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Existence results for a mixed boundary value problem

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Abstract. In the present paper, we obtain an existence result for a class of mixed boundary value problems for second-order differential equations. A critical point theorem is used, in order to prove the existence of a precise open interval of positive eigenvalues λ, for which the considered problem admits at least one non-trivial classical solution u_λ. It is proved that the norm of u_λ tends to zero as $\lambda \to 0$.

Keywords: mixed boundary value problems, existence results, critical points theory.

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1 Introduction

The aim of this paper is to study the following mixed boundary value problem

$$\begin{cases} -(pu')' + qu = \lambda f(x,u) + g(u) & \text{in }]a,b[, \\ u(a) = u'(b) = 0, \end{cases} \tag{1.1}$$

where $p \in C^1([a,b])$ and $q \in C^0([a,b])$ are positive functions, λ is a positive parameter, $f\colon [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that

(f_1) $\quad |f(x,t)| \le a_1 + a_2|t|^{r-1}, \quad$ a.e. $x \in [a,b]$, $t \in \mathbb{R}$,

where $a_1, a_2 \ge 0$ and $r \in]1,+\infty[$, and $g\colon \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with the Lipschitz constant $L > 0$, i.e.,

$$|g(t_1) - g(t_2)| \le L|t_1 - t_2|$$

for every $t_1, t_2 \in \mathbb{R}$, and $g(0) = 0$.

Our goal here is to obtain some sufficient conditions which imply that the problem (1.1) has at least one classical solution (see Theorem 3.1). We use the variational method and a critical point theorem.

Motivated by the fact that such kind of problems are used to describe a large class of physical phenomena, many authors looked for existence and multiplicity of solutions for

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second-order ordinary differential nonlinear equations, with mixed conditions at the ends. We cite the papers [1–3, 6–10, 14]. For instance, in [7], Bonanno and Tornatore established the existence of infinitely many weak solutions for the mixed boundary value problem

$$\begin{cases} -(pu')' + qu = \lambda f(x,u) & \text{in }]a,b[\,, \\ u(a) = u'(b) = 0, \end{cases}$$

where $p, q \in L^\infty([a,b])$ are such that

$$p_0 := \operatorname*{ess\,inf}_{x\in[a,b]} p(x) > 0, \qquad q_0 := \operatorname*{ess\,inf}_{x\in[a,b]} q(x) \geq 0,$$

$f\colon [a,b]\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function and λ is a positive parameter.

We also refer the reader to the paper [11] in which, by means of an abstract critical points result of Ricceri [13], existence of at least three solutions for the following two-point boundary value problem

$$\begin{cases} u'' + (\lambda f(t,u) + g(u))h(t,u') = \mu p(t,u)h(t,u') & \text{in }]a,b[\,, \\ u(a) = u(b) = 0, \end{cases}$$

where λ and μ are positive parameters, $f\colon [a,b]\times\mathbb{R}\to\mathbb{R}$ is continuous, $g\colon\mathbb{R}\to\mathbb{R}$ is Lipschitz continuous with $g(0)=0$, $h\colon [a,b]\times\mathbb{R}\to\mathbb{R}$ is bounded, continuous, with $m := \inf h > 0$, and $p\colon [a,b]\times\mathbb{R}\to\mathbb{R}$ is L^1-Carathéodory, are ensured.

The paper is organized as follows. In Section 2 we introduce our abstract framework and we give some notations. In Section 3 we prove the main result (Theorem 3.1), while Section 4 is devoted to some consequences and remarks on the results of the paper. Here we give an application of the results (Example 4.7).

2 Preliminaries

In order to prove our main result, that is Theorem 3.1, we report here the result obtained in [5] (see [5, Theorem 3.1 and Remark 3.3]).

Theorem 2.1. *Let X be a reflexive real Banach space, let $\Phi, \Psi\colon X\to\mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is strongly continuous, sequentially weakly lower semicontinuous and coercive in X and Ψ is sequentially weakly upper semicontinuous in X. Let I_λ be the functional defined as $I_\lambda := \Phi - \lambda\Psi$, $\lambda\in\mathbb{R}$, and for any $r > \inf_X \Phi$ let φ be the function defined as*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\left(\sup_{v\in\Phi^{-1}(]-\infty,r[)} \Psi(v)\right) - \Psi(u)}{r - \Phi(u)}. \tag{2.1}$$

Then, for any $r > \inf_X \Phi$ and any $\lambda \in \,]0, 1/\varphi(r)[$, the restriction of the functional I_λ to $\Phi^{-1}(]-\infty,r[)$ admits a global minimum, which is a critical point (precisely a local minimum) of I_λ in X.

Now, let $f\colon [a,b]\times\mathbb{R}\to\mathbb{R}$ be a continuous function and $g\colon\mathbb{R}\to\mathbb{R}$ be a Lipschitz continuous function with the Lipschitz constant $L > 0$, i.e.,

$$|g(t_1) - g(t_2)| \leq L|t_1 - t_2|$$

for every $t_1, t_2 \in \mathbb{R}$, and $g(0) = 0$.

Put

$$F(x,t) := \int_0^t f(x,\xi)d\xi, \qquad G(t) := -\int_0^t g(\xi)d\xi$$

for all $x \in [a,b]$ and $t \in \mathbb{R}$. Denote

$$X := \left\{u \in W^{1,2}([a,b]) : u(a) = 0\right\};$$

the usual norm in X is defined by

$$\|u\|_X := \left(\int_a^b (u(x))^2 dx + \int_a^b (u'(x))^2 dx\right)^{1/2}.$$

For every $u, v \in X$, we define

$$\langle u, v\rangle := \int_a^b p(x)u'(x)v'(x)dx + \int_a^b q(x)u(x)v(x)dx. \tag{2.2}$$

Clearly, (2.2) defines an inner product on X whose corresponding norm is

$$\|u\| := \left(\int_a^b p(x)(u'(x))^2 dx + \int_a^b q(x)(u(x))^2 dx\right)^{1/2}.$$

Then, it is easy to see that the norm $\|\cdot\|$ on X is equivalent to $\|\cdot\|_X$. In fact, put

$$p_0 := \min_{x\in[a,b]} p(x) > 0, \qquad q_0 := \min_{x\in[a,b]} q(x) > 0, \qquad m := \min\{p_0, q_0\} > 0,$$

and

$$p_1 := \max_{x\in[a,b]} p(x), \qquad q_1 := \max_{x\in[a,b]} q(x), \qquad M := \max\{p_1, q_1\}.$$

Then, we have

$$m^{1/2}\|u\|_X \le \|u\| \le M^{1/2}\|u\|_X, \qquad \forall u \in X.$$

In the following, we will use $\|\cdot\|$ instead of $\|\cdot\|_X$ on X. Note that X is a reflexive real Banach space.

By standard regularity results, since f is a continuous function, $p \in C^1([a,b])$ and $q \in C^0([a,b])$, then weak solutions of problem (1.1) belong to $C^2([a,b])$, thus they are classical solutions.

It is well known that the embedding $X \hookrightarrow C^0([a,b])$ is compact and

$$\|u\|_\infty \le \sqrt{\frac{b-a}{p_0}}\|u\| \tag{2.3}$$

for all $u \in X$ (see, e.g., [15]).

Fixing $r \in [1, +\infty[$, from the Sobolev embedding theorem, there exists a positive constant c_r such that

$$\|u\|_{L^r([a,b])} \le c_r\|u\|_X \le \frac{c_r}{\sqrt{m}}\|u\|, \qquad \forall u \in X, \tag{2.4}$$

and, in particular, the embedding $X \hookrightarrow L^r([a,b])$ is compact.

Suppose that the Lipschitz constant $L > 0$ of the function g satisfies

$$L < \max\left\{q_0, \frac{p_0}{(b-a)^2}\right\}. \tag{2.5}$$

Consider the energy functional $I_\lambda \colon X \to \mathbb{R}$ associated to (1.1) defined as follows

$$I_\lambda(u) := \Phi(u) - \lambda\Psi(u), \qquad \forall u \in X,$$

where

$$\Phi(u) := \frac{1}{2}\|u\|^2 + \int_a^b G(u(x))dx$$

and

$$\Psi(u) := \int_a^b F(x, u(x))dx.$$

Lemma 2.2. *Let the functional Φ be defined as above. Then we have the following estimates for every $u \in X$:*

$$\frac{q_0 - L}{2q_0}\|u\|^2 \le \Phi(u) \le \frac{q_0 + L}{2q_0}\|u\|^2, \tag{2.6}$$

$$\frac{p_0 - L(b-a)^2}{2p_0}\|u\|^2 \le \Phi(u) \le \frac{p_0 + L(b-a)^2}{2p_0}\|u\|^2. \tag{2.7}$$

Proof. Since g is Lipschitz continuous and satisfies $g(0) = 0$, we have

$$|g(t)| \le L|t|, \qquad \forall t \in \mathbb{R},$$

and so,

$$|G(t)| \le L\int_0^t |\xi|d\xi = \frac{L}{2}t^2, \qquad \forall t \in \mathbb{R}.$$

Therefore, condition $q_0 > 0$ implies that

$$\begin{aligned}\left|\int_a^b G(u(x))dx\right| &\le \frac{L}{2}\int_a^b (u(x))^2dx \\ &\le \frac{L}{2q_0}\int_a^b q(x)(u(x))^2dx \\ &\le \frac{L}{2q_0}\|u\|^2,\end{aligned}$$

for every $u \in X$, and thus (2.6) follows.

On the other hand, the inequality (2.3) yields

$$\begin{aligned}\left|\int_a^b G(u(x))dx\right| &\le \frac{L}{2}\int_a^b (u(x))^2dx \\ &\le \frac{L(b-a)^2}{2p_0}\|u\|^2,\end{aligned}$$

for every $u \in X$. Therefore, we deduce (2.7). The proof is complete. □

By the condition (2.5) and Lemma 2.2 we deduce that Φ is coercive.

By standard arguments, we have that Φ is Gâteaux differentiable and sequentially weakly lower semicontinuous and its Gâteaux derivative is the functional $\Phi'(u) \in X^*$, given by

$$\Phi'(u)(v) = \int_a^b p(x)u'(x)v'(x)dx + \int_a^b q(x)u(x)v(x)dx - \int_a^b g(u(x))v(x)dx$$

for every $v \in X$. Furthermore, the differential $\Phi' \colon X \to X^*$ is a Lipschitzian operator. Indeed, for any $u, v \in X$, there holds

$$\begin{aligned}
\|\Phi'(u) - \Phi'(v)\|_{X^*} &= \sup_{\|w\| \le 1} |\langle \Phi'(u) - \Phi'(v), w\rangle| \\
&\le \sup_{\|w\| \le 1} |\langle u - v, w\rangle| + \sup_{\|w\| \le 1} \int_a^b |g(u(x)) - g(v(x))|\, |w(x)| dx \\
&\le \sup_{\|w\| \le 1} \|u - v\|\, \|w\| \\
&\quad + \sup_{\|w\| \le 1} \left(\int_a^b |g(u(x)) - g(v(x))|^2 \right)^{1/2} \left(\int_a^b |w(x)|^2 \right)^{1/2}.
\end{aligned}$$

Recalling that g is Lipschitz continuous and the embedding $X \hookrightarrow L^2([a,b])$ is compact, the claim is true. In particular, we derive that Φ is continuously differentiable.

On the other hand, the fact that X is compactly embedded into $C^0([a,b])$ implies that the functional Ψ is well defined, continuously Gâteaux differentiable and with compact derivative, whose Gâteaux derivative is given by

$$\Psi'(u)(v) = \int_a^b f(x, u(x))v(x)dx$$

for every $v \in X$. Hence Ψ is sequentially weakly (upper) continuous (see [16, Corollary 41.9]).

Fixing the real parameter λ, a function $u \colon [a,b] \to \mathbb{R}$ is said to be a *weak solution* of problem (1.1) if $u \in X$ and

$$\int_a^b p(x)u'(x)v'(x)dx + \int_a^b q(x)u(x)v(x)dx - \lambda \int_a^b f(x, u(x))v(x)dx - \int_a^b g(u(x))v(x)dx = 0$$

for all $v \in X$.

Hence, the critical points of I_λ are exactly the weak (classical) solutions of (1.1).

In conclusion, we cite a recent monograph by Kristály, Rădulescu and Varga [12] as a general reference on variational methods adopted here.

3 Main results

Put

$$\alpha := \begin{cases} \dfrac{2p_0}{p_0 - L(b-a)^2}, & \text{if } q_0 < \dfrac{p_0}{(b-a)^2}, \\[2ex] \dfrac{2q_0}{q_0 - L}, & \text{if } q_0 \ge \dfrac{p_0}{(b-a)^2}. \end{cases}$$

The main result in this paper is the following.

Theorem 3.1. *Let $f\colon [a,b]\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying condition (f_1). In addition, if $f(x,0)=0$ for a.e. $x\in[a,b]$, assume also that*

(f_2) *there exist a non-empty open set $D\subseteq\,]a,b[$ and a set $B\subseteq D$ of positive Lebesgue measure such that*

$$\limsup_{t\to 0^+}\frac{\inf_{x\in B}F(x,t)}{t^2}=+\infty,\quad\text{and}\quad \liminf_{t\to 0^+}\frac{\inf_{x\in D}F(x,t)}{t^2}>-\infty.$$

Then, there exists a positive number $\lambda^\star$ given by

$$\lambda^\star := r\sup_{\gamma>0}\left(\frac{\gamma}{ra_1c_1\left(\frac{\alpha}{m}\right)^{1/2}+a_2c_r^r\left(\frac{\alpha}{m}\right)^{r/2}\gamma^{r-1}}\right),$$

such that, for every $\lambda\in\,]0,\lambda^\star[$, problem (1.1) admits at least one non-trivial classical solution $u_\lambda\in X$. Moreover,

$$\lim_{\lambda\to 0^+}\|u_\lambda\|=0,$$

and the function $\lambda\mapsto I_\lambda(u_\lambda)$ is negative and strictly decreasing in $]0,\lambda^\star[$.

Proof. We prove the result for the case $q_0<p_0/(b-a)^2$. The proof for the case $q_0\geq p_0/(b-a)^2$ is similar.

Fix $\lambda\in\,]0,\lambda^\star[$. Our aim is to apply Theorem 2.1 with the Sobolev space X and the functionals Φ and Ψ introduced in Section 2. As given in Section 2, Φ and Ψ satisfy the regularity assumptions of Theorem 2.1. Clearly, $\inf_{u\in X}\Phi(u)=0$. Owing to (f_1), one has that

$$F(x,\xi)\leq a_1|\xi|+\frac{a_2}{r}|\xi|^r, \tag{3.1}$$

for any $(x,\xi)\in[a,b]\times\mathbb{R}$.

Since $0<\lambda<\lambda^\star$, there exists $\bar\gamma>0$ such that

$$\lambda<\frac{r\bar\gamma}{ra_1c_1\left(\frac{2p_0}{m(p_0-L(b-a)^2)}\right)^{1/2}+a_2c_r^r\left(\frac{2p_0}{m(p_0-L(b-a)^2)}\right)^{r/2}\bar\gamma^{r-1}}=:\lambda^\star_{\bar\gamma}. \tag{3.2}$$

Now, set $r\in\,]0,+\infty[$ and consider the function

$$\chi(\rho):=\frac{\sup\limits_{v\in\Phi^{-1}(]-\infty,\rho[)}\Psi(v)}{\rho}.$$

Taking into account (3.1) it follows that

$$\Psi(v)=\int_a^b F(x,v(x))dx\leq a_1\|v\|_{L^1([a,b])}+\frac{a_2}{r}\|v\|^r_{L^r([a,b])}.$$

Then, due to (2.7), we get

$$\|u\|<\left(\frac{2p_0\rho}{p_0-L(b-a)^2}\right)^{1/2}, \tag{3.3}$$

for every $u\in X$ such that $\Phi(u)<\rho$.

Now, from (2.4) and by using (3.3), one has

$$\Psi(v)<a_1c_1\left(\frac{2p_0}{m(p_0-L(b-a)^2)}\right)^{1/2}\rho^{1/2}+a_2\frac{c_r^r}{r}\left(\frac{2p_0}{m(p_0-L(b-a)^2)}\right)^{r/2}\rho^{r/2},$$

for every $v \in X$ such that $\Phi(v) < \rho$. Hence

$$\sup_{v\in\Phi^{-1}(]-\infty,\rho[)} \Psi(v) \le a_1 c_1 \left(\frac{2p_0}{m(p_0 - L(b-a)^2)}\right)^{1/2} \rho^{1/2} + a_2 \frac{c_r^r}{r} \left(\frac{2p_0}{m(p_0 - L(b-a)^2)}\right)^{r/2} \rho^{r/2}.$$

Then

$$\chi(\rho) \le a_1 c_1 \left(\frac{2p_0}{m(p_0 - L(b-a)^2)}\right)^{1/2} \rho^{-1/2} + a_2 \frac{c_r^r}{r} \left(\frac{2p_0}{m(p_0 - L(b-a)^2)}\right)^{r/2} \rho^{r/2-1}.$$

In particular, we deduce that

$$\chi(\bar{\gamma}^2) \le a_1 c_1 \left(\frac{2p_0}{m(p_0 - L(b-a)^2)}\right)^{1/2} \bar{\gamma}^{-1} + a_2 \frac{c_r^r}{r} \left(\frac{2p_0}{m(p_0 - L(b-a)^2)}\right)^{r/2} \bar{\gamma}^{r-2}. \tag{3.4}$$

At this point, observe that

$$\varphi(\bar{\gamma}^2) = \inf_{u\in\Phi^{-1}(]-\infty,\bar{\gamma}^2[)} \frac{\left(\sup_{v\in\Phi^{-1}(]-\infty,\bar{\gamma}^2[)} \Psi(v)\right) - \Psi(u)}{\bar{\gamma}^2 - \Phi(u)} \le \chi(\bar{\gamma}^2),$$

taking into account that $0_X \in \Phi^{-1}(]-\infty,\bar{\gamma}^2[)$ and $\Phi(0_X) = \Psi(0_X) = 0$.

In conclusion, bearing in mind (3.2), the above inequality together with (3.4) yields

$$\begin{aligned}\varphi(\bar{\gamma}^2) &\le \chi(\bar{\gamma}^2)\\ &\le a_1 c_1 \left(\frac{2p_0}{m(p_0 - L(b-a)^2)}\right)^{1/2} \bar{\gamma}^{-1} + a_2 \frac{c_r^r}{r} \left(\frac{2p_0}{m(p_0 - L(b-a)^2)}\right)^{r/2} \bar{\gamma}^{r-2}\\ &< \frac{1}{\lambda}.\end{aligned}$$

In other words,

$$\begin{aligned}\lambda \in &\left]0, \frac{r\bar{\gamma}}{r a_1 c_1 \left(\frac{2p_0}{m(p_0-L(b-a)^2)}\right)^{1/2} + a_2 c_r^r \left(\frac{2p_0}{m(p_0-L(b-a)^2)}\right)^{r/2} \bar{\gamma}^{r-1}}\right[\\ &\subseteq]0, 1/\varphi(\bar{\gamma}^2)[.\end{aligned}$$

By Theorem 2.1, there exists a function $u_\lambda \in \Phi^{-1}(]-\infty,\bar{\gamma}^2[)$ such that

$$I'_\lambda(u_\lambda) = \Phi'(u_\lambda) - \lambda\Psi'(u_\lambda) = 0,$$

and, in particular, u_λ is a global minimum of the restriction of I_λ to $\Phi^{-1}(]-\infty,\bar{\gamma}^2[)$.

Now, we have to show that for any $\lambda \in]0,\lambda^\star[$ the solution u_λ is not the trivial zero function. If $f(\cdot,0) \neq 0$, then it easily follows that $u_\lambda \not\equiv 0$ in X, since the trivial function does not solve problem (1.1).

Let us consider the case when $f(\cdot,0) = 0$ and let us fix $\lambda \in]0,\lambda^\star[$. We will prove that the function u_λ cannot be trivial in X. To this end, let us show that

$$\limsup_{\|u\|\to 0^+} \frac{\Psi(u)}{\Phi(u)} = +\infty. \tag{3.5}$$

For this, due to (f_2), we can fix a sequence $\{\xi_n\} \subset \mathbb{R}^+$ converging to zero and two constants σ and κ (with $\sigma > 0$) such that

$$\lim_{n\to+\infty} \frac{\inf_{x\in B} F(x,\xi_n)}{\xi_n^2} = +\infty,$$

and

$$\inf_{x\in D} F(x,\xi) \geq \kappa\xi^2,$$

for every $\xi \in [0,\sigma]$.

Now, fix a set $C \subset B$ of positive measure and a function $v \in C_0^\infty([a,b]) \subset X$ such that:

i) $v(x) \in [0,1]$, for every $x \in [a,b]$;

ii) $v(x) = 1$, for every $x \in C$;

iii) $v(x) = 0$, for every $x \in]a,b[\setminus D$.

Finally, fix $M > 0$ and consider a real positive number η with

$$M < \frac{2p_0}{p_0 + L(b-a)^2} \frac{\eta \operatorname{meas}(C) + \kappa \int_{D\setminus C} v(x)^2\,dx}{\|v\|^2}.$$

Then, there is $\nu \in \mathbb{N}$ such that $\xi_n < \sigma$ and

$$\inf_{x\in B} F(x,\xi_n) \geq \eta\xi_n^2,$$

for every $n > \nu$.

Now, for every $n > \nu$, by the properties of the function v (that is, $0 \leq \xi_n v(x) < \sigma$ for n sufficiently large), one has

$$\begin{aligned}\frac{\Psi(\xi_n v)}{\Phi(\xi_n v)} &= \frac{\int_C F(x,\xi_n)\,dx + \int_{D\setminus C} F(x,\xi_n v(x))\,dx}{\Phi(\xi_n v)} \\ &\geq \frac{2p_0}{p_0 + L(b-a)^2} \frac{\eta \operatorname{meas}(C) + \kappa \int_{D\setminus C} v(x)^2\,dx}{\|v\|^2} > M.\end{aligned}$$

Since M could be taken arbitrarily large, it follows that

$$\lim_{n\to+\infty} \frac{\Psi(\xi_n v)}{\Phi(\xi_n v)} = +\infty,$$

from which (3.5) clearly follows. Hence, there exists a sequence $\{w_n\} \subset X$ strongly converging to zero, such that, for every n sufficiently large, $w_n \in \Phi^{-1}(]-\infty, \bar{\gamma}^2[)$, and

$$I_\lambda(w_n) = \Phi(w_n) - \lambda\Psi(w_n) < 0.$$

Since u_λ is a global minimum of the restriction of I_λ to $\Phi^{-1}(]-\infty, \bar{\gamma}^2[)$, we conclude that

$$I_\lambda(u_\lambda) < 0 = I_\lambda(0_X), \tag{3.6}$$

so that u_λ is not trivial in X.

Moreover, from (3.6) we easily see that the map

$$]0, \lambda^\star[\ni \lambda \mapsto I_\lambda(u_\lambda)$$

is negative.

Now, we claim that

$$\lim_{\lambda\to 0^+} \|u_\lambda\| = 0.$$

Indeed, bearing in mind that Φ is a coercive functional and that $u_\lambda \in \Phi^{-1}(]-\infty, \bar{\gamma}^2[)$, for every $\lambda \in]0, \lambda^\star_{\bar{\gamma}}[$, we obtain

$$\|u_\lambda\| < \left(\frac{2p_0}{p_0 - L(b-a)^2}\right)^{1/2} \bar{\gamma}.$$

As a consequence, using the growth condition (f_1) together with the property (2.4), it follows that

$$\begin{aligned}
\left|\int_a^b f(x,u_\lambda(x))u_\lambda(x)\,dx\right| &\le a_1\|u_\lambda\|_{L^1([a,b])} + a_2\|u_\lambda\|^r_{L^r([a,b])}\\
&\le \frac{a_1c_1}{m^{1/2}}\|u_\lambda\| + \frac{a_2c_r^r}{m^{r/2}}\|u_\lambda\|^r \qquad (3.7)\\
&< a_1c_1\left(\frac{2p_0}{m(p_0 - L(b-a)^2)}\right)^{1/2} + a_2c_r^r\left(\frac{2p_0}{m(p_0 - L(b-a)^2)}\right)^{r/2}\\
&=: M_{\bar{\gamma}},
\end{aligned}$$

for every $\lambda \in]0, \lambda^\star_{\bar{\gamma}}[$.

Since u_λ is a critical point of I_λ, then $I'_\lambda(u_\lambda)(v) = 0$, for any $v \in X$ and every $\lambda \in]0, \lambda^\star_{\bar{\gamma}}[$. In particular, $I'_\lambda(u_\lambda)(u_\lambda) = 0$, that is

$$\Phi'(u_\lambda)(u_\lambda) = \lambda \int_a^b f(x, u_\lambda(x))u_\lambda(x)\,dx, \qquad (3.8)$$

for every $\lambda \in]0, \lambda^\star_{\bar{\gamma}}[$. Hence, from (2.3), (3.7) and (3.8), it follows that

$$0 \le \frac{p_0 - L(b-a)^2}{2p_0}\|u_\lambda\|^2 \le \Phi'(u_\lambda)(u_\lambda) < \lambda M_{\bar{\gamma}},$$

for every $\lambda \in]0, \lambda^\star_{\bar{\gamma}}[$. Letting $\lambda \to 0^+$, we get $\lim_{\lambda\to 0^+} \|u_\lambda\| = 0$, as claimed.

Finally, we have to show that the map $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $]0, \lambda^\star[$. For this, we observe that for any $u \in X$, one has

$$I_\lambda(u) = \lambda\left(\frac{\Phi(u)}{\lambda} - \Psi(u)\right). \qquad (3.9)$$

Now, let us fix $0 < \lambda_1 < \lambda_2 < \lambda^\star_{\bar{\gamma}}$ and let u_{λ_i} be the global minimum of the functional I_{λ_i} restricted to $\Phi^{-1}(]-\infty, \bar{\gamma}^2[)$ for $i = 1, 2$. Also, let

$$m_{\lambda_i} := \left(\frac{\Phi(u_{\lambda_i})}{\lambda_i} - \Psi(u_{\lambda_i})\right) = \inf_{v\in\Phi^{-1}(]-\infty,\bar{\gamma}^2[)}\left(\frac{\Phi(v)}{\lambda_i} - \Psi(v)\right),$$

for every $i = 1, 2$.

Clearly, the negativity of the map $\lambda \mapsto I_\lambda(u_\lambda)$ in $]0, \lambda^\star[$ together with (3.9) and the positivity of λ imply that

$$m_{\lambda_i} < 0, \quad \text{for } i = 1, 2. \qquad (3.10)$$

Moreover,

$$m_{\lambda_2} \le m_{\lambda_1}, \tag{3.11}$$

thanks to $0 < \lambda_1 < \lambda_2$ and $\Phi \ge 0$ by Lemma 2.2. Then, by (3.9)–(3.11) and again by the fact that $0 < \lambda_1 < \lambda_2$, we get that

$$I_{\lambda_2}(u_{\lambda_2}) = \lambda_2 m_{\lambda_2} \le \lambda_2 m_{\lambda_1} < \lambda_1 m_{\lambda_1} = I_{\lambda_1}(u_{\lambda_1}),$$

so that the map $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $]0, \lambda^\star[$, which completes the proof. □

Remark 3.2. Theorem 3.1 can be also obtained applying Theorem 2.3 of [4], which directly ensures that the local minimum is non-zero.

4 Additional results and comments

In this section we give some consequences, remarks and an example.

Remark 4.1. By direct computation, it follows that the parameter $\lambda^\star$ in Theorem 3.1 can be expressed as

$$\lambda^\star = \begin{cases} +\infty, & \text{if } 1 < r < 2, \\ \dfrac{2m}{\alpha a_2 c_2^2}, & \text{if } r = 2, \\ \dfrac{r\widetilde{\gamma}_{\max}}{r a_1 c_1 \left(\frac{\alpha}{m}\right)^{1/2} + a_2 c_r^r \left(\frac{\alpha}{m}\right)^{r/2} \widetilde{\gamma}_{\max}^{r-1}}, & \text{if } 2 < r < +\infty, \end{cases}$$

where

$$\widetilde{\gamma}_{\max} := \left(\frac{m}{\alpha}\right)^{1/2} \left(\frac{r a_1 c_1}{a_2 c_r^r (r-2)}\right)^{1/(r-1)}.$$

Remark 4.2. From the above expressions, it follows that if the term f is sublinear at infinity (i.e., $r \in]1,2[$ in (f_1)), Theorem 3.1 ensures that, for all $\lambda > 0$, problem (1.1) admits at least one nontrivial classical solution.

Remark 4.3. We observe that if $f(x,0) = 0$ for a.e. $x \in [a,b]$, Theorem 3.1 is a bifurcation result. Indeed, in this setting, it follows that the trivial solution solves problem (1.1) for every parameter λ. Hence, $\lambda = 0$ is a bifurcation point for problem (1.1), in the sense that the pair $(0,0)$ belongs to the closure of the set

$$\{(u_\lambda, \lambda) \in X \times]0, +\infty[\, : u_\lambda \text{ is a nontrivial classical solution of (1.1)}\}$$

in the space $X \times \mathbb{R}$.

Indeed, by Theorem 3.1 we have that

$$\|u_\lambda\| \to 0 \quad \text{as } \lambda \to 0^+.$$

Hence, there exist two sequences $\{u_j\}_{j\in\mathbb{N}}$ in X and $\{\lambda_j\}_{j\in\mathbb{N}}$ in $\mathbb{R}^+$ (here $u_j := u_{\lambda_j}$) such that

$$\lambda_j \to 0^+ \quad \text{and} \quad \|u_j\| \to 0,$$

as $j \to +\infty$.

Moreover, for any $\lambda_1, \lambda_2 \in]0, \lambda^\star[$, with $\lambda_1 \ne \lambda_2$, the solutions u_{λ_1} and u_{λ_2} given by Theorem 3.1 are different, thanks to the fact that the map

$$]0, \lambda^\star[\ni \lambda \mapsto I_\lambda(u_\lambda)$$

is strictly decreasing.

The next result is an immediate consequence of Remark 4.1.

Corollary 4.4. *Let $f\colon [a,b]\times\mathbb{R}\to\mathbb{R}$ be a continuous function with $f(x,0)=0$ for a.e. $x\in[a,b]$, satisfying the following subcritical growth condition*

$$|f(x,t)|\leq a_1+a_2|t|^{r-1},\qquad \text{a.e. } x\in[a,b],\ t\in\mathbb{R}, \tag{4.1}$$

where $a_1,a_2\geq 0$ and $r\in\,]2,+\infty[$. Further, assume that there exists a non-empty open set $B\subseteq\,]a,b[$ such that

$$\lim_{t\to 0^+}\frac{\inf_{x\in B}F(x,t)}{t^2}=+\infty. \tag{4.2}$$

Then, there exists a positive number $\lambda^\star$ given by

$$\lambda^\star:=\frac{m(r-2)}{\alpha a_1 c_1(r-1)}\left(\frac{ra_1c_1}{a_2c_r^r(r-2)}\right)^{1/(r-1)},$$

such that, for every $\lambda\in\,]0,\lambda^\star[$, problem (1.1) admits at least one nontrivial classical solution $u_\lambda\in X$. Moreover,

$$\lim_{\lambda\to 0^+}\|u_\lambda\|=0,$$

and the function $\lambda\to I_\lambda(u_\lambda)$ is negative and strictly decreasing in $]0,\lambda^\star[$.

We state an example on the following special case of our results.

Theorem 4.5. *Let $f\colon\mathbb{R}\to\mathbb{R}$ be a continuous function such that $f(0)=0$, and*

$$\lim_{t\to 0^+}\frac{f(t)}{t}=+\infty,\qquad \lim_{|t|\to+\infty}\frac{f(t)}{|t|^s}=0,$$

for some $0\leq s<+\infty$. Then, there exists $\lambda^\star>0$ such that, for every $\lambda\in\,]0,\lambda^\star[$, the following autonomous mixed problem

$$\begin{cases}-(pu')'+qu=\lambda f(u)+g(u) & \text{in }]a,b[,\\ u(a)=u'(b)=0,\end{cases}$$

admits at least one nontrivial classical solution $u_\lambda\in X$. Moreover,

$$\lim_{\lambda\to 0^+}\|u_\lambda\|=0,$$

and the mapping

$$\lambda\mapsto\Phi(u_\lambda)-\lambda\int_a^b\left(\int_0^{u_\lambda(x)}f(x,t)dt\right)dx$$

is negative and strictly decreasing in $]0,\lambda^\star[$.

Proof. The conclusion follows immediately from Theorem 3.1. Indeed, if

$$\lim_{t\to 0^+}\frac{f(t)}{t}=+\infty,$$

then, we have

$$\lim_{t\to 0^+}\frac{F(t)}{t^2}=+\infty,$$

and condition (f_2) holds true. Moreover, hypothesis

$$\lim_{|t|\to+\infty}\frac{f(t)}{|t|^s}=0,$$

where $0\leq s<+\infty$, implies the growth condition (f_1). ☐

Remark 4.6. We observe that if f is a non-negative function, our results guarantee that the attained solution is non-negative. To this end, let u_0 be a solution of problem (1.1). Arguing by contradiction, assume that the set $A := \{x \in [a,b] : u_0(x) < 0\}$ is non-empty and of positive Lebesgue measure. Put $\bar{v}(x) := \min\{0, u_0(x)\}$ for all $x \in [a,b]$. Clearly, $\bar{v} \in X$ and, taking into account that u_0 is a weak solution and by choosing $v = \bar{v}$, one has

$$\int_a^b p(x)u_0'(x)\bar{v}'(x)dx + \int_a^b q(x)u_0(x)\bar{v}(x)dx - \lambda\int_a^b f(x,u_0(x))\bar{v}(x)dx - \int_a^b g(u_0(x))\bar{v}(x)dx = 0,$$

that is,

$$\int_A p(x)|u_0'(x)|^2dx + \int_A q(x)|u_0(x)|^2dx - \int_A g(u_0(x))u_0(x)dx \le 0.$$

On the other hand, if $q_0 < p_0/(b-a)^2$, then

$$\frac{p_0 - L(m(A))^2}{p_0}\|u_0\|^2_{W^{1,2}(A)} \le \int_A p(x)|u_0'(x)|^2dx + \int_A q(x)|u_0(x)|^2dx - \int_A g(u_0(x))u_0(x)dx,$$

where $m(A)$ is the Lebesgue measure of the set A, and if $q_0 \ge p_0/(b-a)^2$, then

$$\frac{q_0 - L}{q_0}\|u_0\|^2_{W^{1,2}(A)} \le \int_A p(x)|u_0'(x)|^2dx + \int_A q(x)|u_0(x)|^2dx - \int_A g(u_0(x))u_0(x)dx.$$

Hence, $u_0 \equiv 0$ on A which is absurd. So, it follows that u_0 is non-negative.

The next example deals with a nonlinearity f has vanishes at zero. The existence of one nontrivial solution for the mixed problem involving the map f is achieved by using Corollary 4.4.

Example 4.7. Consider the following problem

$$\begin{cases} -(2e^x u')' + u(e^x - 1) = \lambda f(x,u) & \text{in }]0,1[, \\ u(0) = u'(1) = 0, \end{cases} \tag{4.3}$$

where $f(x,u) := \alpha(x)|u|^{h-2}u + \beta(x)|u|^{l-2}u$ and $\alpha, \beta\colon [0,1] \to \mathbb{R}$ are two continuous positive and bounded functions, and $1 < h < 2 < l$. Then, for every $\lambda \in]0, \lambda^\star[$, where

$$\lambda^\star := \frac{l-2}{8(l-1)\max\{\|\alpha\|_\infty, \|\beta\|_\infty\}}\left(\frac{lc_1}{c_l^l(l-2)}\right)^{1/(l-1)},$$

problem (4.3) admits at least one nontrivial classical solution

$$u_\lambda \in Y := \{u \in W^{1,2}([0,1]) : u(0) = 0\}.$$

Moreover, by

$$\|u_\lambda\| := \left(\int_0^1 2e^x(u_\lambda'(x))^2dx + \int_0^1 e^x(u_\lambda(x))^2dx\right)^{1/2},$$

we have

$$\lim_{\lambda\to 0^+}\|u_\lambda\| = 0,$$

and the function

$$\lambda \mapsto \frac{1}{2}\|u_\lambda\|^2 + \int_0^1 u_\lambda(x)dx - \lambda\int_0^1\left(\int_0^{u_\lambda(x)} f(x,t)dt\right)dx$$

is negative and strictly decreasing in $]0,\lambda^\star[$.

To prove this, we can apply Corollary 4.4 with

$$f(x,t) := \alpha(x)|t|^{h-2}t + \beta(x)|t|^{l-2}t,$$
$$p(x) := 2e^x, \qquad q(x) := e^x, \qquad g(t) := t,$$

for every $(x,t) \in [0,1] \times \mathbb{R}$. In fact, $f(x,0) = 0$ for a.e. $x \in [0,1]$ and it is easy to verify that

$$|f(x,t)| \leq 2\max\{\|\alpha\|_\infty, \|\beta\|_\infty\}\left(1+|t|^{l-1}\right), \quad \text{a.e. } x \in [0,1],\ t \in \mathbb{R}.$$

Then, condition (4.1) holds. Moreover, a direct computation shows that

$$\lim_{t\to 0^+} \frac{\inf_{x\in B} F(x,t)}{t^2} \geq \frac{\inf_{x\in B}\alpha(x)}{h}\left(\lim_{t\to 0^+}\frac{1}{t^{2-h}}\right) = +\infty,$$

where $B \subseteq]0,1[$ is an arbitrary non-empty open set. Hence, assumption (4.2) is verified and the conclusion follows.

Remark 4.8. We point out that the energy functional I_λ associated to problem (4.3) is unbounded from below. Indeed, fix $u \in X \setminus \{0\}$ and let $\tau \in \mathbb{R}$. We have

$$\begin{aligned} I_\lambda(\tau u) &= \Phi(\tau u) - \lambda\int_0^1\left(\int_0^{\tau u(x)} f(x,t)dt\right)dx \\ &\leq \frac{3}{4}\|u\|^2 - \lambda\frac{\tau^h \inf_{x\in[0,1]}\alpha(x)}{h}\|u\|^h_{L^h([0,1])} - \lambda\frac{\tau^l \inf_{x\in[0,1]}\beta(x)}{l}\|u\|^l_{L^l([0,1])} \to -\infty, \end{aligned}$$

as $\tau \to +\infty$, bearing in mind that $h < 2 < l$.

Hence, since the functional I_λ is not coercive, the classical direct method result cannot be applied to the case treated in Example 4.7.

Remark 4.9. We note that, applying Theorem 2.1, we have the relevant result of Theorem 3.1 for the following mixed boundary value problem with a complete equation

$$\begin{cases} -(\bar{p}u')' + \bar{n}u' + \bar{q}u = \lambda f(x,u) + g(u) & \text{in }]a,b[, \\ u(a) = u'(b) = 0, \end{cases} \tag{4.4}$$

where $\bar{p} \in C^1([a,b])$ and $\bar{q}, \bar{n} \in C^0([a,b])$ such that $\bar{p}$ and $\bar{q}$ are positive functions, λ is a positive parameter, $f\colon [a,b]\times\mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$|f(x,t)| \leq e^{R(x)}\left(a_1 + a_2|t|^{r-1}\right), \quad \text{a.e. } x \in [a,b],\ t \in \mathbb{R},$$

where $a_1, a_2 \geq 0$ and $r \in]1,+\infty[$ and R is a primitive of $\bar{n}/\bar{p}$, while $g\colon \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with the Lipschitz constant $L > 0$ satisfying

$$L < \max\left\{\min_{x\in[a,b]} e^{-R(x)}\bar{q}(x), \frac{\min_{x\in[a,b]} e^{-R(x)}\bar{p}(x)}{(b-a)^2}\right\},$$

and $g(0) = 0$.

In fact, since the solutions of problem (4.4) are solutions of the problem

$$\begin{cases} -(e^{-R}\bar{p}u')' + e^{-R}\bar{q}u = (\lambda f(x,u) + g(u))\,e^{-R} & \text{in }]a,b[, \\ u(a) = u'(b) = 0, \end{cases}$$

we can state and prove a result for problem (4.4) similar to Theorem 3.1.

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