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Sturm comparison theorems via Picone-type inequalities for some nonlinear elliptic type equations with damped terms

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Abstract. In this paper, we establish a Picone-type inequality for a class of some nonlinear elliptic type equations with damped terms, and obtain Sturmian comparison theorems using the Picone-type inequality. As an application by using comparison theorem oscillation result and Wirtinger-type inequality are given.

Keywords: Picone-type inequality, elliptic equations, Sobolev space, half-linear equations, oscillation criteria, Wirtinger-type inequality.

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1 Introduction

Since the pioneering work of Sturm [27] in 1836, Sturmian comparison theorems have been derived for differential equations of various types. In order to obtain Sturmian comparison theorems for ordinary differential equations of second order, Picone [25] established an identity, known as the Picone identity. In the latter years, Jaroš and Kusano [15] derived a Picone-type identity for half-linear differential equations of second order. They also developed Sturmian theory for both forced and unforced half-linear and quasilinear equations based on this identity. Since Picone identities play an important role in the study of qualitative theory of differential equations, establishing Picone identities has become a popular research topic. We refer the reader to Kreith [20, 21], Swanson [28, 29] for Picone identities and Sturmian comparison theorems for linear elliptic equations and to Allegretto [3], Allegretto and Huang [4, 5], Bognár and Došlý [9], Dunninger [12], Kusano, Jaroš and Yoshida [22], Yoshida [32, 31, 30] for Picone identities, Sturmian comparison and/or oscillation theorems for half-linear elliptic equations. In particular, we mention the paper [12] by Dunninger which seems to be the first paper dealing with Sturmian comparison theorems for half-linear elliptic equations.

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Recently, Yoshida [35] established Sturmian comparison and oscillation theorems for quasilinear undamped elliptic operators with mixed nonlinearities in the following forms,

$$\ell(u) := \sum_{k=1}^{m} \nabla \cdot \left(a_k(x) |\sqrt{a_k(x)} \nabla u|^{\alpha-1} \nabla u \right) + c(x)|u|^{\alpha-1}u,$$
$$L(v) := \sum_{k=1}^{m} \nabla \cdot \left(A_k(x) |\sqrt{A_k(x)} \nabla v|^{\alpha-1} \nabla v \right) + g(x,v)$$

where $a_k(x), A_k(x)$ are matrices and

$$g(x,v) = C(x)|v|^{\alpha-1}v + \sum_{i=1}^{\ell} D_i(x)|v|^{\beta_i-1}v + \sum_{j=1}^{m} E_j(x)|v|^{\gamma_j-1}v.$$

Most of the work in the literature deals with the Sturmian comparison results for elliptic equations that contain undamped terms. In this paper, we establish Sturmian comparison theorems for a pair of damped elliptic operators p and P defined by

$$p(u) := \nabla \cdot \left(a(x)|\nabla u|^{\alpha-1}\nabla u\right) + (\alpha+1)|\nabla u|^{\alpha-1} b(x) \cdot \nabla u + c(x)|u|^{\alpha-1}u, \tag{1.1}$$
$$P(v) := \nabla \cdot \left(A(x)|\nabla v|^{\alpha-1}\nabla v\right) + (\alpha+1)|\nabla v|^{\alpha-1} B(x) \cdot \nabla v + g(x,v), \tag{1.2}$$

where $|\cdot|$ denotes the Euclidean length, $\alpha > 0$ is a constant, $\nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)^T$, (the superscript T denotes the transpose). It is assumed that $\beta_i > \alpha > \gamma_j > 0$ $(i = 1, 2, \ldots, \ell;$ $j = 1, 2, \ldots, m)$. To the best of our knowledge, damped elliptic operators such as $p(u)$ and $P(v)$ defined as above have not been studied.

Note that the principal part of (1.1) and (1.2) are reduced to the p-Laplacian $\nabla \cdot \left(|\nabla u|^{p-2}\nabla u\right)$, $(p = \alpha + 1)$. We know that a variety of physical phenomena are modeled by equations involving the p-Laplacian [2, 7, 8, 23, 24, 26]. We refer the reader to Diaz [11] for detailed references on physical background of the p-Laplacian.

We organize this paper as follows. In Section 2, we establish a Picone-type inequality. In Section 3, we present comparison results for the equations $p(u) = 0$ and $P(v) = 0$ and in Section 4, as an application we conclude some oscillation results and give a Wirtinger-type inequality.

2 Picone-type inequalities

In this section, we establish a Picone-type inequality for the coupled operators p and P defined by (1.1) and (1.2) respectively. Let G be a bounded domain in R^n with piecewise smooth boundary ∂G, and assume that $a(x) \in C(\bar{G}, R^+)$, $A(x) \in C(\bar{G}, R^+)$, $b(x) \in C(\bar{G}, R^n)$, $B(x) \in C(\bar{G}, R^n)$, $c(x) \in C(\bar{G}, R)$, $C(x) \in C(\bar{G}, R)$, $D_i(x) \in C(\bar{G}, [0,\infty))$, $E_j(x) \in C(\bar{G}, [0,\infty))$, $(i = 1, 2, \ldots, \ell;\ j = 1, 2, \ldots, m)$.

The domain $\mathcal{D}_p(G)$ of p is defined to be the set of all functions u of class $C^1(\bar{G}, R)$ with the property that $a(x)|\nabla u|^{\alpha-1}\nabla u \in C^1(G, R^n) \cap C(\bar{G}, R^n)$. The domain $\mathcal{D}_P(G)$ of P is defined similarly.

Let $N = \min\{\ell, m\}$ and

$$H(\beta, \alpha, \gamma; D(x), E(x)) = \frac{\beta-\gamma}{\alpha-\gamma} \left(\frac{\beta-\alpha}{\alpha-\gamma}\right)^{\frac{\alpha-\beta}{\beta-\gamma}} (D(x))^{\frac{\alpha-\gamma}{\beta-\gamma}} (E(x))^{\frac{\beta-\alpha}{\beta-\gamma}}.$$

We will need the following lemmas, in order to prove our results.

Lemma 2.1 ([22, Lemma 2.1]). *The inequality*

$$|X|^{\alpha+1} + \alpha|Y|^{\alpha+1} - (\alpha+1)|Y|^{\alpha-1}X \cdot Y \geq 0.$$

is valid for any $X \in R^n$ and $Y \in R^n$, where the equality holds if and only if $X = Y$.

Lemma 2.2 ([32, Lemma 8.3.2]). *Let $F(x) \in C(G, R^+)$ satisfy $F(x) > \alpha > 0$. Then the inequality*

$$|\nabla u - uw(x)|^{\alpha+1} \leq \frac{F(x)}{F(x)-\alpha}|\nabla u|^{\alpha+1} + \frac{|F(x)w(x)|^{\alpha+1}}{F(x)-\alpha}|u|^{\alpha+1}$$

holds for any function $u \in C^1(G, R)$ and any n-vector function $w(x) \in C(G, R^n)$.

Theorem 2.3 (Picone-type inequality). *Let $F(x) \in C(G, R^+)$ satisfying $F(x) > \alpha$. If $u \in \mathcal{D}_p(G)$, $v \in \mathcal{D}_P(G)$ and $v \neq 0$ in G (that is, v has no zero in G), then the following Picone-type inequality holds:*

$$\begin{aligned}
&\nabla \cdot \left(\frac{u}{\varphi(v)} \left[\varphi(v) a(x) |\nabla u|^{\alpha-1} \nabla u - \varphi(u) A(x) |\nabla v|^{\alpha-1} \nabla v \right] \right) \\
&\quad \geqslant \left(a(x) - \alpha|b(x)| - A(x)\frac{F(x)}{F(x)-\alpha} \right) |\nabla u|^{\alpha+1} \\
&\qquad + \left(C_1(x) - c(x) - |b(x)| - A(x)\frac{|F(x)B(x)/A(x)|^{\alpha+1}}{F(x)-\alpha} \right) |u|^{\alpha+1} \\
&\qquad + A(x) \left[\left| \nabla u - \frac{uB(x)}{A(x)} \right|^{\alpha+1} + \alpha \left| \frac{u}{v} \nabla v \right|^{\alpha+1} - (\alpha+1) \left(\nabla u - \frac{uB(x)}{A(x)} \right) \cdot \Phi\Big(\frac{u}{v}\nabla v\Big) \right] \\
&\qquad + \frac{u}{\varphi(v)} \left(\varphi(v) p(u) - \varphi(u) P(v) \right),
\end{aligned} \tag{2.1}$$

where $\varphi(s) = |s|^{\alpha-1}s$, $s \in R$, $\Phi(\xi) = |\xi|^{\alpha-1}\xi$, $\xi \in R^n$ and

$$C_1(x) = C(x) + \sum_{i=1}^{N} H(\beta_i, \alpha_i, \gamma_i; D_i(x), E_i(x)).$$

Proof. We easily see that

$$\begin{aligned}
\nabla \cdot \left(ua(x)|\nabla u|^{\alpha-1}\nabla u \right) &= a(x)|\nabla u|^{\alpha+1} - c(x)|u|^{\alpha+1} \\
&\quad + up(u) - (\alpha+1)ub(x) \cdot \Phi(\nabla u).
\end{aligned} \tag{2.2}$$

We observe that the following identity holds:

$$\begin{aligned}
&- \nabla \cdot \left(u\varphi(u)\frac{A(x)|\nabla v|^{\alpha-1}\nabla v}{\varphi(v)} \right) \\
&\quad = -A(x) \left| \nabla u - \frac{uB(x)}{A(x)} \right|^{\alpha+1} \\
&\qquad + A(x) \left[\left| \nabla u - \frac{uB(x)}{A(x)} \right|^{\alpha+1} + \alpha \left| \frac{u}{v} \nabla v \right|^{\alpha+1} - (\alpha+1) \left(\nabla u - \frac{uB(x)}{A(x)} \right) \cdot \Phi\Big(\frac{u}{v}\nabla v\Big) \right] \\
&\qquad + \frac{u\varphi(u)}{\varphi(v)} g(x, v) - \frac{u\varphi(u)}{\varphi(v)} P(v).
\end{aligned} \tag{2.3}$$

We combine (2.2) with (2.3) to obtain the following:

$$\begin{aligned}\nabla\cdot&\left(\frac{u}{\varphi(v)}\left[\varphi(v)a(x)|\nabla u|^{\alpha-1}\nabla u-\varphi(u)A(x)|\nabla v|^{\alpha-1}\nabla v\right]\right)\\&=a(x)|\nabla u|^{\alpha+1}-c(x)|u|^{\alpha+1}-(\alpha+1)ub(x)\cdot\Phi(\nabla u)-A(x)\left|\nabla u-\frac{uB(x)}{A(x)}\right|^{\alpha+1}\\&\quad+A(x)\left[\left|\nabla u-\frac{uB(x)}{A(x)}\right|^{\alpha+1}+\alpha\left|\frac{u}{v}\nabla v\right|^{\alpha+1}-(\alpha+1)\left(\nabla u-\frac{uB(x)}{A(x)}\right)\cdot\Phi\left(\frac{u}{v}\nabla v\right)\right]\\&\quad+\frac{u\varphi(u)}{\varphi(v)}g(x,v)+\frac{u}{\varphi(v)}[\varphi(v)p(u)-\varphi(u)P(v)].\end{aligned}\tag{2.4}$$

Using Young's inequality we have,

$$\begin{aligned}\frac{u\varphi(u)}{\varphi(v)}g(x,v)&\geq C(x)|u|^{\alpha+1}+\left(\sum_{i=1}^{N}H(\beta_i,\alpha_i,\gamma_i;D_i(x),E_i(x))\right)|u|^{\alpha+1}\\&=C_1(x)|u|^{\alpha+1}\end{aligned}\tag{2.5}$$

and

$$(\alpha+1)ub(x)\cdot\Phi(\nabla u)\leq|b(x)|\left(|u|^{\alpha+1}+\alpha|\nabla u|^{\alpha+1}\right).\tag{2.6}$$

From Lemma 2.2, we can write

$$\left|\nabla u-\frac{uB(x)}{A(x)}\right|^{\alpha+1}\leq\frac{F(x)}{F(x)-\alpha}|\nabla u|^{\alpha+1}+\frac{\left|F(x)\frac{B(x)}{A(x)}\right|^{\alpha+1}}{F(x)-\alpha}|u|^{\alpha+1}.\tag{2.7}$$

We combine (2.5)–(2.7) with (2.4) to obtain the desired inequality (2.1). □

Theorem 2.4. *If $v\in\mathcal{D}_P(G)$, and $v\neq 0$ in G, then the following inequality holds for any $u\in C^1(G,R)$:*

$$\begin{aligned}-\nabla\cdot&\left(\frac{u\varphi(u)}{\varphi(v)}A(x)|\nabla v|^{\alpha-1}\nabla v\right)\\&\geq-A(x)\left|\nabla u-\frac{uB(x)}{A(x)}\right|^{\alpha+1}\\&\quad+A(x)\left[\left|\nabla u-\frac{uB(x)}{A(x)}\right|^{\alpha+1}+\alpha\left|\frac{u}{v}\nabla v\right|^{\alpha+1}-(\alpha+1)\left(\nabla u-\frac{uB(x)}{A(x)}\right)\cdot\Phi\left(\frac{u}{v}\nabla v\right)\right]\\&\quad+C_1(x)|u|^{\alpha+1}-\frac{u\varphi(u)}{\varphi(v)}P(v),\end{aligned}\tag{2.8}$$

where $\varphi(s)$, $\Phi(\xi)$ and $C_1(x)$ are defined as in Theorem 2.3.

Proof. Combining (2.3) with (2.5) yields the desired inequality (2.8). □

3 Sturmian comparison theorems

In this section we present some Sturmian comparison results on the basis of the Picone-type inequality obtained in Section 2.

Theorem 3.1 (Sturmian comparison theorem). *Let $F(x) \in C(G, R^+)$ satisfy $F(x) > \alpha$. If there exists a nontrivial solution $u \in \mathcal{D}_p(G)$ of $p(u) = 0$ such that $u = 0$ on ∂G and*

$$V(u) := \int_G \left[\left(a(x) - \alpha|b(x)| - A(x)\frac{F(x)}{F(x)-\alpha} \right) |\nabla u|^{\alpha+1} + \left(C_1(x) - c(x) - |b(x)| - A(x)\frac{|F(x)B(x)/A(x)|^{\alpha+1}}{F(x)-\alpha} \right) |u|^{\alpha+1} \right] dx \geq 0 \tag{3.1}$$

then every solution $v \in \mathcal{D}_P(G)$ of $P(v) = 0$ must vanish at some point of $\bar{G}$.

Proof. Suppose that, contrary to our claim there exists a solution $v \in \mathcal{D}_P(G)$ of $P(v) = 0$ satisfying $v \neq 0$ on $\bar{G}$. We integrate (2.1) over G and then apply the divergence theorem to obtain

$$0 \geq V(u) + \int_G A(x) \left[\left| \nabla u - \frac{uB(x)}{A(x)} \right|^{\alpha+1} + \alpha \left| \frac{u}{v}\nabla v \right|^{\alpha+1} - (\alpha+1) \left(\nabla u - \frac{uB(x)}{A(x)} \right) \cdot \Phi\left(\frac{u}{v}\nabla v \right) \right] dx \geq 0 \tag{3.2}$$

and therefore

$$\int_G A(x) \left[\left| \nabla u - \frac{uB(x)}{A(x)} \right|^{\alpha+1} + \alpha \left| \frac{u}{v}\nabla v \right|^{\alpha+1} - (\alpha+1) \left(\nabla u - \frac{uB(x)}{A(x)} \right) \cdot \Phi\left(\frac{u}{v}\nabla v \right) \right] dx = 0. \tag{3.3}$$

From Lemma 2.1, we see that

$$\nabla u - \frac{uB(x)}{A(x)} \equiv \frac{u}{v}\nabla v \ \ or \ \ \nabla\left(\frac{u}{v}\right) - \frac{B(x)}{A(x)}\frac{u}{v} \equiv 0 \ \text{ in } \ G, \tag{3.4}$$

then it follows from a result of Jaroš, Kusano and Yoshida [17] that

$$\frac{u}{v} = C_0 e^{\alpha(x)} \ \text{ on } \ \bar{G} \tag{3.5}$$

for some constant C_0 and some continuous function $\alpha(x)$. Since $u = 0$ on ∂G, we see that $C_0 = 0$, which contradicts the fact that u is nontrivial. The proof is complete. □

Corollary 3.2. *Let $F(x) \in C(G, R^+)$ satisfy $F(x) > \alpha$. Assume that*

$$a(x) \geq \alpha|b(x)| + A(x)\frac{F(x)}{F(x)-\alpha} \tag{3.6}$$

and

$$C_1(x) \geq c(x) + |b(x)| + A(x)\frac{\left|F(x)\frac{B(x)}{A(x)}\right|^{\alpha+1}}{F(x)-\alpha} \tag{3.7}$$

in G. If there exists a nontrivial solution $u \in \mathcal{D}_p(G)$ of $p(u) = 0$ such that $u = 0$ on ∂G, then every solution $v \in \mathcal{D}_P(G)$ of $P(v) = 0$ must vanish at some point of $\bar{G}$.

Theorem 3.3. *If there exists a nontrivial function $u \in C^1(\bar{G}, R)$ such that $u = 0$ on ∂G and*

$$M(u) := \int_G \left\{ A(x) \left| \nabla u - \frac{uB(x)}{A(x)} \right|^{\alpha+1} - C_1(x)|u|^{\alpha+1} \right\} dx \le 0 \tag{3.8}$$

then every solution $v \in \mathcal{D}_P(G)$ of $P(v) = 0$ must vanish at some point of G unless $u = C_0 e^{\alpha(x)} v$, where $C_0 \neq 0$ is a constant and $\nabla \alpha(x) = \frac{B(x)}{A(x)}$ in G.

Proof. Suppose that there exists a solution $v \in \mathcal{D}_P(G)$ of $P(v) = 0$ satisfying $v \neq 0$ in G. Since $\partial G \in C^1$, $u \in C^1(\bar{G}, R)$ and $u = 0$ on ∂G, we find that u belongs to the Sobolev space $W_0^{1,\alpha+1}(G)$ which is the closure in the norm

$$\|w\| := \left(\int_G \left[|w|^{\alpha+1} + |\nabla w|^{\alpha+1} \right] dx \right)^{\frac{1}{\alpha+1}} \tag{3.9}$$

of the class $C_0^\infty(G)$ of infinitely differentiable functions with compact supports in G [1, 13]. Then there is a sequence u_k of functions in $C_0^\infty(G)$ converging to u in the norm (3.9). Integrating (2.8) with $u = u_k$ over G, then applying the divergence theorem, we have

$$\begin{aligned} M(u_k) \ge \int_G A(x) \Bigg[\left| \nabla u_k - \frac{u_k B(x)}{A(x)} \right|^{\alpha+1} + \alpha \left| \frac{u_k}{v} \nabla v \right|^{\alpha+1} \\ - (\alpha+1) \left(\nabla u_k - \frac{u_k B(x)}{A(x)} \right) \cdot \Phi\left(\frac{u_k}{v} \nabla v \right) \Bigg] dx \ge 0. \end{aligned} \tag{3.10}$$

We first claim that $\lim_{k\to+\infty} M(u_k) = M(u) = 0$. Since $A(x)$, $C(x)$, $D(x)$ and $E(x)$ are bounded on $\bar{G}$, there exists a constant $K_1 > 0$ such that

$$A(x) \le K_1 \quad \text{and} \quad |C_1(x)| \le K_1. \tag{3.11}$$

It is easy to check that

$$\begin{aligned} |M(u_k) - M(u)| \le K_1 \int_G \left| \left| \nabla u_k - \frac{u_k B(x)}{A(x)} \right|^{\alpha+1} - \left| \nabla u - \frac{uB(x)}{A(x)} \right|^{\alpha+1} \right| dx \\ + K_1 \int_G \left| |u_k|^{\alpha+1} - |u|^{\alpha+1} \right| dx. \end{aligned} \tag{3.12}$$

From the mean value theorem we see that

$$\begin{aligned} &\left| \left| \nabla u_k - \frac{u_k B(x)}{A(x)} \right|^{\alpha+1} - \left| \nabla u - \frac{uB(x)}{A(x)} \right|^{\alpha+1} \right| \\ &\quad \le (\alpha+1) \left(\left| \nabla u_k - \frac{u_k B(x)}{A(x)} \right| + \left| \nabla u - \frac{uB(x)}{A(x)} \right| \right)^\alpha \left| \nabla(u_k - u) + \frac{B(x)}{A(x)}(u_k - u) \right| \\ &\quad \le (\alpha+1) \left(|\nabla u_k| + |\nabla u| + \frac{|B(x)|}{A(x)}|u_k| + \frac{|B(x)|}{A(x)}|u| \right)^\alpha \left(|\nabla(u_k - u)| + \frac{|B(x)|}{A(x)}|u_k - u| \right). \end{aligned}$$

Since also $B(x)$ is bounded on $\bar{G}$, then there is a constant K_2 such that $\frac{|B(x)|}{A(x)} \le K_2$ on $\bar{G}$. Let us take $K_3 = \max\{1, K_2\}$. From the above inequality we have

$$\begin{aligned} &\left| \left| \nabla u_k - \frac{u_k B(x)}{A(x)} \right|^{\alpha+1} - \left| \nabla u - \frac{uB(x)}{A(x)} \right|^{\alpha+1} \right| \\ &\quad \le (\alpha+1) K_3^{\alpha+1} \left(|\nabla u_k| + |\nabla u| + |u_k| + |u| \right)^\alpha \left(|\nabla(u_k - u)| + |u_k - u| \right). \end{aligned} \tag{3.13}$$

Using (3.13) and applying Hölder's inequality, we get

$$\begin{aligned}\int_G \left| \left|\nabla u_k - \frac{u_k B(x)}{A(x)}\right|^{\alpha+1} - \left|\nabla u - \frac{u B(x)}{A(x)}\right|^{\alpha+1} \right| dx \\ \leq (\alpha+1) K_3^{\alpha+1} \left(\int_G (|\nabla u_k| + |\nabla u| + |u_k| + |u|)^{\alpha+1} dx \right)^{\frac{\alpha}{\alpha+1}} \\ \times \left(\int_G (|\nabla(u_k - u)| + |u_k - u|)^{\alpha+1} dx \right)^{\frac{1}{\alpha+1}} \\ \leq (\alpha+1) K_3^{\alpha+1} \|u_k - u\| \left(\|u_k\| + \|u\|\right)^{\alpha}.\end{aligned} \tag{3.14}$$

Similarly, we obtain

$$\int_G \left| |u_k|^{\alpha+1} - |u|^{\alpha+1} \right| dx \leq (\alpha+1) \left(\|u_k\| + \|u\|\right)^{\alpha} \|u_k - u\|. \tag{3.15}$$

Combining (3.12), (3.14) and (3.15), we have

$$|M(u_k) - M(u)| \leq K_4 \left(\|u_k\| + \|u\|\right)^{\alpha} \|u_k - u\| \tag{3.16}$$

for some positive constant $K_4 = K_4(K_1, K_2, K_3)$ and so that $\lim_{k\to+\infty} M(u_k) = M(u)$. We get from (3.10) that $M(u) \geq 0$ which together with (3.8) implies $M(u) = 0$.

Let $\mathcal{B}$ be an arbitrary ball with $\bar{\mathcal{B}} \subset G$ and define

$$\begin{aligned}Q_{\mathcal{B}}(w) := \int_{\mathcal{B}} A(x) \Bigg[\left|\nabla w - \frac{w B(x)}{A(x)}\right|^{\alpha+1} + \alpha \left|\frac{w}{v}\nabla v\right|^{\alpha+1} \\ - (\alpha+1) \left(\nabla w - \frac{w B(x)}{A(x)}\right) \cdot \Phi\left(\frac{w}{v}\nabla v\right) \Bigg] dx\end{aligned} \tag{3.17}$$

for $w \in C^1(G, R)$.

It is easy to check that

$$0 \leq Q_{\mathcal{B}}(u_k) \leq Q_G(u_k) \leq M(u_k), \tag{3.18}$$

where $Q_G(u_k)$ denotes the right-hand side of (3.17) with $w = u_k$ and with $\mathcal{B}$ replaced by G.

A simple calculation yields

$$\begin{aligned}|Q_{\mathcal{B}}(u_k) - Q_{\mathcal{B}}(u)| \leq K_5 \left(\|u_k\|_{\mathcal{B}} + \|u\|_{\mathcal{B}}\right)^{\alpha} \|u_k - u\|_{\mathcal{B}} + K_6 \left(\|u_k\|_{\mathcal{B}}\right)^{\alpha} \|u_k - u\|_{\mathcal{B}} \\ + K_7 \|\varphi(u_k) - \varphi(u)\|_{L^q_{(\mathcal{B})}} \|u\|_{\mathcal{B}},\end{aligned} \tag{3.19}$$

where $q = \frac{\alpha+1}{\alpha}$, the constants K_5, K_6 and K_7 are independent of k and the subscript $\mathcal{B}$ indicates the integrals involved in the norm (3.9) are to be taken over $\mathcal{B}$ instead of G. It is known that the Nemitski operator $\varphi\colon L^{\alpha+1}(G) \to L^q(G)$ is continuous [6] and it is clear that $\|u_k - u\|_{\mathcal{B}} \to 0$ as $\|u_k - u\|_G \to 0$.

Therefore, letting $k \to \infty$ in (3.18), we find that $Q_{\mathcal{B}}(u) = 0$. Since $A(x) > 0$ in $\mathcal{B}$, it follows that

$$\left[\left|\nabla u - \frac{u B(x)}{A(x)}\right|^{\alpha+1} + \alpha \left|\frac{u}{v}\nabla v\right|^{\alpha+1} - (\alpha+1) \left(\nabla u - \frac{u B(x)}{A(x)}\right) \cdot \Phi\left(\frac{u}{v}\nabla v\right) \right] \equiv 0 \text{ in } B, \tag{3.20}$$

from which Lemma 2.1 implies that

$$\nabla u - \frac{uB(x)}{A(x)} \equiv \frac{u}{v}\nabla v \ \ or \ \ \nabla\left(\frac{u}{v}\right) - \frac{B(x)}{A(x)}\frac{u}{v} \equiv 0 \ \text{ in } \ \mathcal{B}.$$

Hence we observe that $\frac{u}{v} = C_0 e^{\alpha(x)}$ in $\mathcal{B}$ for some constant C_0 and some continuous function $\alpha(x)$ as in the proof of Theorem 3.1. Since $\mathcal{B}$ is an arbitrary ball with $\bar{\mathcal{B}} \subset G$, we conclude that $\frac{u}{v} = C_0 e^{\alpha(x)}$ in G where $C_0 \neq 0$. □

Corollary 3.4 (Sturmian comparison theorem)**.** *Let $F(x) \in C(G, R^+)$ satisfy $F(x) > \alpha$. If there exists a nontrivial solution $u \in \mathcal{D}_p(G)$ of $p(u) = 0$ for which $u = 0$ on ∂G and (3.1) hold, then every solution $v \in D_P(G)$ of $P(v) = 0$ must vanish at some point of G unless $u = C_0 e^{\alpha(x)} v$, where $C_0 \neq 0$ is a constant and $\nabla\alpha(x) = \frac{B(x)}{A(x)}$ in G.*

Proof. By using (2.2), (2.6), (2.7), (3.8) and Corollary 3.2 we obtain

$$M(u) \leq \int_G \left[\nabla\cdot\left(ua(x)|\nabla u|^{\alpha-1}\nabla u]\right) - up(u)\right] dx = 0.$$

Hence the result follows from Theorem 3.3. □

Remark 3.5. When we take $\alpha = 1$, $b(x) \equiv B(x) \equiv 0$ and $D_i(x) \equiv E_i(x) \equiv 0, (i = 1, 2, \ldots, \ell, j = 1, 2, \ldots, m)$ that is, in the linear elliptic equation case, and $b(x) \equiv B(x) \equiv 0$ and $D_i(x) \equiv E_i(x) \equiv 0, (i = 1, 2, \ldots, \ell, \ j = 1, 2, \ldots, m)$ that is, in the half-linear elliptic equation case, our results cannot be reduced to the well-known results. Hence our results are indeed a partial extension of the results that are given in the literature. Improvement of our results is left as an open problem to the researchers.

4 Applications

Let Ω be an exterior domain in R^n, that is, $\Omega \supset \{x \in R^n : |x| \geq r_0\}$ for some $r_0 > 0$. We consider the following equations:

$$p(u) = 0 \ \text{ in } \ \Omega \tag{4.1}$$

and

$$P(v) = 0 \ \text{ in } \ \Omega \tag{4.2}$$

where the operators p and P are defined in Section 1 and $a, A \in C(\Omega, R^+)$, $b, B \in C(\Omega, R^n)$, $c, C \in C(\Omega, R)$, $D_i, E_j \in C(\Omega, [0, \infty)), (i = 1, 2, \ldots, \ell; j = 1, 2, \ldots, m)$.

The domain $\mathcal{D}_p(\Omega)$ of p is defined to be the set of all functions u of class $C^1(\Omega, R)$ with the property that $a(x)|\nabla u|^{\alpha-1}\nabla u \in C^1(\Omega, R^n)$. The domain $\mathcal{D}_P(\Omega)$ of P is defined similarly.

A solution $u \in D_p(\Omega)$ of (4.1) (or $v \in \mathcal{D}_P(\Omega)$ of (4.2)) is said to be oscillatory in Ω if it has a zero in Ω_r for any $r > 0$, where

$$\Omega_r = \Omega \cap \{x \in R^n : |x| > r\}.$$

A bounded domain G with $\bar{G} \subset \Omega$ is said to be a nodal domain for the equation (4.1), if there exists a nontrivial function $u \in \mathcal{D}_p(G)$ such that $p(u) = 0$ in G and $u = 0$ on ∂G. The equation (4.1) is called nodally oscillatory in Ω, if (4.1) has a nodal domain contained in Ω_r for any $r > 0$.

Theorem 4.1. *Let $F(x) \in C(G, R^+)$ satisfy $F(x) > \alpha$. Assume that*

$$a(x) \geq \alpha|b(x)| + A(x)\frac{F(x)}{F(x)-\alpha} \tag{4.3}$$

and

$$C_1(x) \geq c(x) + |b(x)| + A(x)\frac{\left|F(x)\frac{B(x)}{A(x)}\right|^{\alpha+1}}{F(x)-\alpha} \tag{4.4}$$

in Ω. If (4.1) is nodally oscillatory in Ω, then every solution $v \in \mathcal{D}_P(G)$ of (4.2) is oscillatory in Ω.

Proof. Since (4.1) in nodally oscillatory in Ω, there exist a nodal domain $G \subset \Omega_r$ for any $r > 0$, and hence there exists a nontrivial function $u \in D_p(G)$ such that $p(u) = 0$ in G and $u = 0$ on ∂G. The conditions (4.3) and (4.4) ensures that $V(u) \geq 0$ is satisfied. From Corollary 3.2 it follows that every solution $v \in \mathcal{D}_P(\Omega)$ of (4.2) vanishes at some point of $\bar{G}$, that is, v must have a zero in Ω_r for any $r > 0$. This implies that v is oscillatory in Ω. □

The following is an immediate consequence of Theorem 4.1 by choosing $F(x) = \alpha + 1$, $b(x) \equiv B(x) \equiv 0$ and $m = 1$.

Corollary 4.2. *If the equation*

$$\nabla \cdot \left(a(x)|\nabla u|^{\alpha-1}\nabla u\right) + \left\{C(x) + \frac{\beta-\gamma}{\alpha-\gamma}\left(\frac{\beta-\alpha}{\alpha-\gamma}\right)^{\frac{\alpha-\beta}{\beta-\gamma}}(D(x))^{\frac{\alpha-\gamma}{\beta-\gamma}}(E(x))^{\frac{\beta-\alpha}{\beta-\gamma}}\right\}|u|^{\alpha-1}u = 0 \tag{4.5}$$

is nodally oscillatory in Ω, then every solution $v \in \mathcal{D}_P(\Omega)$ of the equation

$$\nabla \cdot \left(a(x)|\nabla v|^{\alpha-1}\nabla v\right) + \frac{1}{\alpha+1}g(x, v) = 0$$

is oscillatory in Ω, where $D_1(x) \equiv D(x)$, $E_1(x) \equiv E(x)$, $\alpha_1 \equiv \alpha$, $\gamma_1 \equiv \gamma$.

Various criteria for nodal oscillation can be found in [32]. For example for linear elliptic equations of the form

$$\triangle u + c(x)u = 0, \quad x \in R^2, \tag{4.6}$$

$c(x)$ being a continuous function in R^2, have been given by Kreith and Travis [19]. They showed that (4.6) is nodally oscillatory if

$$\int_{R^2} c(x)\,dx = \infty.$$

Applying this result to the equation (4.5) with $\alpha = 1$, $a(x) \equiv 1$ we have the following result.

Corollary 4.3. *If one of the following holds; either*

$$\int_{R^2} C(x)\,dx = \infty$$

or

$$\int_{R^2} C(x)\,dx \quad \textit{exists, and} \quad \int_{R^2} (D(x))^{\frac{1-\gamma}{\beta-\gamma}}(E(x))^{\frac{\beta-1}{\beta-\gamma}}\,dx = \infty,$$

then the equation (4.5) with $\alpha = 1$, $a(x) \equiv 1$ is nodally oscillatory in Ω.

When we take $\alpha = 1$, $m = 1$, $a(x) \equiv 1$, $C(x) \equiv 0$, Corollaries 4.2–4.3 reduce to Corollaries 3–4 given in [16], respectively.

Inequality (2.8) is utilized to establish Wirtinger-type inequality concerning the elliptic type nonlinear equation $P(v) = 0$. We know that a typical Wirtinger inequality is the following.

Theorem 4.4 ([14]). *If $u(t) \in C^1([a,b])$ and $u(a) = u(b) = 0$ then*

$$\int_a^b u'^2(t)\,dt \geq \left(\frac{\pi}{b-a}\right)^2 \int_a^b u^2(t)\,dt$$

where equality holds if and only if

$$u(t) = k_0 \sin \frac{\pi(t-a)}{b-a}$$

for some constant k_0.

Using Theorem 3.3 , the following Wirtinger-type inequality can be easily obtained.

Theorem 4.5. *Let $\partial G \in C^1$. Assume that there exists a solution v of $\mathcal{D}_P(G)$ of $P(v) = 0$ such that $v \neq 0$ in $\bar{G}$. If $u \in C^1(\bar{G}, R)$ and $u = 0$ on ∂G, then*

$$\int_G A(x) \left| \nabla u - \frac{uB(x)}{A(x)} \right|^{\alpha+1} dx \geq \int_G C_1(x)|u|^{\alpha+1}\,dx. \tag{4.7}$$

Remark 4.6. Note that when we take $B(x) \equiv 0$, we have $0 \leq M(u) = M(c_0 v) = 0$, we observe that $M(u) = 0$. When $B(x) \equiv 0$, $D_i(x) \equiv E_j(x) \equiv 0$, $(i = 1, 2, \ldots, \ell;\ j = 1, 2, \ldots, m)$, Theorem 4.5 gives Corollary 4.2 in [34].

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