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On the uniqueness of limit cycles in discontinuous Liénard-type systems

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Abstract. In this paper, we investigate the uniqueness and stability of limit cycles for a nonlinear Liénard-type differential system with a discontinuity line. By employing a transformation technique and considering the characteristic exponent of the periodic orbit, we give several criteria for the discontinuous planar nonlinear Liénard-type system. An example with different nonlinear functions $H(y)$ is presented to illustrate the obtained results.

Keywords: discontinuity, Liénard-type system, limit cycle, uniqueness.

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1 Introduction

As is well known, the Liénard system is widely used to describe the dynamics appearing in various models (mathematical, physical and mechanical engineering models etc.). Many nonlinear systems can be transformed into the Liénard form by suitable changes [6, 10]. So investigation for the Liénard system is significant from both application and theoretical point of view. Up to now, there have been many achievements on the existence, uniqueness and the number of limit cycles for continuous or even smooth differential system, especially for the Liénard system, see for example [1, 3, 6, 13, 14, 17] and references therein.

In addition, much progress has been made in studying the existence and uniqueness of limit cycles for discontinuous planar differential system, see for example [2, 4, 5, 7–9, 11, 12, 15, 16] and references therein. However, most of the existing papers focus on the investigation for the discontinuous planar piecewise linear differential system [5, 7–9, 15]. For the discontinuous planar nonlinear Liénard system, there are only a few papers. In [11], the authors studied the nonexistence and uniqueness of limit cycles for a discontinuous nonlinear Liénard system. In [12], the number of limit cycles for a discontinuous planar generalized Liénard polynomial differential equation was studied. In [16], the authors studied the number of limit cycles bifurcating from the origin for a class of discontinuous planar Liénard systems. However, on the discontinuous planar nonlinear Liénard-type system, the relevant problems are more complicated which are not easy to be handled due to the nonlinearity of function $H(y)$. To

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the best of our knowledge, there has been no result on the nonlinear Liénard-type system allowing discontinuities.

In this paper, we investigate the uniqueness and stability of limit cycles for a nonlinear Liénard-type differential system with a discontinuity line. We first give some geometrical properties for the discontinuous system. Then by taking a change of variable and considering the characteristic exponent of the periodic orbit, we obtain that the discontinuous planar nonlinear Liénard-type system has at most one stable limit cycle.

The paper is organized as follows. In the next section, we present some preliminaries and geometrical properties for the discontinuous system. In Section 3, we first give several relevant lemmas, then under different hypotheses of the function $H(y)$ we provide several criteria on the uniqueness and stability of limit cycles for the discontinuous planar nonlinear Liénard-type system. In Section 4, an example with different nonlinear functions $H(y)$ is presented to illustrate the obtained results. Conclusion is outlined in Section 5.

2 Preliminaries

Consider the following Liénard-type differential system with a discontinuity line

$$\begin{aligned} \frac{dx}{dt} &= F(x) - H(y), \\ \frac{dy}{dt} &= g(x), \end{aligned} \tag{2.1}$$

where $x \in [a,b]$ with $a \in (-\infty,0), b \in (0,\infty)$, $F(x) = \int_0^x f(s)\,ds$ with $F(0) = 0$, $H(y) \in \mathcal{C}(\mathbb{R},\mathbb{R})$, $yH(y) > 0$ for $y \neq 0$ and $H(+\infty) = +\infty$, and functions $f(x)$, $g(x)$ are given by

$$f(x) = \begin{cases} f_1(x) & \text{for } x < 0, \\ f_2(x) & \text{for } x > 0, \end{cases} \qquad g(x) = \begin{cases} g_1(x) & \text{for } x < 0, \\ 0 & \text{for } x = 0, \\ g_2(x) & \text{for } x > 0, \end{cases} \tag{2.2}$$

satisfying $f_1, g_1 \in \mathcal{C}^1([a,0],\mathbb{R})$ and $f_2, g_2 \in \mathcal{C}^1([0,b],\mathbb{R})$ with $g_1(0) \neq g_2(0)$. The discontinuity line is denoted by Σ_0 of the form

$$\Sigma_0 = \{(x,y) : x = 0,\ -\infty < y < \infty\},$$

then the normal vector to the discontinuity line Σ_0 is $\mathbf{n} = (1,0)^T$.

For system (2.1) with (2.2), the corresponding vector field is as follows

$$V(x,y) = \begin{cases} V_1(x,y) & \text{for } \ (x,y) \in \{(x,y) \in \mathbb{R}^2 : x \in [a,0]\}, \\ V_2(x,y) & \text{for } \ (x,y) \in \{(x,y) \in \mathbb{R}^2 : x \in [0,b]\}, \end{cases} \tag{2.3}$$

where $V_i(x,y) = (F_i(x) - H(y),\ g_i(x))^T$ and $F_i(x) = \int_0^x f_i(s)\,ds$ for $i = 1,2$.

In this paper, we assume that the following hypotheses hold for system (2.1) with (2.2).

$(H1)$ $xg(x) > 0$ for $x \neq 0$.

$(H2)$ $xf(x) > 0$ for $x \neq 0$ and $F(x) = \int_0^x f(s)\,ds$ with $F(\pm\infty) = +\infty$.

$(H3)$ $H'(y) > 0$ for $y > 0$ and $\frac{H(y)}{y}$ is decreasing for $y < 0$.

Obviously, the origin $O(0,0)$ is a unique equilibrium point of (2.1). From $(H2)$ we obtain that the isocline curve $H(y) = F(x)$ is passing through the origin and $F(x) \geq 0$ for $x \in \mathbb{R}$. By $(H3)$ and the inverse function theorem, the derivative $\frac{dH^{-1}(F(x))}{dx} = \frac{f(x)}{H'(y)}$ has the same sign as x for $y > 0$, so the isocline curve $H(y) = F(x)$ passing through the origin is increasing for $x > 0$ and decreasing for $x < 0$ on the (x,y) plane.

Moreover, since $F_1(0) = F_2(0) = 0$ it follows that the horizontal component of the vector field (2.3) is continuous. By Filippov's first order theory [4,5] then the origin O is a unique sliding point on Σ_0 (for any $(0,y) \in \Sigma_0$, if $[\mathbf{n} \cdot V_1(0,y)][\mathbf{n} \cdot V_2(0,y)] \leq 0$ then we speak of the point $(0,y)$ as a sliding point). Therefore, there exists no sliding limit cycle (isolated periodic orbit which has some points in the sliding set (a set of sliding points)) for system (2.1), and then we focus our attention on the crossing limit cycle (isolated periodic orbit which does not share points with the sliding set).

Lemma 2.1. *Let $(H1)$–$(H3)$ hold and suppose that the system* (2.1) *has a periodic orbit. Then for $x > 0$ $(x < 0)$, the periodic orbit intersects the isocline curve $F(x) = H(y)$ only once.*

Proof. It is obvious that the origin O is a unique equilibrium point of (2.1). Since $x' = -H(y)$ for $x = 0$ and $yH(y) > 0$ for $y \neq 0$, it follows that the periodic orbit goes around the origin counterclockwise.

Let Γ be the periodic orbit surrounding the origin, $A(x_A, y_A)$ and $B(x_B, y_B)$ are two points on Γ such that the x-exponent x_A and x_B are the minimum and maximum values, then we have that $x_A < 0 < x_B$. when $x \neq 0$ one has that

$$\frac{dx}{dy} = \frac{F(x) - H(y)}{g(x)}. \tag{2.4}$$

By the vector field of system (2.1), the derivative (2.4) vanishes at the points $A(x_A, y_A)$ and $B(x_B, y_B)$, i.e., $F(x_A) = H(y_A)$, $F(x_B) = H(y_B)$. Moreover, along the curve $H(y) = F(x)$ it follows from $(H1)$ and $(H3)$ that the second derivative $\frac{d^2x}{dy^2} = -\frac{H'(y)}{g(x)}$ has opposite sign as x for $x \neq 0$. So (2.4) vanishes only once for $x > 0$ $(x < 0)$. Correspondingly, the periodic orbit Γ intersects the curve $F(x) = H(y)$ only once for $x > 0$ $(x < 0)$. □

Now we consider a change of variable as follows

$$P = F(x). \tag{2.5}$$

By $(H2)$ then $P(x) \geq 0$ for $x \in \mathbb{R}$ and $P'(x) > 0$ (< 0) for $x > 0$ (< 0). So there exist inverse functions $x_2(P)$ for $x \geq 0$ and $x_1(P)$ for $x \leq 0$ as follows

$$\begin{aligned} x_1 &: [0, F(a)] \to [a, 0] \quad \text{with } F(x_1(P)) = P, \\ x_2 &: [0, F(b)] \to [0, b] \quad \text{with } F(x_2(P)) = P. \end{aligned}$$

Moreover, for $x \neq 0$ it follows from (2.5) that the system (2.1) is transformed into the following differential systems

$$\frac{dy(x_i(P))}{dP} = \frac{g(x_i(P))}{f(x_i(P))[P - H(y)]} \quad \text{for } \; P > 0, \; i = 1, 2. \tag{2.6}$$

For simplicity, denote by $e_i(P) = \frac{g(x_i(P))}{f(x_i(P))}$ then the systems (2.6) can be written as

$$\frac{dy(x_i(P))}{dP} = \frac{e_i(P)}{P - H(y)} \quad \text{for } \quad P > 0, \tag{2.7}$$

satisfying $e_i(P) > 0$ for $P > 0,\ i = 1, 2$. By the inverse function theorem then the isocline curve $P = H(y)$ is increasing on the positive half (P, y) plane with $P > 0$.

$(H4)$ Assume that there exist two limits

$$\lim_{x\to 0^-} \frac{g(x)}{f(x)} = \lim_{P\to 0^+} e_1(P) = l_1, \qquad \lim_{x\to 0^+} \frac{g(x)}{f(x)} = \lim_{P\to 0^+} e_2(P) = l_2,$$

satisfying $0 \le l_2 \le l_1 < \infty$, and $l_2 \le l_1$ implies that $e_2(P) < e_1(P)$ for $0 < P$ sufficiently small.

Note that the systems (2.7) can be continuously extended to $P = 0$ if we let $e_i(0) = l_i,\ i = 1, 2$. In this case, the hypothesis $(H4)$ becomes $0 \le e_2(0) \le e_1(0)$ and $e_2(P) < e_1(P)$ for $0 < P$ sufficiently small.

3 Main results

Lemma 3.1. *Let $(H3)$ hold and consider the following differential systems*

$$\begin{aligned} \frac{dP}{dt} &= P - H(y), \\ \frac{dy}{dt} &= h_i(P), \end{aligned} \tag{3.1}$$

and let the functions $y_i : [c, d] \to \mathbb{R},\ i = 1, 2$ denote two solutions of systems (3.1). Assume that

$$0 < h_1(P) < h_2(P) \quad \text{for} \ \ P \in (c, d). \tag{3.2}$$

For $P - H(y) > 0$ one has that

(i) when $y_1(c) \le y_2(c)$ then $y_1(P) < y_2(P)$ for $P \in (c, d]$;

(ii) when $y_1(d) \ge y_2(d)$ then $y_1(P) > y_2(P)$ for $P \in [c, d)$.

For $P - H(y) < 0$ one has that

(i) when $y_1(c) \ge y_2(c)$ then $y_1(P) > y_2(P)$ for $P \in (c, d]$;

(ii) when $y_1(d) \le y_2(d)$ then $y_1(P) < y_2(P)$ for $P \in [c, d)$.

Proof. Let $y_1(P)$ and $y_2(P)$ for $P \in [c, d]$ be the solutions of systems (3.1). Since $y_1(c) \le y_2(c)$, by the properties of autonomous systems one has that $y_1(P) \le y_2(P)$ for all $P \in [c, d]$. We first show that

$$H(y_1(P)) \le H(y_2(P)) \quad \text{for} \ \ P \in [c, d]. \tag{3.3}$$

There are three possible cases as follows.

- If $0 < y_1(P) \le y_2(P)$, by $H'(y) > 0$ for $y > 0$ one has that $0 < H(y_1(P)) \le H(y_2(P))$.
- If $y_1(P) \le 0 \le y_2(P)$, it is obvious that $H(y_1(P)) \le 0 \le H(y_2(P))$ due to $H(0) = 0$ and $yH(y) > 0$ for $y \ne 0$.
- If $y_1(P) \le y_2(P) < 0$, since $\frac{H(y)}{y}$ is decreasing for $y < 0$ it follows that $\frac{H(y_1(P))}{y_1(P)} \ge \frac{H(y_2(P))}{y_2(P)}$, which together with $0 < \frac{y_2(P)}{y_1(P)} \le 1$ imply that $H(y_1(P)) \le H(y_2(P)) < 0$.

So (3.3) holds. Moreover, it follows from (3.2) that for $P - H(y) > 0$,

$$\frac{dy_1(P)}{dP} = \frac{h_1(P)}{P - H(y_1(P))} \leq \frac{h_1(P)}{P - H(y_2(P))} < \frac{h_2(P)}{P - H(y_2(P))} = \frac{dy_2(P)}{dP}.$$

This implies that the difference function $y_2(P) - y_1(P)$ is strictly increasing for $P \in [c, d]$. So $y_2(P) > y_1(P)$ for $P \in (c, d]$ and then the statement (i) holds.

For the purpose of the statement (ii), we suppose on the contrary that there exists $\widetilde{P} \in [c, d)$ such that $y_1(\widetilde{P}) \leq y_2(\widetilde{P})$. By (i) then $y_1(P) < y_2(P)$ for $P \in (\widetilde{P}, d]$, particularly $y_1(d) < y_2(d)$. It is a contradiction and so the statement (ii) holds.

By a similar analysis for the case $P - H(y) < 0$, the statements hold. □

Lemma 3.2. *Assume that* $(H1)$*–*$(H3)$ *hold for system* (2.1) *and let* Γ *be a periodic orbit surrounding the origin. Then one has that*

$$\iint_\Delta \operatorname{div} V \, dx \, dy = 0,$$

where Δ *denotes a region surrounded by* Γ *and* $\operatorname{div} V$ *denotes the divergence of the vector field* V*. Moreover, if* $\oint_\Gamma \operatorname{div} V dt < 0$ (> 0) *then* Γ *is a stable (an unstable) limit cycle.*

Proof. Let Δ_-, Γ^- and Δ_+, Γ^+ be parts of Δ and Γ contained in $x < 0$ and $x > 0$ respectively. $M(0, y_M)$ and $N(0, y_N)$ denote two intersections between Γ with the discontinuity line Σ_0 satisfying $y_M < 0 < y_N$. Since $f_1, g_1 \in \mathcal{C}^1([a, 0], \mathbb{R})$ and $f_2, g_2 \in \mathcal{C}^1([0, b], \mathbb{R})$, it follows that

$$\iint_\Delta \operatorname{div} V \, dx \, dy = \iint_{\Delta_-} f_1(x) \, dx \, dy + \iint_{\Delta_+} f_2(x) \, dx \, dy.$$

By Green's formula, we obtain that

$$\iint_{\Delta_-} f_1(x) \, dx \, dy = \int_{\Gamma^- \cup \overline{MN}} -g(x) \, dx + (F(x) - H(y)) \, dy = \int_N^M H(y) \, dy,$$

$$\iint_{\Delta_+} f_2(x) \, dx \, dy = \int_{\Gamma^+ \cup \overline{NM}} -g(x) \, dx + (F(x) - H(y)) \, dy = \int_M^N H(y) \, dy,$$

where $\overline{MN}$ denotes an oriented segment from the point M to N and $\overline{NM}$ is similar. The proof of the stability is similar to the one in [17]. □

Lemma 3.3. *Let* $(H1)$*–*$(H4)$ *hold for system* (2.1)*. Then a necessary condition for the existence of periodic orbits is that the equation* $e_1(P) = e_2(P)$ *has at least one solution* P_0*,* $P_0 \in (0, \min\{F(a), F(b)\})$*.*

Proof. Let Γ be a periodic orbit surrounding the origin O, which is presented by $y = y(x)$ for $x_A < x < x_B$ (see Figure 3.1). $M(0, y_M)$ and $N(0, y_N)$ denote two intersections between Γ with Σ_0 satisfying $y_M < 0$ and $y_N > 0$, and the lower trajectory arc $\widehat{AMB}$ is presented by $y = \widetilde{y}(x)$, the upper trajectory arc $\widehat{BNA}$ is presented by $y = \overline{y}(x)$. Then $\Gamma = \widehat{AMB} \cup \widehat{BNA}$ and $\widetilde{y}(x) < H^{-1}(F(x)) < \overline{y}(x)$ for $x_A < x < x_B$. Moreover, let $\Gamma_1(P)$ be the trajectory arc $\widehat{NAM}$ as follows

$$\Gamma_1(P) = \begin{cases} \widetilde{y}(x_1(P)) & \text{for } \widetilde{y}(x_1) \leq y_A, \\ \overline{y}(x_1(P)) & \text{for } \overline{y}(x_1) \geq y_A, \end{cases}$$

and $\Gamma_2(P)$ be the trajectory arc $\widehat{MBN}$ as follows

$$\Gamma_2(P) = \begin{cases} \widetilde{y}(x_2(P)) & \text{for } \widetilde{y}(x_2) \leq y_B, \\ \overline{y}(x_2(P)) & \text{for } \overline{y}(x_2) \geq y_B, \end{cases}$$

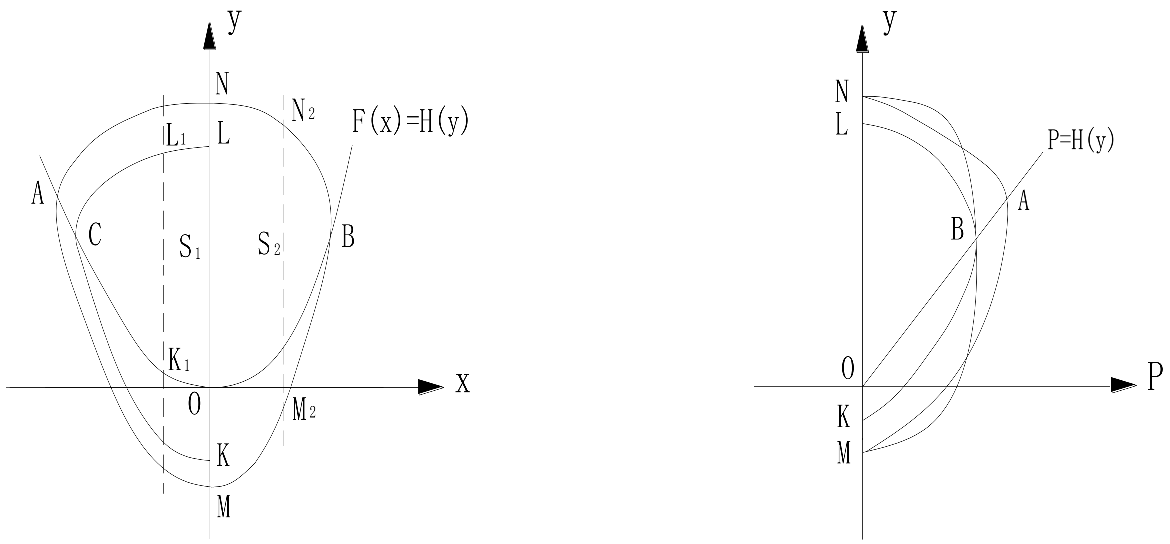

Figure 3.1: Left graph shows the periodic orbit of (2.1) on the (x, y) plane, right graph is the corresponding trajectory arcs on the (P, y) plane with $P > 0$.

satisfying $\widetilde{y}(x_1(0)) = \widetilde{y}(x_2(0))$ and $\overline{y}(x_1(0)) = \overline{y}(x_2(0))$. Then $\Gamma = \Gamma_1(P) \cup \Gamma_2(P)$ for $0 \le P \le \max\{H(y_A), H(y_B)\}$.

We first show that the trajectory arc $\Gamma_1(P)$ intersects with $\Gamma_2(P)$. Consider the following differential systems

$$\begin{cases} \frac{dP}{dt} = P - H(y), \\ \frac{dy}{dt} = e_2(P) \quad \text{for } \ P > 0, \end{cases} \qquad \begin{cases} \frac{dP}{dt} = -P - H(y), \\ \frac{dy}{dt} = -e_1(-P) \quad \text{for } \ P < 0, \end{cases} \tag{3.4}$$

where the right system is a symmetry system of $\frac{dP}{dt} = P - H(y)$, $\frac{dy}{dt} = e_1(P)$ for $P > 0$. It is easy to see that the systems (3.4) have a counterclockwise periodic orbit $\widetilde{\Gamma}$, which is constituted by trajectory arcs $\Gamma_2(P)$ and $\widetilde{\Gamma}_1(P)$, where $\widetilde{\Gamma}_1(P)$ denotes the symmetry trajectory arc of $\Gamma_1(P)$ with respect to the discontinuity line Σ_0.

From (3.4) then $\operatorname{div}V(P, y) = \operatorname{sgn}(P)$, where $V(P, y) = (|P| - H(y), e(P))^T$ satisfying $e(P) = e_2(P)$ for $P > 0$ and $e(P) = -e_1(-P)$ for $P < 0$. So by Lemma 3.2 we have that

$$\iint_{\Delta} \operatorname{sgn}(P)\, dP\, dy = 0, \tag{3.5}$$

where Δ is a region surrounded by the periodic orbit $\widetilde{\Gamma}$. Denote by $\iint_{\Delta} \operatorname{sgn}(P)\, dP\, dy = -\Delta_1 + \Delta_2$, and Δ_1 (Δ_2) is an area of Δ in the left (right) hand side of Σ_0. Then (3.5) implies that the trajectory arc $\Gamma_1(P)$ intersects with $\Gamma_2(P)$ at least once.

Now we claim that there exists at least one solution P_0 with $P_0 \in (0, \min\{F(a), F(b)\})$ such that $e_2(P_0) = e_1(P_0)$. Otherwise, by $(H4)$ then $e_2(P) < e_1(P)$ for all $P \in (0, \min\{F(a), F(b)\}]$. Moreover, since $\widetilde{y}(x_1(0)) = \widetilde{y}(x_2(0))$ and $\overline{y}(x_1(0)) = \overline{y}(x_2(0))$, it follows from Lemma 3.1 that

$$\widetilde{y}(x_2(P)) < \widetilde{y}(x_1(P)), \qquad \overline{y}(x_1(P)) < \overline{y}(x_2(P)) \quad \text{for } \ P \in (0, \min\{F(a), F(b)\}].$$

This implies that $\Gamma_1(P)$ does not intersect with $\Gamma_2(P)$, it is a contradiction. So the equation $e_2(P) = e_1(P)$ has at least one solution P_0 for $P_0 \in (0, \min\{F(a), F(b)\})$. $\square$

Theorem 3.4. *Let $(H1)$–$(H4)$ hold. Assume that the equation $e_1(P) = e_2(P)$ has a unique zero P_0 with $P_0 \in (0, \min\{F(a), F(b)\})$ and positive function $\frac{e_1(P)}{P}$ is decreasing for $P \in (0, F(a))$. Then the system* (2.1) *has at most one periodic orbit, and it is a unique stable limit cycle if it exists.*

Proof. We first show that $H(y_A) > H(y_B)$. By Lemma 3.3 then $\Gamma_1(P)$ intersects with $\Gamma_2(P)$ at least one point. Since P_0 is a unique zero for $e_1(P) = e_2(P)$, it follows from $(H4)$ and Lemma 3.1 that there exists a unique $\delta_1 > P_0$ such that in region $P - H(y) > 0$, $\widetilde{y}(x_1(0)) = \widetilde{y}(x_2(0))$ implies that

$$\begin{cases} \widetilde{y}(x_2(P)) < \widetilde{y}(x_1(P)) & \text{for } 0 < P < \delta_1, \\ \widetilde{y}(x_2(P)) > \widetilde{y}(x_1(P)) & \text{for } \delta_1 < P < \min\{H(y_A), H(y_B)\}, \end{cases} \tag{3.6}$$

where $P = \delta_1$ is the unique intersection between $\widetilde{y}(x_1(P))$ and $\widetilde{y}(x_2(P))$. Similarly, there exists a unique $\delta_2 > P_0$ such that in region $P - H(y) < 0$, $\overline{y}(x_1(0)) = \overline{y}(x_2(0))$ implies

$$\begin{cases} \overline{y}(x_2(P)) > \overline{y}(x_1(P)) & \text{for } 0 < P < \delta_2, \\ \overline{y}(x_2(P)) < \overline{y}(x_1(P)) & \text{for } \delta_2 < P < \min\{H(y_A), H(y_B)\}, \end{cases} \tag{3.7}$$

where $P = \delta_2$ is the unique intersection between $\overline{y}(x_1(P))$ and $\overline{y}(x_2(P))$. Therefore there are only two intersections between $\Gamma_1(P)$ and $\Gamma_2(P)$ for $0 < P < \max\{H(y_A), H(y_B)\}$, which implies that the inequalities (3.6)–(3.7) are satisfied only for $H(y_A) > H(y_B)$.

Choose a point $C(x_C, y_C)$ (see Figure 3.1) satisfying $F(x_C) = H(y_C)$ for $x_A < x_C < 0$ such that $H(y_C) = H(y_B) > 0$. Consider an orbit Γ_0 of system (2.1) passing through the point C, which is presented by $y = y_0(x)$ for $x_C < x < 0$. Let $L(0, y_L)$ and $K(0, y_K)$ be two intersections between Γ_0 with Σ_0 satisfying $y_L > 0$ and $y_K < 0$. By the properties of autonomous systems then $y_M < y_K$ and $y_N > y_L$. Moreover, define the orbit $\Gamma_0(P)$ for $0 \le P \le H(y_C)$ as follows

$$\Gamma_0(P) = \begin{cases} \widetilde{y}_0(x_1(P)) & \text{for } \widetilde{y}_0(x_1) \le y_C, \\ \overline{y}_0(x_1(P)) & \text{for } \overline{y}_0(x_1) \ge y_C, \end{cases}$$

where $\widetilde{y}_0(x_1)$ is the lower trajectory arc $\widehat{CK}$ and $\overline{y}_0(x_1)$ is the upper trajectory arc $\widehat{LC}$.

Since $\widetilde{y}(x_2(0)) < \widetilde{y}_0(x_1(0))$ and $\widetilde{y}(x_2(H(y_B))) = \widetilde{y}_0(x_1(H(y_C)))$, it follows from Lemma 3.1 that $\widetilde{y}(x_2(P)) < \widetilde{y}_0(x_1(P))$ for $0 \le P < H(y_B)$. We consider three possible cases as follows.

- If $0 \le \widetilde{y}(x_2(P)) < \widetilde{y}_0(x_1(P))$, by $H'(y) > 0$ for $y > 0$ then $0 \le H(\widetilde{y}(x_2(P))) < H(\widetilde{y}_0(x_1(P)))$.
- If $\widetilde{y}(x_2(P)) < \widetilde{y}_0(x_1(P)) \le 0$, since $\frac{H(y)}{y}$ is decreasing for $y < 0$ it follows that $\frac{H(\widetilde{y}(x_2(P)))}{\widetilde{y}(x_2(P))} > \frac{H(\widetilde{y}_0(x_1(P)))}{\widetilde{y}_0(x_1(P))}$, furthermore $H(\widetilde{y}(x_2(P))) < H(\widetilde{y}_0(x_1(P))) \le 0$ due to $0 < \frac{\widetilde{y}_0(x_1(P))}{\widetilde{y}(x_2(P))} < 1$.
- If $\widetilde{y}(x_2(P)) < 0 < \widetilde{y}_0(x_1(P))$, then $H(\widetilde{y}(x_2(P))) < 0 < H(\widetilde{y}_0(x_1(P)))$ is obvious due to $yH(y) > 0$ for $y \neq 0$.

So we have that

$$H(\widetilde{y}(x_2(P))) < H(\widetilde{y}_0(x_1(P))) \quad \text{for } 0 \le P < H(y_B). \tag{3.8}$$

Similarly, by $\overline{y}(x_2(0)) > \overline{y}_0(x_1(0)) > 0$ and $\overline{y}(x_2(H(y_B))) = \overline{y}_0(x_1(H(y_C))) > 0$, then it follows from Lemma 3.1 and $H'(y) > 0$ for $y > 0$ that

$$H(\overline{y}(x_2(P))) > H(\overline{y}_0(x_1(P))) \quad \text{for } 0 \le P < H(y_B). \tag{3.9}$$

On the other hand, since the equation $e_1(P) = e_2(P)$ has a unique solution $P_0 \in (0, F(x_B))$, we assume that the system

$$F(x_1) = F(x_2), \qquad \frac{g(x_1)}{f(x_1)} = \frac{g(x_2)}{f(x_2)},$$

has a unique pair of solution $(x_1, x_2) = (s_1, s_2)$ with $x_C < s_1 < 0 < s_2 < x_B$. Moreover, let the periodic orbit Γ be presented by $\{(x(t), y(t))\}$ and the orbit Γ_0 be presented by $\{(x_0(t), y_0(t))\}$. In Figure 3.1, $M_2(s_2, y_{M_2})$ and $N_2(s_2, y_{N_2})$ denote two intersections between Γ with the line $x = s_2$, $K_1(s_1, y_{K_1})$ and $L_1(s_1, y_{L_1})$ denote two intersections between Γ_0 with the line $x = s_1$.

Now for the purpose of the uniqueness, we compute the characteristic exponent ρ of the periodic orbit Γ as follows

$$\rho = \oint_\Gamma \operatorname{div} V\, dt = \oint_\Gamma f(x(t))\, dt,$$

where the integral is counterclockwise. Denote by $\rho = \oint_\Gamma f(x(t))\, dt = I + J$ with

$$I = \int_{\widehat{MBN}} f(x(t))\, dt + \int_{\widehat{LCK}} f(x_0(t))\, dt, \qquad J = \int_{\widehat{NAM}} f(x(t))\, dt - \int_{\widehat{LCK}} f(x_0(t))\, dt.$$

We first compute the integral $I = I_1 + I_2 + I_3 + I_4$, where

$$\begin{aligned} I_1 &= \int_{\widehat{MM_2}} f(x(t))\, dt + \int_{\widehat{K_1K}} f(x_0(t))\, dt = \int_0^{s_2} \frac{f(x)\, dx}{F(x) - H(\widetilde{y}(x))} + \int_{s_1}^0 \frac{f(x)\, dx}{F(x) - H(\widetilde{y}_0(x))} \\ &= \int_0^{P_0} \frac{dP}{P - H(\widetilde{y}(x_2(P)))} + \int_{P_0}^0 \frac{dP}{P - H(\widetilde{y}_0(x_1(P)))} \\ &= \int_0^{P_0} \frac{[H(\widetilde{y}(x_2(P))) - H(\widetilde{y}_0(x_1(P)))]\, dP}{[P - H(\widetilde{y}(x_2(P)))][P - H(\widetilde{y}_0(x_1(P)))]}, \end{aligned}$$

$$\begin{aligned} I_2 &= \int_{\widehat{M_2B}} f(x(t))\, dt + \int_{\widehat{CK_1}} f(x_0(t))\, dt = \int_{s_2}^{x_B} \frac{f(x)\, dx}{F(x) - H(\widetilde{y}(x))} + \int_{x_C}^{s_1} \frac{f(x)\, dx}{F(x) - H(\widetilde{y}_0(x))} \\ &= \lim_{\widetilde{P} \to H(y_B)} \left(\int_{P_0}^{\widetilde{P}} \frac{dP}{P - H(\widetilde{y}(x_2(P)))} - \int_{P_0}^{\widetilde{P}} \frac{dP}{P - H(\widetilde{y}_0(x_1(P)))} \right) \\ &= \lim_{\widetilde{P} \to H(y_B)} \int_{P_0}^{\widetilde{P}} \frac{[H(\widetilde{y}(x_2(P))) - H(\widetilde{y}_0(x_1(P)))]\, dP}{[P - H(\widetilde{y}(x_2(P)))][P - H(\widetilde{y}_0(x_1(P)))]}, \end{aligned}$$

then it follows from (3.8) that $I_1 < 0$ and $I_2 < 0$. Similarly, by (3.9) then

$$I_3 = \int_{\widehat{BN_2}} f(x(t))\, dt + \int_{\widehat{L_1C}} f(x_0(t))\, dt < 0, \qquad I_4 = \int_{\widehat{N_2N}} f(x(t))\, dt + \int_{\widehat{LL_1}} f(x_0(t))\, dt < 0.$$

Furthermore, we have that $I = I_1 + I_2 + I_3 + I_4 < 0$.

For the integral $J = J_1 + J_2$ with $J_1 = \int_{\widehat{NA}} f(x(t))\, dt - \int_{\widehat{LC}} f(x_0(t))\, dt$ and $J_2 = \int_{\widehat{AM}} f(x(t))\, dt - \int_{\widehat{CK}} f(x_0(t))\, dt$, we only consider J_1, J_2 is similar and so omitted. It follows that

$$J_1 = \int_0^{H(y_A)} \frac{dP}{P - H(\overline{y}(x_1(P)))} - \int_0^{H(y_B)} \frac{dP}{P - H(\overline{y}_0(x_1(P)))}.$$

Let

$$H(\overline{Y}(P)) = \frac{H(y_A)}{H(y_B)} H\left(\overline{y}_0 \left(x_1 \left(\frac{H(y_B)}{H(y_A)} P \right) \right) \right), \tag{3.10}$$

for $P \in (0, H(y_A))$, then one has that

$$J_1 = \int_0^{H(y_A)} \frac{dP}{P - H(\overline{y}(x_1(P)))} - \int_0^{H(y_A)} \frac{dP}{P - H(\overline{Y}(P))}$$
$$= \int_0^{H(y_A)} \frac{[H(\overline{y}(x_1(P))) - H(\overline{Y}(P))]\, dP}{[P - H(\overline{y}(x_1(P)))][P - H(\overline{Y}(P))]}.$$

It is easy to see that $H(\overline{Y}(P))$ is a solution of the differential equation $\frac{dH(y)}{dP} = \frac{\widehat{e_1}(P)}{P - H(H(y))}$ with $\widehat{e_1}(P) = \frac{H(y_A)}{H(y_B)} e_1(\frac{H(y_B)}{H(y_A)} P)$. Since $\frac{e_1(P)}{P}$ is decreasing for $P \in (0, F(a))$ and $\frac{H(y_A)}{H(y_B)} > 1$, it follows that $e_1(\frac{H(y_B)}{H(y_A)} P) > \frac{H(y_B)}{H(y_A)} e_1(P)$ and then $\widehat{e_1}(P) > e_1(P)$ for $P \in (0, H(y_A))$. Moreover, for $P = H(y_A)$ in (3.10) one has that

$$H(\overline{Y}(H(y_A))) = \frac{H(y_A)}{H(y_B)} H\left(\overline{y}_0\left(x_1\left(\frac{H(y_B)}{H(y_A)} H(y_A)\right)\right)\right) = H(y_A) = H(\overline{y}(x_1(H(y_A)))).$$

So by Lemma 3.1 then $H(\overline{Y}(P)) > H(\overline{y}(x_1(P)))$ for $P \in (0, H(y_A))$. Furthermore, $J_1 < 0$ and then $\rho = I + J < 0$. This means that the periodic orbit Γ is a unique stable limit cycle, since it is impossible to coexist two consecutive stable periodic orbits.

Therefore, the system (2.1) has at most one periodic orbit, and it is a unique stable limit cycle if it exists. □

If we replace the hypothesis $(H3)$ with the following $(H3)^*$, then we have Theorem 3.5.

$(H3)^*$ $H'(y) > 0$ for $y \neq 0$.

Theorem 3.5. *Let $(H1)$–$(H2)$, $(H3)^*$ and $(H4)$ hold. Assume that the equation $e_1(P) = e_2(P)$ has a unique zero P_0, $P_0 \in (0, \min\{F(a), F(b)\})$ and positive function $\frac{e_1(P)}{P}$ is decreasing for $P \in (0, F(a))$. Then the system (2.1) has at most one periodic orbit, and it is a unique stable limit cycle if it exists.*

Proof. By $(H3)^*$ and $yH(y) > 0$ for $y \neq 0$, it is easily observed that the geometrical properties of system (2.1) and Lemmas 3.1–3.3 are satisfied.

For the uniqueness of limit cycles of system (2.1), the main difference with Theorem 3.4 lies in the inequalities (3.8)–(3.9). These are satisfied due to $(H3)^*$ and so the conclusion holds. □

If the hypothesis $(H3)$ is replaced with the following $(H3)^{**}$, then we have Theorem 3.6.

$(H3)^{**}$ $H'(y) > 0$ for $y \in \mathbb{R}$.

Theorem 3.6. *Let $(H1)$–$(H2)$, $(H3)^{**}$ and $(H4)$ hold. Assume that the equation $e_1(P) = e_2(P)$ has a unique zero P_0, $P_0 \in (0, \min\{F(a), F(b)\})$ and the positive function $\frac{e_1(P)}{P}$ is decreasing for $P \in (0, F(a))$. Then the system (2.1) has at most one periodic orbit, and it is a unique stable limit cycle if it exists.*

Proof. By $(H3)^{**}$ it is obvious that the geometrical properties and Lemmas 3.1–3.3 are satisfied. So with the similar way to Theorem 3.4 the conclusion holds. □

4 Example

Example 4.1. Consider the following discontinuous Liénard-type differential system

$$\begin{cases} x' = -x - H(y), \\ y' = 2x - 1 \quad \text{for } x < 0; \end{cases} \qquad \begin{cases} x' = \frac{1}{2}x - H(y), \\ y' = x \quad \text{for } x \geq 0. \end{cases} \tag{4.1}$$

It is easy to see that the discontinuity line $\Sigma_0 = \{(x, y) : x = 0, -\infty < y < \infty\}$, functions $f(x)$ and $g(x)$ are given by

$$f(x) = \begin{cases} -1, & x < 0, \\ \frac{1}{2}, & x \geq 0; \end{cases} \qquad g(x) = \begin{cases} 2x - 1, & x < 0, \\ x, & x \geq 0. \end{cases}$$

So the hypotheses $(H1)$–$(H2)$ hold.

Case 1. The function $H(y)$ in (4.1) is given by

$$H(y) = \begin{cases} y^2 + y & \text{for } y \geq 0, \\ y & \text{for } -1 \leq y \leq 0, \\ ye^{-(y+1)} & \text{for } y \leq -1. \end{cases}$$

It is easily obtained that $H \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ with $H(0) = 0$, $yH(y) > 0$ for $y \neq 0$ and $H'(y) = 2y + 1 > 0$ for $y > 0$, $\frac{H(y)}{y}$ is decreasing for $y < 0$. So the hypothesis $(H3)$ holds.

On the other hand, by some simple computations then $e_2(P) = 4P$ and $e_1(P) = 1 + 2P$. Furthermore, $l_2 = \lim_{P \to 0^+} e_2(P) = 0$ and $l_1 = \lim_{P \to 0^+} e_1(P) = 1$, which imply that $e_2(P) < e_1(P)$ for $0 < P$ sufficiently small. So the hypothesis $(H4)$ holds. Moreover, the equation $e_2(P) = e_1(P)$ has a unique solution $P_0 = \frac{1}{2}$ satisfying $e_2(P) < e_1(P)$ for $0 < P < \frac{1}{2}$, $e_2(P) > e_1(P)$ for $P > \frac{1}{2}$, and the positive function $\frac{e_1(P)}{P} = \frac{1}{P} + 2$ is decreasing for $P > 0$.

Therefore, by Theorem 3.4 the discontinuous system (4.1) has at most one stable limit cycle.

Case 2. Let the function $H(y)$ in (4.1) be $H(y) = y^{\frac{3}{5}}$.

Then $H(0) = 0$, $yH(y) = y^{\frac{8}{5}} > 0$ for $y \neq 0$ and $H'(y) = \frac{3}{5}y^{-\frac{2}{5}} > 0$ for $y \neq 0$, so the hypothesis $(H3)^*$ holds. Therefore, by Theorem 3.5 the discontinuous system (4.1) has at most one stable limit cycle.

Remark 4.2. From cases 1–2, we note that $(H3)$ does not contain $(H3)^*$ and vice versa. This implies that conditions of Theorem 3.4 do not contain ones in Theorem 3.5 and vice versa.

Remark 4.3. It is easy to see that $(H3)^{**}$ is stronger than $(H3)^*$, that is, conditions of Theorem 3.6 is stronger than the ones in Theorem 3.5. However, when $H(y) = y$ in the system (2.1), Theorem 3.6 in this paper is in accord with Theorem 3 in [11].

5 Conclusion

In this paper, we have investigated the uniqueness and stability of limit cycles for a nonlinear Liénard-type differential system with a discontinuity line. Firstly, we have given some geometrical properties for the discontinuous system. Secondly, by taking a change of variable and verifying the characteristic exponent of the periodic orbit, we have obtained that the discontinuous planar nonlinear Liénard-type system has at most one stable limit cycle. Finally, we have given an example with different nonlinearity functions $H(y)$ to illustrate the obtained results. This implies that the hypothesis $(H3)$ does not contain $(H3)^*$ and vice versa.

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References

[1] C. Attanayake, D. Senaratne, A. Kodippili, Existence of a moving attractor for semi-linear parabolic equations, *Electron. J. Qual. Theory Differ. Equ.* **2013**, No. 68, 1–11. MR3145035

[2] V. Carmona, S. Fernández-García, E. Freire, F. Torres, Melnikov theory for a class of planar hybrid systems, *Phys. D* **248**(2013), 44–54. MR3028948; url

[3] G. Chang, T. Zhang, M. Han, On the number of limit cycles of a class of polynomial systems of Liénard type, *J. Math. Anal. Appl.* **408**(2013), 775–780. MR3085071; url

[4] A. Filippov, *Differential equations with discontinuous righthand sides*, Kluwer Academic Publishers Group, Dordrecht, 1988 MR1028776; url

[5] E. Freire, E. Ponce, F. Torres, Canonical discontinuous planar piecewise linear systems, *SIAM J. Appl. Dyn. Syst.* **11**(2012), 181–211. MR2902614; url

[6] J. Giné, J. Llibre, Weierstrass integrability in Liénard differential systems, *J. Math. Anal. Appl.* **377**(2011), 362–369. MR2754835; url

[7] F. Giannakopoulos, K. Pliete, Planar systems of piecewise linear differential equations with a line of discontinuity, *Nonlinearity* **14**(2001), 1611–1632. MR1867095; url

[8] S. Huan, X. Yang, Existence of limit cycles in general planar piecewise linear systems of saddle-saddle dynamics, *Nonlinear Anal.* **92**(2013), 82–95. MR3091110; url

[9] S. Huan, X. Yang, On the number of limit cycles in general planar piecewise linear systems of node-node types, *J. Math. Anal. Appl.* **411**(2014), 340–353. MR3118489; url

[10] M. Han, H. Zang, T. Zhang, A new proof to Bautin's theorem, *Chaos Solitons Fractals* **31**(2007), 218–223. MR2263281; url

[11] J. Llibre, E. Ponce, F. Torres, On the existence and uniqueness of limit cycles in Liénard differential equations allowing discontinuities, *Nonlinearity* **21**(2008), 2121–2142. MR2430665; url

[12] J. Llibre, A. Mereu, Limit cycles for discontinuous generalized Liénard polynomial differential equations, *Electron. J. Differential Equations* **2013**, No. 195, 1–8. MR3104971

[13] J. Llibre, M. Ordóñez, E. Ponce, On the existence and uniqueness of limit cycles in planar continuous piecewise linear systems without symmetry, *Nonlinear Anal. Real World Appl.* **14**(2013), 2002–2012. MR3043136; url

[14] J. Sun, Y. Zhang, A necessary and sufficient condition for the oscillation of solutions of Liénard type system with multiple singular points, *Appl. Math. Mech. (English Ed.)* **18**(1997), 1205–1210. MR1613599

[15] S. Shui, X. Zhang, J. Li, The qualitative analysis of a class of planar Filippov systems, *Nonlinear Anal.* **73**(2010), 1277–1288. MR2661225; url

[16] Y. Xiong, M. Han, Hopf bifurcation of limit cycles in discontinuous Liénard systems, *Abstr. Appl. Anal.* (2012), Art. ID 690453, 1–27. MR2975271; url

[17] Z. Zhang, T. Ding, W. Huang, Z. Dong, *Qualitative theory of differential equations*, Beijing, Science Press, 2006. MR1175631