T)) - \int_0^T g_2(\tau, x, \psi(\tau), \dot{\psi}(\tau)) d\tau, \\ \tilde{L}(x) &= \|d_1(x)\| + \|\tilde{d}_1(x)\| + \|d_2(x)\| + \|\tilde{d}_2(x)\| + T \left[\max_{t \in [0, T]} \|h_1(t, x)\| + \max_{t \in [0, T]} \|h_2(t, x)\| \right], \\ l_0(x) &= \max \left\{ \max_{t \in [0, T]} l_1(t, x) + \max_{t \in [0, T]} l_2(t, x), \tilde{L}(x) \right\}, \\ \alpha(x) &= \max_{t \in [0, T]} \|A(t, x)\|, \\ \rho_1(x) &= \max(K, \alpha(x)K + 1) l_0(x), \\ \rho_2(x) &= \max(K, \alpha(x)K + 1) \max \left\{ \max_{t \in [0, T]} \|\tilde{f}^{(0)}(t, x)\|, \|\tilde{g}^{(0)}(x)\| \right\}, \\ \rho_3(x) &= \rho_2(x) \exp \left\{ x \max_{x \in [0, \omega]} \rho_1(x) \right\}. \end{aligned}$$

A classical solution to problem (1.1)–(1.3) will be sought as a solution to problem (2.1)–(2.3). To find the solution to problem (2.1)–(2.3) we propose the following algorithm.

Step 0. On solving the one-parametered family of boundary value problem with integral condition (2.1), (2.2) for $u(t, x) = \psi(t)$ and $w(t, x) = \dot{\psi}(t)$, for all $(t, x) \in \bar{\Omega}$ we find $v^{(0)}(t, x)$. From integral relations (2.3) for $v(t, x) = v^{(0)}(t, x)$ and $\frac{\partial v(t, x)}{\partial x} = \frac{\partial v^{(0)}(t, x)}{\partial x}$ we determine $u^{(0)}(t, x)$ and $w^{(0)}(t, x)$ for all $(t, x) \in \bar{\Omega}$.

Step 1. On solving the one-parametered family of boundary value problem with integral condition (2.1), (2.2) for $u(t, x) = u^{(0)}(t, x)$ and $w(t, x) = w^{(0)}(t, x)$, we find $v^{(1)}(t, x)$ for all $(t, x) \in \bar{\Omega}$. From integral relations (2.3) for $v(t, x) = v^{(1)}(t, x)$ and $\frac{\partial v(t, x)}{\partial x} = \frac{\partial v^{(1)}(t, x)}{\partial x}$ we determine $u^{(1)}(t, x)$ and $w^{(1)}(t, x)$ for all $(t, x) \in \bar{\Omega}$.

Continuing this process, at the m -th step we find $v^{(m)}(t, x)$, $u^{(m)}(t, x)$ and $w^{(m)}(t, x)$ for all $(t, x) \in \bar{\Omega}$, where $m = 0, 1, 2, \dots$

The following statement provides conditions of feasibility and convergence of the algorithm, which also ensure the existence of a unique classical solution to problem (1.1)–(1.3).

Theorem 2.2. *Let*

- (i) *assumptions a)–c) hold for the functions $f(t, x, u, w)$, $g_1(x, u_1, u_2, w_1, w_2)$, $g_2(t, x, u, w)$;*
- (ii) *the one-parametered family of boundary value problems with integral condition (2.4), (2.5) be well-posed with the constant K ;*
- (iii) $\int_0^\omega \rho_3(\zeta) d\zeta \leq \rho$.

Then the sequence of triples $\{v^{(m)}(t, x), u^{(m)}(t, x), w^{(m)}(t, x)\}$, constructed according to the above-indicated algorithm, converges uniformly to the unique solution $\{v^(t, x), u^*(t, x), w^*(t, x)\}$ to problem (2.1)–(2.3) for $m \rightarrow \infty$ and $v^* \in S(\psi(t), \rho)$, $u^* \in S(\psi(t), \rho)$, $w^* \in S(\psi(t), \rho)$.*

Proof. Consider problem (2.1)–(2.3). Let us use the method of successive approximations and the above-given algorithm. Take $\psi(t)$ and $\dot{\psi}(t)$ as initial approximations of functions $u(t, x)$ and $w(t, x)$, respectively. Determine the function $v^{(0)}(t, x)$ from the problem

$$\frac{\partial v}{\partial t} = A(t, x)v + \tilde{f}^{(0)}(t, x), \quad (2.6)$$

$$P(x)v(0, x) + S(x)v(T, x) + \int_0^T L(\tau, x)v(\tau, x)d\tau = \tilde{g}^{(0)}(x), \quad x \in [0, \omega]. \quad (2.7)$$

Problem (2.6), (2.7) is a one-parametered family of boundary value problems with integral condition for a system of ordinary differential equations. This problem has been studied in [8] and solved by parameterization method [10]. Necessary and sufficient conditions for the unique solvability and well-posedness of one-parametered family of boundary value problems with integral condition (2.4), (2.5) were established in terms of initial data. Estimate of the solution to this problem was obtained via the data.

By assumption (ii) of the theorem, problem (2.6), (2.7) is well-posed. It follows that problem (2.6), (2.7) has the unique solution $v^{(0)}(t, x)$, and for the solution the following estimate holds:

$$\max_{t \in [0, T]} \|v^{(0)}(t, x)\| \leq K \max \left(\max_{t \in [0, T]} \|\tilde{f}^{(0)}(t, x)\|, \|\tilde{g}^{(0)}(x)\| \right).$$

Its derivative $\frac{\partial v^{(0)}(t, x)}{\partial t}$ satisfies the inequality

$$\max_{t \in [0, T]} \left\| \frac{\partial v^{(0)}(t, x)}{\partial t} \right\| \leq [\alpha(x)K + 1] \max \left(\max_{t \in [0, T]} \|\tilde{f}^{(0)}(t, x)\|, \|\tilde{g}^{(0)}(x)\| \right).$$

In using the integral relations (2.3) we find $u^{(0)}(t, x)$ and $w^{(0)}(t, x)$:

$$u^{(0)}(t, x) = \psi(t) + \int_0^x v^{(0)}(t, \xi) d\xi, \quad w^{(0)}(t, x) = \dot{\psi}(t) + \int_0^x \frac{\partial v^{(0)}(t, \xi)}{\partial t} d\xi.$$

The following inequalities hold:

$$\begin{aligned} \max_{t \in [0, T]} \|u^{(0)}(t, x) - \psi(t)\| &\leq \int_0^x \max_{t \in [0, T]} \|v^{(0)}(t, \xi)\| d\xi, \\ \max_{t \in [0, T]} \|w^{(0)}(t, x) - \dot{\psi}(t)\| &\leq \int_0^x \max_{t \in [0, T]} \left\| \frac{\partial v^{(0)}(t, \xi)}{\partial t} \right\| d\xi. \end{aligned}$$

Then

$$\begin{aligned} &\max \left(\max_{t \in [0, T]} \|u^{(0)}(t, x) - \psi(t)\|, \max_{t \in [0, T]} \|w^{(0)}(t, x) - \dot{\psi}(t)\| \right) \\ &\leq \int_0^x \max \left(\max_{t \in [0, T]} \|v^{(0)}(t, \xi)\|, \max_{t \in [0, T]} \left\| \frac{\partial v^{(0)}(t, \xi)}{\partial t} \right\| \right) d\xi \\ &\leq \int_0^x \max(K, \alpha(\xi)K + 1) \max \left(\max_{t \in [0, T]} \|\tilde{f}^{(0)}(t, \xi)\|, \|\tilde{g}^{(0)}(\xi)\| \right) d\xi = \int_0^x \rho_2(\xi) d\xi. \end{aligned}$$

Suppose that $u^{(m-1)}(t, x)$ and $w^{(m-1)}(t, x)$ are known. The m -th approximation of function $v(t, x)$, i.e. $v^{(m)}(t, x)$ is to be found from problem (2.1), (2.2), when $w(t, x) = w^{(m-1)}(t, x)$, $u(t, x) = u^{(m-1)}(t, x)$, $m = 1, 2, \dots$

$$\frac{\partial v^{(m)}}{\partial t} = A(t, x)v^{(m)} + f(t, x, u^{(m-1)}(t, x), w^{(m-1)}(t, x)), \quad (2.8)$$

$$\begin{aligned} &P(x)v^{(m)}(0, x) + S(x)v^{(m)}(T, x) + \int_0^T L(\tau, x)v^{(m)}(\tau, x) d\tau \\ &= -g_1(x, u^{(m-1)}(0, x), u^{(m-1)}(T, x), w^{(m-1)}(0, x), w^{(m-1)}(T, x)) \\ &\quad - \int_0^T g_2(\tau, x, u^{(m-1)}(\tau, x), w^{(m-1)}(\tau, x)) d\tau, \quad x \in [0, \omega]. \end{aligned} \quad (2.9)$$

By assumption (ii) of the theorem, problem (2.8), (2.9) is well-posed. It follows that problem (2.8), (2.9) has the unique solution $v^{(m)}(t, x)$, and the following estimate holds for the solution:

$$\max_{t \in [0, T]} \|v^{(m)}(t, x)\| \leq K \max \left(\max_{t \in [0, T]} \|\tilde{f}^{(m-1)}(t, x)\|, \|\tilde{g}^{(m-1)}(x)\| \right), \quad (2.10)$$

where

$$\begin{aligned} \tilde{f}^{(m-1)}(t, x) &= f(t, x, u^{(m-1)}(t, x), w^{(m-1)}(t, x)), \\ \tilde{g}^{(m-1)}(x) &= -g_1(x, u^{(m-1)}(0, x), u^{(m-1)}(T, x), w^{(m-1)}(0, x), w^{(m-1)}(T, x)) \\ &\quad - \int_0^T g_2(\tau, x, u^{(m-1)}(\tau, x), w^{(m-1)}(\tau, x)) d\tau. \end{aligned}$$

Its derivative $\frac{\partial v^{(m)}(t, x)}{\partial t}$ satisfy the inequality

$$\max_{t \in [0, T]} \left\| \frac{\partial v^{(m)}(t, x)}{\partial t} \right\| \leq [\alpha(x)K + 1] \max \left(\max_{t \in [0, T]} \|\tilde{f}^{(m-1)}(t, x)\|, \|\tilde{g}^{(m-1)}(x)\| \right). \quad (2.11)$$

By the above-found $v^{(m)}(t, x)$ we determine the m -th approximation $u(t, x)$ and $w(t, x)$ from integral relations (2.3):

$$u^{(m)}(t, x) = \psi(t) + \int_0^x v^{(m)}(t, \xi) d\xi, \quad w^{(m)}(t, x) = \dot{\psi}(t) + \int_0^x \frac{\partial v^{(m)}(t, \xi)}{\partial t} d\xi. \quad (2.12)$$

The functions $u^{(m)}(t, x)$ and $w^{(m)}(t, x)$ satisfy the following inequalities:

$$\begin{aligned} \max_{t \in [0, T]} \|u^{(m)}(t, x) - \psi(t)\| &\leq \int_0^x \max_{t \in [0, T]} \|v^{(m)}(t, \xi)\| d\xi, \\ \max_{t \in [0, T]} \|w^{(m)}(t, x) - \dot{\psi}(t)\| &\leq \int_0^x \max_{t \in [0, T]} \left\| \frac{\partial v^{(m)}(t, \xi)}{\partial t} \right\| d\xi. \end{aligned}$$

Taking into account estimates (2.10) and (2.11), we obtain

$$\begin{aligned} &\max \left(\max_{t \in [0, T]} \|u^{(m)}(t, x) - \psi(t)\|, \max_{t \in [0, T]} \|w^{(m)}(t, x) - \dot{\psi}(t)\| \right) \\ &\leq \int_0^x \max \left(\max_{t \in [0, T]} \|v^{(m)}(t, \xi)\|, \max_{t \in [0, T]} \left\| \frac{\partial v^{(m)}(t, \xi)}{\partial t} \right\| \right) d\xi \\ &\leq \int_0^x \max [K, \alpha(\xi)K + 1] \max \left(\max_{t \in [0, T]} \|\tilde{f}^{(m-1)}(t, \xi)\|, \|\tilde{g}^{(m-1)}(\xi)\| \right) d\xi \leq \int_0^x \rho_3(\xi) d\xi, \end{aligned}$$

i.e. $v^{(m)} \in S(\psi(t), \rho)$, $u^{(m)} \in S(\psi(t), \rho)$, $w^{(m)} \in S(\psi(t), \rho)$.

Introduce the following notations for the differences of successive approximations

$$\begin{aligned} \Delta v^{(m)}(t, x) &= v^{(m+1)}(t, x) - v^{(m)}(t, x), \\ \Delta u^{(m)}(t, x) &= u^{(m+1)}(t, x) - u^{(m)}(t, x), \\ \Delta w^{(m)}(t, x) &= w^{(m+1)}(t, x) - w^{(m)}(t, x). \end{aligned}$$

Using the well-posedness of problem (2.4), (2.5) we easily establish the following estimates:

$$\begin{aligned} &\max_{t \in [0, T]} \|\Delta v^{(m)}(t, x)\| \\ &\leq Kl_0(x) \max \left(\max_{t \in [0, T]} \|\Delta u^{(m-1)}(t, x)\|, \max_{t \in [0, T]} \|\Delta w^{(m-1)}(t, x)\| \right), \end{aligned} \quad (2.13)$$

$$\begin{aligned} &\max_{t \in [0, T]} \left\| \frac{\partial \Delta v^{(m)}(t, x)}{\partial t} \right\| \\ &\leq (\alpha(x)K + 1) l_0(x) \max \left(\max_{t \in [0, T]} \|\Delta u^{(m-1)}(t, x)\|, \max_{t \in [0, T]} \|\Delta w^{(m-1)}(t, x)\| \right). \end{aligned} \quad (2.14)$$

Similarly, from integral relations (2.3) we obtain

$$\max_{t \in [0, T]} \|\Delta u^{(m)}(t, x)\| \leq \int_0^x \max_{t \in [0, T]} \|\Delta v^{(m)}(t, \xi)\| d\xi, \quad (2.15)$$

$$\max_{t \in [0, T]} \|\Delta w^{(m)}(t, x)\| \leq \int_0^x \max_{t \in [0, T]} \left\| \frac{\partial \Delta v^{(m)}(t, \xi)}{\partial t} \right\| d\xi. \quad (2.16)$$

Taking into account estimates (2.13), (2.14), estimates (2.15), (2.16) yield

$$\begin{aligned} &\max \left(\max_{t \in [0, T]} \|\Delta u^{(m)}(t, x)\|, \max_{t \in [0, T]} \|\Delta w^{(m)}(t, x)\| \right) \\ &\leq \int_0^x \rho_1(\xi) \max \left(\max_{t \in [0, T]} \|\Delta u^{(m-1)}(t, \xi)\|, \max_{t \in [0, T]} \|\Delta w^{(m-1)}(t, \xi)\| \right) d\xi. \end{aligned} \quad (2.17)$$

Inequality (2.17) is satisfied for any $m = 1, 2, \dots$. By substituting successively the relevant differences into the right-hand side of inequality (2.17) we have

$$\max \left(\max_{t \in [0, T]} \|\Delta u^{(m)}(t, x)\|, \max_{t \in [0, T]} \|\Delta w^{(m)}(t, x)\| \right) \leq \frac{1}{(m-1)!} \int_0^x [\xi \max_{\xi_1 \in [0, \omega]} \rho_1(\xi_1)]^{m-1} \rho_2(\xi) d\xi.$$

Hence it follows that the functional sequences $u^{(m)}(t, x)$, $w^{(m)}(t, x)$ converge uniformly to the functions $u^*(t, x)$, $w^*(t, x)$ on $\bar{\Omega}$ as long as $m \rightarrow \infty$. Given $m \rightarrow \infty$ relations (2.13), (2.14) lead to the uniform convergence of sequences $v^{(m)}(t, x)$, $\frac{\partial v^{(m)}(t, x)}{\partial t}$ on $\bar{\Omega}$ to the functions $v^*(t, x)$, $\frac{\partial v^*(t, x)}{\partial t}$, respectively.

The above-determined triple of functions $\{v^*(t, x), u^*(t, x), w^*(t, x)\}$ is the solution to problem (2.1)–(2.3), and it satisfies the inequalities:

$$\begin{aligned} \max \left(\max_{t \in [0, T]} \|u^*(t, x) - \psi(t)\|, \max_{t \in [0, T]} \|w^*(t, x) - \psi(t)\| \right) &\leq \int_0^\omega \rho_3(\xi) d\xi \leq \rho, \\ \max \left(\max_{t \in [0, T]} \|v^*(t, x)\|, \max_{t \in [0, T]} \left\| \frac{\partial v^*(t, x)}{\partial t} \right\| \right) &\leq \int_0^\omega \rho_3(\xi) d\xi \leq \rho, \end{aligned}$$

i.e. $v^* \in S(\psi(t), \rho)$, $u^* \in S(\psi(t), \rho)$, $w^* \in S(\psi(t), \rho)$.

Uniqueness of the solution to problem (2.1)–(2.3) can be easily proved by contradiction.

The proof of the assertions of Theorem 2.2 is complete. \square

The equivalence of problems (2.1)–(2.3) and (1.1)–(1.3) is established by our next theorem.

Theorem 2.3. *Suppose that assumptions (i)–(iii) of Theorem 2.2 hold.*

Then the nonlinear boundary value problem with integral condition for a system of hyperbolic equations (1.1)–(1.3) has the unique classical solution $u^(t, x)$, belonging to $S(\psi(t), \rho)$.*

We illustrate the assertions of our theorems by the examples below.

Example 2.4. Consider the following boundary value problem with integral condition for the two-dimensional system of on $[0, 1] \times [0, \omega]$:

$$\frac{\partial^2 u}{\partial t \partial x} = \frac{1}{3} \begin{pmatrix} 0 & 2+t \\ 2+t^2 & 0 \end{pmatrix} \frac{\partial u}{\partial x} + f\left(t, x, u, \frac{\partial u}{\partial t}\right), \quad (2.18)$$

$$u(t, 0) = 0, \quad t \in [0, 1], \quad (2.19)$$

$$\frac{1}{2} \frac{\partial u(0, x)}{\partial x} + \int_0^1 \frac{\partial u(\tau, x)}{\partial x} d\tau = 1, \quad x \in [0, \omega], \quad (2.20)$$

where

$$u(t, x) = \begin{pmatrix} u_1(t, x) \\ u_2(t, x) \end{pmatrix}, \quad f\left(t, x, u, \frac{\partial u}{\partial t}\right) = \begin{pmatrix} \frac{1}{10} \left\{ \frac{1}{3} \left(\frac{\partial u_1}{\partial t} \right)^3 + \frac{1}{2} u_1^2 + \sin(x+t) \right\} \\ \frac{1}{10} \left\{ \frac{1}{3} \left(\frac{\partial u_2}{\partial t} \right)^3 + \frac{1}{2} u_2^2 + \cos(x+t) \right\} \end{pmatrix}. \quad (2.21)$$

Here $g_1 = -\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $g_2 = 0$.

Let us check the fulfillment of conditions of Theorem 2.2. Condition *a*) holds on the set $G_0(0, 0, \rho)$ with $l_1 = \frac{1}{10\rho^2}$, $l_2 = \frac{1}{10\rho}$. Conditions *b*) and *c*) are satisfied. The one-parametered

family of boundary value problems with integral condition, corresponding to (2.18)–(2.20), looks as follows:

$$\frac{\partial v}{\partial t} = \frac{1}{3} \begin{pmatrix} 0 & 2+t \\ 2+t^2 & 0 \end{pmatrix} v + F(t, x), \quad (2.22)$$

$$\frac{1}{2}v(0, x) + \int_0^1 v(\tau, x) d\tau = 1, \quad x \in [0, \omega]. \quad (2.23)$$

We use the results of Section 2 of [8]. For $h = 1$ (i.e. $N = 1$) and $\nu = 1$, the 2×2 matrix $Q_1(1, x)$ is invertible, $\|[Q_1(1, x)]^{-1}\| \leq 0.896$, and $q_1(1, x) = 0.896 \cdot [e - 1 - 1] = 0.64 < 1$. Then according to [8, Theorem 2], problem (2.22), (2.23) is well-posed with the constant $K = 107$. This leads to the fulfillment of condition (ii). For $\rho_1 = 10.8 \cdot (\frac{1}{\rho^2} + \frac{1}{\rho})$ and $\rho_2 = 108$, we have

$$\int_0^\omega \rho_3(\xi) d\xi = \int_0^\omega 108 \cdot e^{10.8(\frac{1}{\rho^2} + \frac{1}{\rho})\xi} d\xi = \frac{10}{\frac{1}{\rho^2} + \frac{1}{\rho}} \left(e^{10.8(\frac{1}{\rho^2} + \frac{1}{\rho})\omega} - 1 \right).$$

If the numbers $\omega > 0$ and $\rho > 0$ satisfies the inequality $10.8(\frac{1}{\rho^2} + \frac{1}{\rho})\omega < \ln(\frac{11}{10} + \frac{1}{10\rho})$, then condition (iii) also holds. For example, we can take $\omega = 1/10$, $\rho = 12$, or $\omega = 1/2$, $\rho = 59$.

Thus, all conditions of Theorem 2.2 are satisfied. Consequently, problem (2.18)–(2.20) has a unique classical solution $u^*(t, x)$, which can be found by our algorithm, and this solution belongs to $S(0, \rho)$.

Example 2.5. Consider the following boundary value problem with integral condition for the two-dimensional system of on $[0, 1] \times [0, 1]$:

$$\frac{\partial^2 u}{\partial t \partial x} = \frac{1}{3} \begin{pmatrix} 0 & 1+t+x \\ 1+t^2+x & 0 \end{pmatrix} \frac{\partial u}{\partial x} + f\left(t, x, u, \frac{\partial u}{\partial t}\right), \quad (2.24)$$

$$u(t, 0) = 0, \quad t \in [0, 1], \quad (2.25)$$

$$\frac{1}{2} \frac{\partial u(0, x)}{\partial x} + \int_0^1 \frac{\partial u(\tau, x)}{\partial x} d\tau = 1, \quad x \in [0, 1]. \quad (2.26)$$

Let us check the fulfillment of conditions of Theorem 2.2. Condition *a*) holds on the set $G_0(0, 0, \rho)$ with $l_1 = \frac{1}{10\rho^2}$, $l_2 = \frac{1}{10\rho}$. Conditions *b*) and *c*) are satisfied. The one-parametered family of boundary value problems with integral condition, corresponding to (2.24)–(2.26), looks as follows:

$$\frac{\partial v}{\partial t} = \frac{1}{3} \begin{pmatrix} 0 & 1+t+x \\ 1+t^2+x & 0 \end{pmatrix} v + F(t, x), \quad (2.27)$$

$$\frac{1}{2}v(0, x) + \int_0^1 v(\tau, x) d\tau = 1, \quad x \in [0, 1]. \quad (2.28)$$

We use the results of Section 2 of [8]. For $h = 1$ (i.e. $N = 1$) and $\nu = 1$, the 2×2 matrix $Q_1(1, x)$ is invertible, $\|[Q_1(1, x)]^{-1}\| \leq 0.8954$, and $q_1(1, x) = 0.8954 \cdot [e - 2] = 0.6447 < 1$. Then according to [8, Theorem 2], problem (2.27), (2.28) is well-posed with the constant $K = 46$. This leads to the fulfillment of condition (ii). For $\rho_1 = 4.7 \cdot (\frac{1}{\rho^2} + \frac{1}{\rho})$ and $\rho_2 = 47$, we have

$$\int_0^\omega \rho_3(\xi) d\xi = \int_0^1 47 \cdot e^{4.7(\frac{1}{\rho^2} + \frac{1}{\rho})\xi} d\xi = \frac{10}{\frac{1}{\rho^2} + \frac{1}{\rho}} \left(e^{4.7(\frac{1}{\rho^2} + \frac{1}{\rho})} - 1 \right).$$

If the number $\rho > 0$ satisfies inequality $4.7(\frac{1}{\rho^2} + \frac{1}{\rho}) < \ln(\frac{11}{10} + \frac{1}{10\rho})$, then condition (iii) also holds. For example, we can take $\rho = 50$.

Thus, all conditions of Theorem 2.2 are satisfied. Consequently, problem (2.24)–(2.26) has a unique classical solution $u^*(t, x)$, which can be found by our algorithm, and this solution belongs to $S(0, \rho)$.

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