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Periodic solutions of second-order systems with subquadratic convex potential

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Abstract. In this paper, we investigate the existence of periodic solutions for the second order systems at resonance:

$$\begin{cases} \ddot{u}(t) + m^2\omega^2 u(t) + \nabla F(t, u(t)) = 0 & \text{a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases}$$

where $m > 0$, the potential $F(t, x)$ is convex in x and satisfies some general subquadratic conditions. The main results generalize and improve Theorem 3.7 in J. Mawhin and M. Willem [Critical point theory and Hamiltonian systems, Springer-Verlag, New York, 1989].

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1 Introduction and main results

Consider the second order Hamiltonian systems

$$\begin{cases} \ddot{u}(t) + m^2\omega^2 u(t) + \nabla F(t, u(t)) = 0 & \text{a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases} \tag{1.1}$$

where $T > 0$, $\omega = 2\pi/T$ and $m > 0$ is an integer. The potential $F\colon [0, T] \times R^N \to R$ satisfies the following assumption:

(A) $F(t, x)$ is measurable in t for every $x \in R^N$ and continuously differentiable in x for a.e. $t \in [0, T]$, and there exist $a \in C(R^+, R^+)$, $b \in L^1(0, T; R^+)$ such that

$$|F(t, x)| \le a(|x|)b(t), \qquad |\nabla F(t, x)| \le a(|x|)b(t)$$

for all $x \in R^N$ and a.e. $t \in [0, T]$.

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If $m = 0$, the non-resonant second order Hamiltonian systems have been extensively investigated during the past two decades. Different solvability hypotheses on the potential are given, such as: the convexity conditions (see [6, 8, 12, 13]); the coercivity conditions (see [1, 5, 10]); the subquadratic conditions (including the sublinear nonlinearity case, see [7, 9, 11–14, 16, 18]); the superquadratic conditions (see [3, 7, 17, 18, 21]) and the asymptotically quadratic conditions (see [19, 21, 24]).

Using the variational principle of Clarke and Ekeland together with an approximate argument of H. Brézis [2], Mawhin and Willem [6] proved an existence theorem for semilinear equations of the form $Lu = \nabla F(x, u)$, where L is a noninvertible linear selfadjoint operator and F is convex with respect to u and satisfies a suitable asymptotic quadratic growth condition. This result was applied to periodic solutions of first order Hamiltonian systems with convex potential. In [5], the authors considered the second order systems (1.1) with $m = 0$. They proved that when the potential F satisfies the following assumptions:

(A') $F(t, x)$ is measurable in t for every $x \in R^N$, and continuously differentiable and convex in x for a.e. $t \in [0, T]$;

(A_1) There exists $l \in L^4(0, T; R^N)$ such that

$$(l(t), x) \leq F(t, x), \qquad \forall x \in R^N \text{ and a.e. } t \in [0, T];$$

(A'_2) There exist $\alpha \in (0, \omega^2)$ and $\gamma \in L^2(0, T; R^+)$ such that

$$F(t, x) \leq \frac{1}{2}\alpha|x|^2 + \gamma(t), \qquad \forall x \in R^N \text{ and a.e. } t \in [0, T];$$

(A'_3) $\int_0^T F(t, x)\, dt \to +\infty$ as $|x| \to \infty$, $x \in R^N$;

then problem (1.1) has at least one solution, see [5, Theorem 3.5]. This result was slightly improved in Tang [8] by relaxing the integrability of l and γ. In [12], Tang and Wu dealt with the (β, γ)-subconvex case, i.e.,

$$F(t, \beta(x + y)) \leq \gamma(F(t, x) + F(t, y)), \qquad \forall x, y \in R^N \text{ and a.e. } t \in [0, T] \tag{1.2}$$

for some $\gamma > 0$. Under assumptions (A), (A'_3) and (1.2) and the subquadratic condition: there exist $0 < \mu < 2$ and $M > 0$ such that

$$(\nabla F(t, x), x) \leq \mu F(t, x), \qquad \forall |x| \geq M \text{ and a.e. } t \in [0, T],$$

they obtained the existence result by taking advantage of Rabinowitz's saddle point theorem. Recently, Tang and Wu [13] extended a theorem established by A. C. Lazer, E. M. Landesman and D. R. Meyers [4] on the existence of critical points without compactness assumptions, using the reduction method, the perturbation argument and the least action principle. As a main application, they successively studied the existence of periodic solutions of problem (1.1) ($m = 0$) with subquadratic convex potential, with subquadratic $\mu(t)$-convex potential and with subquadratic $k(t)$-concave potential, which unifies and significantly generalizes some earlier results in [5, 8, 15, 22, 23] obtained by other methods.

If $m \neq 0$, it is a resonance case. Using the dual least action principle and the perturbation technique, Mawhin and Willem [5] also obtained the following theorem.

Theorem A ([5, Theorem 3.7]). *Suppose that $F(t,x)$ satisfies conditions (A'), (A_1) and the following:*

(A_2) *There exist $\alpha \in (0,(2m+1)\omega^2)$ and $\gamma \in L^2(0,T;R^+)$ such that*

$$F(t,x) \le \frac{1}{2}\alpha|x|^2 + \gamma(t), \qquad \forall x \in R^N \text{ and a.e. } t \in [0,T].$$

(A_3) $\displaystyle\int_0^T F(t, a\cos m\omega t + b\sin m\omega t)\,dt \to +\infty \quad \text{as } |a|+|b| \to \infty, \quad a,b \in R^N.$

Then problem (1.1) has at least one solution in H_T^1, where

$$H_T^1 = \left\{ u\colon [0,T] \to R^N \;\middle|\; \begin{array}{l} u \text{ is absolutely continuous,} \\ u(0) = u(T) \text{ and } \dot u \in L^2(0,T;R^N) \end{array} \right\}$$

is a Hilbert space with the norm defined by

$$\|u\| = \left(\int_0^T |u(t)|^2\,dt + \int_0^T |\dot u(t)|^2\,dt \right)^{1/2}.$$

Motivated by the works mentioned above, in this paper, we are interested in problem (1.1), where the potential is convex and satisfies conditions which are more general than (A_2). Applying the abstract critical point theory established in [13], we prove some existence results, which generalize Theorem A and complement the results in [13]. The main results are the following theorems.

Theorem 1.1. *Suppose that assumption (A) holds and $F(t,x)$ is convex in x for a.e. $t \in [0,T]$. Assume that (A_3) holds and:*

(A_4) *There exists $\gamma \in L^1(0,T;R^+)$ such that*

$$F(t,x) \le \frac{2m+1}{2}\omega^2|x|^2 + \gamma(t) \tag{1.3}$$

for all $x \in R^N$ and a.e. $t \in [0,T]$, and

$$\operatorname{meas}\left\{ t \in [0,T] \;\middle|\; F(t,x) - \frac{2m+1}{2}\omega^2|x|^2 \to -\infty \quad \text{as } |x| \to \infty \right\} > 0. \tag{1.4}$$

Then problem (1.1) has at least one solution in H_T^1.

Remark 1.2. Theorem 1.1 extends Theorem A, since (A_4) is weaker than (A_2) and assumption (A) holds for functions F in Theorem A (see [13, Remark 1.3] for a proof). There are functions F which match our setting but not satisfying Theorem A. For example, let

$$F(t,x) = \frac{2m+1}{2}\omega^2\left(|x|^2 - (1+|x|^2)^{\frac{3}{4}}\right) + (l(t),x),$$

where $l \in L^3(0,T;R^N) \backslash L^\infty(0,T;R^N)$. Then by Young's inequality, one has

$$\begin{aligned} -\frac{2m+1}{2}\omega^2(1+|x|^2)^{\frac{3}{4}} + (l(t),x) &\le -\frac{2m+1}{2}\omega^2|x|^{\frac{3}{2}} + |l(t)||x| \\ &\le -\frac{2m+1}{2}\omega^2|x|^{\frac{3}{2}} \\ &\quad + \frac{2m+1}{2}\left(\omega^{\frac{4}{3}}|x|\right)^{\frac{3}{2}} + \frac{2m+1}{4}\left(\frac{4}{3(2m+1)}\right)^3\omega^{-4}|l(t)|^3 \\ &\le \frac{16}{27(2m+1)^2}\omega^{-4}|l(t)|^3 \end{aligned}$$

for all $x \in R^N$ and a.e. $t \in [0,T]$. Thus F satisfies (1.3) with $\gamma(t) = \frac{16}{27(2m+1)^2}\omega^{-4}|l(t)|^3$. Evidently, (A_3) and (1.4) are satisfied, and $F(t,\cdot)$ is convex because

$$f(x) := g(h(x))$$

is convex by the fact that

$$g(s) := (s - (1+s)^{\frac{3}{4}}), \qquad s > 0$$

is convex and increasing, and

$$h(x) := |x|^2, \qquad x \in R^N$$

is convex. Hence F satisfies all the conditions of Theorem 1.1. But it does not satisfy Theorem A, for (A_2) does not hold.

Theorem 1.1 yields immediately the following corollary.

Corollary 1.3. *The conclusion of Theorem 1.1 remains valid if we replace (A_4) by*

(A_5) $F(t,x) - \dfrac{2m+1}{2}\omega^2|x|^2 \to -\infty$ *as* $|x| \to \infty$ *for a.e.* $t \in [0,T]$.

Remark 1.4. It is easy to see that (A_5) is weaker than (A_2). So Corollary 1.3 also generalizes Theorem A.

Corollary 1.5. *The conclusion of Theorem 1.1 remains valid if we replace (A_4) by*

(A_6) *There exist $\alpha \in L^\infty(0,T;R^+)$ with* meas $\{t \in [0,T] : \alpha(t) < (2m+1)\omega^2\} > 0$ *and* $\alpha(t) \le (2m+1)\omega^2$ *for a.e. $t \in [0,T]$, and $\gamma \in L^1(0,T;R^+)$ such that*

$$F(t,x) \le \frac{1}{2}\alpha(t)|x|^2 + \gamma(t) \quad \text{for all } x \in R^N \text{ and a.e. } t \in [0,T]. \tag{1.5}$$

Remark 1.6. Corollary 1.5 also generalizes Theorem A. There are functions F satisfying our Corollary 1.5 and not satisfying Theorem A and Corollary 1.3. For example, let

$$F(t,x) = \frac{1}{2}\beta(t)|x|^2 + (l(t),x),$$

where $\beta \in L^\infty(0,T;R^+)$ with $\beta(t) \le (2m+1)\omega^2$ for a.e. $t \in [0,T]$, $\int_0^T \beta(t)dt > 0$,

$$\text{meas}\,\{t \in [0,T] : \beta(t) < (2m+1)\omega^2\} > 0,$$

and $l \in L^\infty(0,T;R^N)$ with $|l(t)| \le \frac{1}{2}((2m+1)\omega^2 - \beta(t))$ for a.e. $t \in [0,T]$. Then one has

$$F(t,x) \le \frac{1}{2}\beta(t)|x|^2 + |l(t)||x| \le \frac{1}{2}(\beta(t) + |l(t)|)|x|^2 + \frac{1}{2}|l(t)|,$$

which is just (1.5) with $\alpha = \beta(t) + |l(t)|$ and $\gamma = |l(t)|/2$. Hence F satisfies Corollary 1.5. But in the case that meas $\{t \in [0,T] : \beta(t) = (2m+1)\omega^2\} > 0$, F does not satisfy the conditions of Theorem A and Corollary 1.3.

Theorem 1.7. *Suppose that assumption (A) holds and $F(t,x)$ is convex in x for a.e. $t \in [0,T]$. Assume that (A_3) holds and the following condition is fulfilled.*

(A_7) *There exists* $\alpha \in L^\infty(0,T;R^+)$ *with* meas $\{t \in [0,T] \,|\alpha(t) < (2m+1)\omega^2\} > 0$ *and* $\alpha(t) \le (2m+1)\omega^2$ *for a.e.* $t \in [0,T]$ *such that*

$$\limsup_{|x|\to\infty} |x|^{-2} F(t,x) \le \frac{1}{2}\alpha(t) \quad \text{uniformly for a.e. } t \in [0,T].$$

Then problem (1.1) *has at least one solution in* H_T^1.

Remark 1.8. The conditions (A_6) and (A_7) are not equivalent in general. There are functions F satisfying (A_7) but not (A_6). For example, let

$$F(t,x) = \frac{1}{2}\mu(t)|x|^2 + |x|^{\frac{3}{2}}, \qquad \forall x \in R^N \text{ and a.e. } t \in [0,T],$$

where $\mu \in L^1(0,T;R)$ with $\mu(t) \le (2m+1)\omega^2$ for a.e. $t \in [0,T]$, $\int_0^T \mu(t)\,dt > 0$, and meas $\{t \in [0,T] : \mu(t) < \omega^2\} > 0$. Then (A_7) holds with $\alpha = \mu^+(t)$. But F does not satisfy (A_6) if meas $\{t \in [0,T] : \mu(t) = \omega^2\} > 0$. On the other hand, there are functions F satisfying (A_6) but not (A_7). For example, let

$$F(t,x) = \frac{1}{3}t^{-\frac{1}{8}}\left(\sqrt{2m+1}\,\omega|x|\right)^{\frac{3}{2}}, \qquad \forall x \in R^N \text{ and a.e. } t \in [0,T].$$

By Young's inequality, one has

$$F(t,x) \le \frac{1}{3}\left(\frac{3}{4}\left(\sqrt{2m+1}\,\omega|x|\right)^2 + \frac{(t^{-\frac{1}{8}})^4}{4}\right) = \frac{(2m+1)\omega^2}{4}|x|^2 + \frac{t^{-\frac{1}{2}}}{12},$$

which is just (1.5) with $\alpha = (2m+1)\omega^2/2$ and $\gamma = t^{-\frac{1}{2}}/12$. However, $F(t,x)$ does not satisfy (A_7), because

$$\limsup_{|x|\to\infty} \frac{\frac{1}{3}t^{-\frac{1}{8}}\left(\sqrt{2m+1}\,\omega|x|\right)^{\frac{3}{2}}}{|x|^2} \le \frac{(2m+1)\omega^2}{4}$$

does not uniformly hold for a.e. $t \in [0,T]$.

Remark 1.9. Theorem 1.7 generalizes Theorem A. There are functions F satisfying our Theorem 1.7 and not satisfying Theorems A and 1.1. For example, let

$$F(t,x) = \frac{1}{2}\alpha(t)|x|^2 + |x|^{\frac{3}{2}} + (l(t),x),$$

where $\alpha \in L^\infty(0,T;R^+)$ with $\alpha(t) \le (2m+1)\omega^2$ for a.e. $t \in [0,T]$, $\int_0^T \alpha(t)\,dt > 0$,

$$\text{meas}\,\{t \in [0,T] : \alpha(t) < (2m+1)\omega^2\} > 0,$$

and $l \in L^\infty(0,T;R^N)$. Then F satisfies all the conditions of Theorem 1.7. But obviously F does not satisfy Theorems A and 1.1.

Theorem 1.10. *Suppose that assumption* (A) *holds and* $F(t,x)$ *is convex in* x *for a.e.* $t \in [0,T]$. *Assume that* (A_3) *holds and:*

(A_8) *There exist* $\alpha \in L^1(0,T;R^+)$ *with* $\int_0^T \alpha(t)\,dt < \frac{12(2m+1)}{T(m+1)^2}$ *and* $\gamma \in L^1(0,T;R^+)$ *such that*

$$F(t,x) \le \frac{1}{2}\alpha(t)|x|^2 + \gamma(t), \qquad \forall x \in R^N \text{ and a.e. } t \in [0,T]. \tag{1.6}$$

Then problem (1.1) *has at least one solution in* H_T^1.

Remark 1.11. There are functions F satisfying our Theorem 1.10 and not satisfying the results mentioned above. For example, let

$$F(t,x) = \frac{1}{2}\beta(t)|x|^2 + (l(t),x),$$

where $\beta \in L^1(0,T;R^+)$ with $0 < \int_0^T \beta(t)\,dt < \frac{12(2m+1)}{T(m+1)^2}$ and $l \in L^2(0,T;R^N)$. Then one has

$$\begin{aligned} F(t,x) &\le \frac{1}{2}\beta(t)|x|^2 + |l(t)||x| \\ &\le \frac{1}{2}\left(\beta(t) + \frac{12(2m+1) - T(m+1)^2|\beta|_1}{2T^2(m+1)^2}\right)|x|^2 + \frac{T^2(m+1)^2}{12(2m+1) - T(m+1)^2|\beta|_1}|l(t)|^2, \end{aligned}$$

which is just (1.6) with

$$\alpha = \beta(t) + \frac{12(2m+1) - T(m+1)^2|\beta|_1}{2T^2(m+1)^2} \quad \text{and} \quad \gamma = \frac{T^2(m+1)^2}{12(2m+1) - T(m+1)^2|\beta|_1}|l(t)|^2.$$

Thus F satisfies all the conditions of Theorem 1.10. But in the case that

$$\text{meas}\left\{t \in [0,T] : \beta(t) > (2m+1)\omega^2\right\} > 0,$$

F does not satisfy the conditions of Theorems A, 1.1 and 1.7.

2 Proofs of the theorems

Under assumption (A), the energy functional associated to problem (1.1) given by

$$\varphi(u) = -\frac{1}{2}\int_0^T |\dot{u}(t)|^2\,dt + \frac{m^2\omega^2}{2}\int_0^T |u(t)|^2\,dt + \int_0^T F(t,u(t))\,dt$$

is continuously differentiable and weakly upper semi-continuous on H_T^1. Furthermore,

$$\langle \varphi'(u), v\rangle = -\int_0^T (\dot{u}(t),\dot{v}(t))\,dt + m^2\omega^2\int_0^T (u(t),v(t))\,dt + \int_0^T (\nabla F(t,u(t)),v(t))\,dt$$

for all $u,v \in H_T^1$, and φ' is weakly continuous. It is well known that the weak solutions of problem (1.1) correspond to the critical points of φ (see [5]).

For $u \in \widetilde{H}_T^1 \triangleq \left\{u \in H_T^1 : \int_0^T u(t)\,dt = 0\right\}$, we have

$$\|u\|_\infty \le \frac{T}{12}\int_0^T |\dot{u}(t)|^2\,dt \quad \text{(Sobolev's inequality)},$$

which implies that

$$\|u\|_\infty \le C\|u\|, \qquad \forall u \in H_T^1 \tag{2.1}$$

for some $C > 0$, where $\|u\|_\infty = \max_{t\in[0,T]}|u(t)|$ (see [5, Proposition 1.3]).

We recall an abstract critical point theorem which will be used in the sequel.

Proposition 2.1 ([13, Theorem 1.1]). *Suppose that V and W are reflexive Banach spaces, $\varphi \in C^1(V \times W, R)$, $\varphi(v, \cdot)$ is weakly upper semi-continuous for all $v \in V$ and $\varphi(\cdot, w) \colon V \to R$ is convex for all $w \in W$, that is,*

$$\varphi(\lambda v_1 + (1-\lambda)v_2, w) \le \lambda\varphi(v_1, w) + (1-\lambda)\varphi(v_2, w)$$

for all $\lambda \in [0,1]$ and v_1, $v_2 \in V$, $w \in W$, and φ' is weakly continuous. Assume that

$$\varphi(0, w) \to -\infty \quad \text{as } \|w\| \to \infty,$$

and for every $M > 0$,

$$\varphi(v, w) \to +\infty \quad \text{as } \|v\| \to \infty \quad \text{uniformly for } \|w\| \le M.$$

Then φ has at least one critical point.

Proposition 2.2 ([13, Lemma 5.1]). *Assume that H is a real Hilbert space, $f \colon H \times H \to R$ is a bilinear functional. Then $g \colon H \to R$ given by*

$$g(u) = f(u, u), \qquad \forall u \in H$$

is convex if and only if

$$g(u) \ge 0, \qquad \forall u \in H.$$

For $m > 0$, set

$$H_m = \left\{ \sum_{j=0}^{m} (a_j \cos j\omega t + b_j \sin j\omega t) : a_j, b_j \in R^N,\ j = 0, \dots, m \right\},$$

and denote the orthogonal complement of H_m in H_T^1 by $H_m^\perp$. Applying Proposition 2.2, we obtain the following result.

Lemma 2.3. *Assume that $F(t, x)$ is convex in x for a.e. $t \in [0, T]$. Then, for every $w \in H_m^\perp$, $\varphi(v + w)$ is convex in $v \in H_m$.*

Proof. The convexity of $F(t, \cdot)$ implies that $F(t, v + w)$ is convex in $v \in H_m$ for every $w \in H_m^\perp$, and hence $\int_0^T F(t, v + w)dt$ is convex in $v \in H_m$ for every $w \in H_m^\perp$. Notice that

$$-\frac{1}{2}\int_0^T |\dot{v}(t)|^2\,dt + \frac{m^2\omega^2}{2}\int_0^T |v(t)|^2\,dt \ge 0, \qquad \forall v \in H_m.$$

Lemma 2.2 implies that

$$-\frac{1}{2}\int_0^T |\dot{v}(t)|^2\,dt + \frac{m^2\omega^2}{2}\int_0^T |v(t)|^2\,dt$$

is convex in $v \in H_m$. Hence, for each $w \in H_m^\perp$,

$$\begin{aligned}
\varphi(v + w) &= -\frac{1}{2}\int_0^T |\dot{v}(t) + \dot{w}(t)|^2\,dt + \frac{m^2\omega^2}{2}\int_0^T |v(t) + w(t)|^2\,dt + \int_0^T F(t, v(t) + w(t))\,dt \\
&= \left(-\frac{1}{2}\int_0^T |\dot{v}(t)|^2\,dt + \frac{m^2\omega^2}{2}\int_0^T |v(t)|^2\,dt\right) + \int_0^T F(t, v(t) + w(t))\,dt \\
&\quad - \frac{1}{2}\int_0^T |\dot{w}(t)|^2 dt + \frac{m^2\omega^2}{2}\int_0^T |w(t)|^2\,dt
\end{aligned}$$

is convex in $v \in H_m$. This completes the proof. □

Lemma 2.4. *Suppose that assumptions (A) and (A_3) hold and $F(t,x)$ is convex in x for a.e. $t \in [0,T]$. Then for every $M > 0$,*

$$\varphi(v+w) \to +\infty \quad \text{as } \|v\| \to \infty,\ v \in H_m,$$

uniformly for $w \in H_m^\perp$ with $\|w\| \le M$.

Proof. We prove this assertion by contradiction. Suppose that the statement of the theorem does not hold, then there exist $M > 0$, $c_1 > 0$ and two sequences $(v_n) \subset H_m$ and $(w_n) \subset H_m^\perp$ with $\|v_n\| \to \infty$ $(n \to \infty)$ and $\|w_n\| \le M$ for all n such that

$$\varphi(v_n + w_n) \le c_1, \qquad \forall n \in N.$$

For $v \in H_m$, write

$$v = u + a\cos m\omega t + b \sin m\omega t,$$

where $a, b \in R^N$ and

$$u \in H_{m-1} \triangleq \left\{ \sum_{j=0}^{m-1} \left(a_j \cos j\omega t + b_j \sin j\omega t \right) \mid a_j, b_j \in R^N,\ j = 0, 1, \ldots, m-1 \right\}.$$

Define the function $\bar{F}\colon R^{2N} \to R$ by

$$\bar{F}(a,b) = \int_0^T F(t, a\cos m\omega t + b\sin m\omega t)\, dt.$$

It follows from the continuous differentiability and the convexity of $F(t,\cdot)$ that $\bar{F}$ is continuously differentiable and convex on R^{2N}, which yields that $\bar{F}$ is weakly lower semi-continuous on R^{2N}. Using (A_3), one has

$$\bar{F}(a,b) = \int_0^T F(t, a\cos m\omega t + b\sin m\omega t)\, dt \to +\infty \quad \text{as } |a| + |b| \to \infty.$$

Hence, by the least action principle [5, Theorem 1.1], $\bar{F}$ has a minimum at some $(a_0, b_0) \in R^{2N}$ for which

$$\begin{aligned} &\int_0^T (\nabla F(t, a_0 \cos m\omega t + b_0 \sin m\omega t), \cos m\omega t)\, dt \\ &= \int_0^T (\nabla F(t, a_0 \cos m\omega t + b_0 \sin m\omega t), \sin m\omega t)\, dt \\ &= 0. \end{aligned}$$

By the convexity of $F(t,\cdot)$, we obtain

$$\begin{aligned} F(t, v+w) \ge{}& F(t, a_0 \cos m\omega t + b_0 \sin m\omega t) \\ &+ (\nabla F(t, a_0 \cos m\omega t + b_0 \sin m\omega t), u + w + (a - a_0)\cos m\omega t + (b - b_0)\sin m\omega t), \end{aligned}$$

and then, using assumption (A), (2.2) and (2.1),

$$\begin{aligned} \int_0^T F(t, v+w)\, dt \ge{}& \int_0^T F(t, a_0 \cos m\omega t + b_0 \sin m\omega t)\, dt \\ &+ \int_0^T (\nabla F(t, a_0 \cos m\omega t + b_0 \sin m\omega t), u + w)\, dt \\ \ge{}& -\max_{s \in [0, |a_0| + |b_0|]} a(s) \int_0^T b(t)\, dt - \max_{s \in [0, |a_0| + |b_0|]} a(s) \int_0^T b(t) |u + w|\, dt \\ \ge{}& -\max_{s \in [0, |a_0| + |b_0|]} a(s) \int_0^T b(t) dt (1 + \|u\|_\infty + \|w\|_\infty) \\ \ge{}& -c_2 (1 + \|u\|_\infty) \end{aligned}$$

for all $w \in H_m^{\perp}$ with $\|w\| \leq M$, where $c_2 = \max_{s\in[0,|a_0|+|b_0|]} a(s) \int_0^T b(t)dt(1+CM)$. Rewrite $v_n = u_n + a_n \cos m\omega t + b_n \sin m\omega t$, where a_n, $b_n \in R^N$ and $u_n \in H_{m-1}$. Then one has

$$\begin{aligned} c_1 &\geq \varphi(v_n + w_n) \\ &= -\frac{1}{2}\int_0^T |\dot{u}_n|^2\,dt + \frac{m^2\omega^2}{2}\int_0^T |u_n|^2\,dt - \frac{1}{2}\int_0^T |\dot{w}_n|^2\,dt \\ &\quad + \frac{m^2\omega^2}{2}\int_0^T |w_n|^2\,dt + \int_0^T F(t, v_n + w_n)\,dt \\ &\geq \frac{1}{2}(m^2 - (m-1)^2)\omega^2 \int_0^T |u_n|^2\,dt - \frac{M^2}{2} - c_2(1 + \|u_n\|_\infty) \end{aligned}$$

for all n, which implies that (u_n) is bounded by the equivalence of the norms on the finite-dimensional space H_{m-1}. Combining this with assumption (A), the convexity of $F(t,\cdot)$ and (2.1), we obtain

$$\begin{aligned} c_1 &\geq \varphi(v_n + w_n) \\ &\geq -c_3 + \int_0^T F(t, v_n + w_n)\,dt \\ &\geq -c_3 + 2\int_0^T F\left(t, \frac{1}{2}(a_n \cos m\omega t + b_n \sin m\omega t)\right)dt - \int_0^T F(t, -u_n - w_n)\,dt \\ &\geq -c_3 + 2\int_0^T F\left(t, \frac{1}{2}(a_n \cos m\omega t + b_n \sin m\omega t)\right)dt \\ &\quad - \max_{s\in[0, C\|u_n + w_n\|]} a(s)\int_0^T b(t)\,dt, \end{aligned}$$

which yields that the sequences (a_n) and (b_n) are also bounded. This contradicts the fact that $\|v_n\| \to \infty$ as $n \to \infty$. Therefore the conclusion holds. □

Now we are in the position to prove our theorems.

Proof of Theorem 1.1. According to Proposition 2.1, it remains to show that

$$\varphi(w) \to -\infty \quad \text{as } \|w\| \to \infty,\ w \in H_m^{\perp}. \tag{2.2}$$

We follow an argument in [13]. Arguing indirectly, assume that there exists a sequence $(u_n) \subset H_m^{\perp}$ satisfying $\|u_n\| \to \infty$ and

$$\varphi(u_n) \geq c_4, \qquad \forall n \in N \tag{2.3}$$

for some $c_4 \in R$. Write $u_n = a_n\|u_n\| \cos(m+1)\omega t + b_n\|u_n\| \sin(m+1)\omega t + w_n$, where a_n, $b_n \in R^N$ and $w_n \in H_{m+1}^{\perp}$. Then we have, using (1.3),

$$\begin{aligned} c_4 &\leq \varphi(u_n) \\ &\leq -\frac{1}{2}\int_0^T |\dot{u}_n|^2\,dt + \frac{m^2\omega^2}{2}\int_0^T |u_n|^2\,dt + \frac{(2m+1)}{2}\omega^2\int_0^T |u_n|^2\,dt + \int_0^T \gamma(t)\,dt \\ &= -\frac{1}{2}\int_0^T |\dot{w}_n|^2\,dt + \frac{m^2\omega^2}{2}\int_0^T |w_n|^2\,dt + \frac{(2m+1)}{2}\omega^2\int_0^T |w_n|^2\,dt + \int_0^T \gamma(t)\,dt \\ &\leq -\frac{1}{2}\left(1 - \frac{m^2}{(m+2)^2} - \frac{(2m+1)}{(m+2)^2}\right)\int_0^T |\dot{w}_n|^2 dt + \int_0^T \gamma(t)\,dt \\ &= -\frac{2m+3}{2(m+2)^2}\int_0^T |\dot{w}_n|^2\,dt + \int_0^T \gamma(t)\,dt, \end{aligned}$$

which implies that (w_n) is bounded. Taking $v_n = u_n/\|u_n\|$, then $\|v_n\| = 1$, and hence the sequences $\{a_n\}$, $\{b_n\}$ are bounded. Up to a subsequence, we can assume that

$$a_n \to a \quad \text{and} \quad b_n \to b \quad \text{as } n \to \infty$$

for some $a, b \in R^N$. By the boundedness of (w_n), one has $w_n/\|u_n\| \to 0$ as $n \to \infty$. Hence,

$$v_n \to a\cos(m+1)\omega t + b\sin(m+1)\omega t \quad \text{in } H_T^1,$$

and $|a| + |b| \neq 0$, which yields that $v_n(t) \to a\cos(m+1)\omega t + b\sin(m+1)\omega t$ uniformly for a.e. $t \in [0, T]$ by (2.1). Hence $|u_n(t)| \to \infty$ as $n \to \infty$ for a.e. $t \in [0, T]$, because $a\cos(m+1)\omega t + b\sin(m+1)\omega t$ only has finite zeros.

Now set

$$E = \left\{ t \in [0, T] \;\middle|\; F(t, x) - \frac{(2m+1)}{2}\omega^2 |x|^2 \to -\infty \quad \text{as } |x| \to \infty \right\}.$$

It follows from Fatou's lemma (see [20]) that

$$\begin{aligned}
\limsup_{n\to\infty} \varphi(u_n) &\leq \limsup_{n\to\infty} \int_0^T \left[\left(-\frac{(m+1)^2\omega^2}{2} + \frac{m^2\omega^2}{2} \right) |u_n|^2 + F(t, u_n) \right] dt \\
&= \limsup_{n\to\infty} \int_0^T \left(F(t, u_n) - \frac{(2m+1)\omega^2}{2} |u_n|^2 \right) dt \\
&\leq \limsup_{n\to\infty} \int_E \left(F(t, u_n) - \frac{(2m+1)\omega^2}{2} |u_n|^2 \right) dt + \int_0^T \gamma(t)\, dt \\
&= -\infty,
\end{aligned}$$

a contradiction with (2.3).

A combination of (2.2), Lemmas 2.3, 2.4 and Proposition 2.1 shows that φ has at least a critical point. Consequently, problem (1.1) possesses at least one solution in H_T^1 and the proof is completed. □

Proof of Theorem 1.7. First, we claim that there exists a constant $a_0 < \frac{2m+1}{(m+1)^2}$ such that

$$\int_0^T \alpha(t)|u|^2\, dt \leq a_0 \int_0^T |\dot{u}|^2\, dt, \qquad \forall u \in H_m^\perp. \tag{2.4}$$

The proof is similar to the first part of [13, Proof of Theorem 3.2], for the convenience of the readers we sketch it here briefly. Arguing indirectly, we assume that there exists a sequence $(u_n) \subset H_m^\perp$ such that

$$\int_0^T \alpha(t)|u_n|^2\, dt > \left(\frac{2m+1}{(m+1)^2} - \frac{1}{n} \right) \int_0^T |\dot{u}_n|^2\, dt, \qquad \forall n \in N, \tag{2.5}$$

which implies that $u_n \neq 0$ for all n. By the homogeneity of the above inequality, we may assume that $\int_0^T |\dot{u}_n|^2 dt = 1$ and

$$\int_0^T \alpha(t)|u_n|^2\, dt > \frac{2m+1}{(m+1)^2} - \frac{1}{n}, \qquad \forall n \in N. \tag{2.6}$$

It follows from the weak compactness of the unit ball of $H_m^\perp$ that there exists a subsequence, still denoted by (u_n), such that $u_n \rightharpoonup u$ in $H_m^\perp$, $u_n \to u$ in $C(0, T; R^N)$. This, jointly with (2.6), shows that

$$\int_0^T \alpha(t)|u|^2\, dt \geq \frac{2m+1}{(m+1)^2}.$$

Hence

$$\frac{2m+1}{(m+1)^2} \geq \frac{2m+1}{(m+1)^2}\int_0^T |\dot{u}|^2\,dt \geq (2m+1)\omega^2 \int_0^T |u|^2\,dt \geq \int_0^T \alpha(t)|u|^2\,dt \geq \frac{2m+1}{(m+1)^2},$$

and then

$$1 = \int_0^T |\dot{u}|^2\,dt = (m+1)^2\omega^2 \int_0^T |u|^2\,dt$$

and

$$\int_0^T \left((2m+1)\omega^2 - \alpha(t)\right)|u|^2\,dt = 0,$$

which implies that $u = a\cos(m+1)\omega t + b\sin(m+1)\omega t$, a, $b \in R^N$, $u \neq 0$ and $u = 0$ on a positive measure subset. This contradicts the fact that $u = a\cos(m+1)\omega t + b\sin(m+1)\omega t$ only has finite zeros if $u \neq 0$.

It follows from assumptions (A) and (A_7) that, for $\varepsilon \in \left(0, \frac{2m+1}{(m+1)^2} - a_0\right)$, there exists $M_\varepsilon > 0$ such that

$$F(t,x) \leq \frac{1}{2}\left(\alpha(t) + \varepsilon(m+1)^2\omega^2\right)|x|^2 + \max_{s\in[0,M_\varepsilon]} a(s)b(t)$$

for all $x \in R^N$ and a.e. $t \in [0,T]$. Combining this with (2.4), we obtain

$$\begin{aligned}
\varphi(w) &\leq -\frac{1}{2}\int_0^T |\dot{w}|^2 dt + \frac{m^2 w^2}{2}\int_0^T |w|^2\,dt + \frac{1}{2}\int_0^T (\alpha(t) + \varepsilon(m+1)^2\omega^2) w^2\,dt + c_5 \\
&\leq -\frac{1}{2}\left(1 - \frac{m^2}{(m+1)^2} - a_0 - \varepsilon\right)\int_0^T |\dot{w}|^2\,dt + c_5 \\
&\leq -\frac{1}{2}\left(\frac{2m+1}{(m+1)^2} - a_0 - \varepsilon\right)\int_0^T |\dot{w}|^2\,dt + c_5
\end{aligned}$$

for $w \in H_m^\perp$, where $c_5 = \max_{s\in[0,M_\varepsilon]} a(s)\int_0^T b(t)\,dt$, which implies that

$$\varphi(w) \to -\infty \quad \text{as } \|w\| \to \infty \quad \text{on } H_m^\perp,$$

by the equivalence of the L^2-norm of $\dot{w}$ and the H_T^1-norm on $H_m^\perp$. This, jointly with Lemmas 2.3, 2.4 and Proposition 2.1, yields that φ possesses at least one critical point, and hence problem (1.1) has at least one solution in H_T^1. This concludes the proof. □

Proof of Theorem 1.10. By (A_8) and Sobolev's inequality, we have

$$\begin{aligned}
\varphi(w) &\leq -\frac{1}{2}\left(1 - \frac{m^2}{(m+1)^2}\right)\int_0^T |\dot{w}|^2\,dt + \frac{1}{2}\int_0^T \alpha(t)|w|^2\,dt + \int_0^T \gamma(t)\,dt \\
&\leq -\frac{2m+1}{2(m+1)^2}\int_0^T |\dot{w}|^2\,dt + \frac{1}{2}\int_0^T \alpha(t)\,dt \cdot \|w\|_\infty^2 + \int_0^T \gamma(t)\,dt \\
&\leq -\frac{2m+1}{2(m+1)^2}\int_0^T |\dot{w}|^2\,dt + \frac{1}{2}\int_0^T \alpha(t)\,dt \cdot \frac{T}{12}\int_0^T |\dot{w}|^2\,dt + \int_0^T \gamma(t)\,dt \\
&\leq -\frac{1}{2}\left(\frac{2m+1}{(m+1)^2} - \frac{T}{12}\int_0^T \alpha(t)dt\right)\int_0^T |\dot{w}|^2\,dt + \int_0^T \gamma(t)\,dt
\end{aligned}$$

for all $w \in H_m^\perp$. Noting $\int_0^T \alpha(t)\,dt < \frac{12(2m+1)}{T(m+1)^2}$, the last inequality implies that

$$\varphi(w) \to -\infty \quad \text{as } \|w\| \to \infty,\ w \in H_m^\perp.$$

Consequently, Theorem 1.10 follows from Lemmas 2.3, 2.4 and Proposition 2.1. This completes the proof. □

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