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Existence of positive periodic solutions for higher order singular functional difference equations

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Abstract. We study a higher order singular functional difference equation on $\mathbb{Z}$. Sufficient conditions are obtained for the existence of at least one positive periodic solution of the equation. Our proof utilizes the nonlinear alternative of Leray–Schauder.

Keywords: positive periodic solutions, higher order, functional difference equations, singular, nonlinear alternative of Leray–Schauder.

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1 Introduction

Nonlinear difference equations have numerous applications in modeling processes in biology, physics, statistics, and many other areas. For this reason, the existence of positive solutions to these equations is of great interest to many researchers. We refer the reader to [6–12, 15–18] for some recent work on this subject. In this paper, we are concerned with a higher order functional difference equation. To introduce our equation, we let $a \neq 1$, $b \neq 1$ be any fixed positive numbers, and m, k, ω be any fixed positive integers, and for any $u\colon \mathbb{Z} \to \mathbb{R}$, define

$$Lu(n) = u(n+m+k) - au(n+m) - bu(n+k) + abu(n).$$

Here, we study the existence of positive periodic solutions of the higher order functional difference equation

$$Lu(n) = f(n, u(n-\tau(n))) + r(n), \quad n \in \mathbb{Z}, \tag{1.1}$$

where $f\colon \mathbb{Z} \times (0,\infty) \to \mathbb{R}$, $\tau\colon \mathbb{Z} \to \mathbb{Z}$, and $r\colon \mathbb{Z} \to \mathbb{R}$ are ω-periodic on $\mathbb{Z}$, and $f(n,x)$ is continuous in x.

Equation (1.1) with $r(n) \equiv 0$, i.e., the equation

$$Lu(n) = f(n, u(n-\tau(n))), \quad n \in \mathbb{Z},$$

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has been recently studied by Wang and Chen in [18] using Krasnosel'skii's fixed point theorem. When $f(n,x)$ is nonsingular at $x = 0$, sufficient conditions were found there for the existence of positive periodic solutions. In this paper, we will establish a new existence criterion for equation (1.1). The nonlinear term $f(n,x)$ is allowed to be singular at $x = 0$ in (1.1). The proof will employ a nonlinear alternative of Leray–Schauder. Our approach involves examining a one-parameter family of nonsingular problems constructed from a sequence of nonsingular perturbations of f. For each of these nonsingular problems, we will apply the nonlinear alternative of Leray–Schauder to obtain the existence of at least one positive periodic solution. From this sequence of solutions, we will extract a subsequence that converges to a positive periodic solution of (1.1). This type of technique has been successfully used in obtaining positive solutions for several classes of singular problems, see, for example, [1, 2, 6, 12, 13]. Our proofs are partly motivated by these works. Other results on singular problems can be found in [3–5, 7, 14].

As a simple application of our general existence theorem, we also derive some sufficient conditions for the existence of at least one positive periodic solution of the functional difference equation

$$Lu = c(n)(u(n-\tau(n)))^{-\alpha} + \mu d(n)(u(n-\tau(n)))^{\beta} + r(n),\ n \in \mathbb{Z}, \tag{1.2}$$

where $\alpha \geq 0$ and $\beta \geq 0$ are constants, c, d, and r are ω-periodic functions on $\mathbb{Z}$ with $c(n) > 0$ and $d(n) \geq 0$ on $\mathbb{Z}$, and $\mu > 0$ is a parameter.

The remainder of this paper is laid out as follows. In Section 2, we present our assumptions and main results. Some preliminary lemmas as well as the proofs are given in Section 3.

2 Main results

In this paper, for any $c, d \in \mathbb{Z}$ with $c \leq d$, let $[c,d]_{\mathbb{Z}}$ denote the discrete interval $\{c, \dots, d\}$. For the function $r(n)$ given in equation (1.1), define a function $\gamma \colon \mathbb{Z} \to \mathbb{R}$ by

$$\gamma(n) = \sum_{i=1}^{\omega}\sum_{j=1}^{\omega} G(i,j) r(n_{ij}), \tag{2.1}$$

where

$$G(i,j) = \frac{a^{\omega}b^{\omega}a^{-i}b^{-j}}{(1-a^{\omega})(1-b^{\omega})}, \quad i,j \in [1,\omega]_{\mathbb{Z}}, \tag{2.2}$$

and

$$n_{ij} = n + (i-1)k + (j-1)m. \tag{2.3}$$

Let

$$\delta_1 = \frac{a^{\omega}b^{\omega}}{|(1-a^{\omega})(1-b^{\omega})|} \min\{a^{-1}, a^{-\omega}\} \cdot \min\{b^{-1}, b^{-\omega}\}$$

and

$$\delta_2 = \frac{a^{\omega}b^{\omega}}{|(1-a^{\omega})(1-b^{\omega})|} \max\{a^{-1}, a^{-\omega}\} \cdot \max\{b^{-1}, b^{-\omega}\}.$$

Then, (2.2) implies that

$$\begin{cases} \delta_1 \leq G(i,j) \leq \delta_2 \quad \text{for } i,j \in [1,\omega]_{\mathbb{Z}} & \text{if } (a-1)(b-1) > 0, \\ \delta_1 \leq -G(i,j) \leq \delta_2 \quad \text{for } i,j \in [1,\omega]_{\mathbb{Z}} & \text{if } (a-1)(b-1) < 0. \end{cases} \tag{2.4}$$

We make the following assumptions.

(H1) $(k,\omega) = (m,\omega) = 1$, where (x,y) denotes the greatest common divisor of x and y;

(H2) either

(a) $(a-1)(b-1) > 0$, $\gamma(n) \geq 0$ on $\mathbb{Z}$, and there exist continuous, nonnegative functions $g(x)$, $h(x)$, and $\phi(n)$ such that $g(x) > 0$ is nonincreasing on $(0,\infty)$, $h(x)/g(x)$ is nondecreasing on $(0,\infty)$, and

$$0 \leq f(n, x + \gamma(n - \tau(n))) \leq \phi(n)(g(x) + h(x)) \quad \text{for } (n,x) \in \mathbb{Z} \times (0,\infty),$$

or

(b) $(a-1)(b-1) < 0$, $\gamma(n) \geq 0$ on $\mathbb{Z}$, and there exist continuous, nonpositive functions $g(x)$, $h(x)$, and nonnegative $\phi(n)$ such that $g(x) < 0$ is nondecreasing on $(0,\infty)$, $h(x)/g(x)$ is nondecreasing on $(0,\infty)$, and

$$0 \geq f(n, x + \gamma(n - \tau(n))) \geq \phi(n)(g(x) + h(x)) \quad \text{for } (n,x) \in \mathbb{Z} \times (0,\infty);$$

(H3) for each $q > 0$, there exists a continuous function $\psi_q(n)$ such that either

(i) $\psi_q(n)$ is nonnegative, $\psi_q(n) > 0$ for some $n \in [1,\omega]_{\mathbb{Z}}$, and

$$f(n, x + \gamma) \geq \psi_q(n) \quad \text{for } (n,x) \in [1,\omega]_{\mathbb{Z}} \times (0,q],$$

if (H2)(a) holds, or

(ii) $\psi_q(n)$ is nonpositive, $\psi_q(n) < 0$ for some $n \in [1,\omega]_{\mathbb{Z}}$, and

$$f(n, x + \gamma) \leq \psi_q(n) \quad \text{for } (n,x) \in [1,\omega]_{\mathbb{Z}} \times (0,q],$$

if (H2)(b) holds;

(H4) there exists $R > 0$ such that

$$R > \delta_2 \omega g(\delta R) \left(1 + \frac{h(R)}{g(R)}\right) \sum_{i=1}^{\omega} \phi(i),$$

where $\delta = \delta_1/\delta_2 \in (0,1]$.

Now, we state our main results.

Theorem 2.1. *Assume that (H1)–(H4) hold. Then equation (1.1) has at least one positive periodic solution $y(n)$ satisfying $y(n) > \gamma(n)$ on $[1,\omega]_{\mathbb{Z}}$ and $0 < \max_{n\in[1,\omega]_{\mathbb{Z}}} |y(n) - \gamma(n)| < R$.*

As a consequence of Theorem 2.1, we have the following corollary.

Corollary 2.2. *Assume that (H1) holds, $(a-1)(b-1) > 0$, and $\gamma(n) \geq 0$ on $\mathbb{Z}$. Then, we have*

(i) if $\beta < 1$, then equation (1.2) has at least one positive periodic solution for each $\mu \in (0,\infty)$;

(ii) if $\beta \geq 1$, then there exists $\bar{\mu} > 0$ such that equation (1.2) has at least one positive periodic solution for each $\mu \in (0,\bar{\mu})$.

3 Proofs of the main results

Throughout this section, let X be the set of all real ω-periodic functions on $\mathbb{Z}$. Then, equipped with the maximum norm $\|u\| = \max_{n\in[1,\omega]_{\mathbb{Z}}} |u(n)|$, X is a Banach space.

Lemma 3.1 below can be proved using [18, Lemma 2.1].

Lemma 3.1. *Assume (H1) holds. Then for any $h \in X$, $u(n)$ is a periodic solution of the equation*

$$Lu(n) = h(n), \quad n \in \mathbb{Z},$$

if and only if $u(n)$ is a solution of the summation equation

$$u(n) = \sum_{i=1}^{\omega}\sum_{j=1}^{\omega} G(i,j)h(n_{ij}),$$

where $G(i,j)$ and n_{ij} are given by (2.2) and (2.3) respectively.

We refer the reader to [1, Theorem 1.2.3] for the following version of the well known nonlinear alternative of Leray–Schauder.

Lemma 3.2. *Let K be a convex subset of a normed linear space X, and let Ω be a bounded open subset with $\tilde{p} \in \Omega$. Then every compact map $N\colon \overline{\Omega} \to K$ has at least one of the following properties:*

(i) N has at least one fixed point in $\overline{\Omega}$;

(ii) there is $u \in \partial\Omega$ and $\lambda \in (0,1)$ such that $u = (1-\lambda)\tilde{p} + \lambda N u$.

For any $h \in X$, it is easy to see that

$$\sum_{i=1}^{\omega}\sum_{j=1}^{\omega} h(n_{ij}) = \omega \sum_{i=1}^{\omega} h(i).$$

We will use this identity in the proof of Theorem 2.1.

Now, we are ready to prove our results.

Proof of Theorem 2.1. We show the case where (H2)(a) holds. The proof for (H2)(b) is similar. Let $\Omega = \{u \in X : \|u\| < R\}$, where R is given in (H4). We first observe that, to prove the theorem, it suffices to show that the equation

$$Lu(n) = f(n, u(n-\tau(n)) + \gamma(n-\tau(n))), \quad n \in \mathbb{Z}. \tag{3.1}$$

has a positive periodic solution $u \in \Omega$ satisfying $0 < \|u\| < R$. If fact, if this is true, we let $y(n) = u(n) + \gamma(n)$. Then, $y(n) > \gamma(n)$ on $[1,\omega]_{\mathbb{Z}}$, $0 < \|y-\gamma\| < R$, and

$$\begin{aligned} Ly(n) &= Lu(n) + L\gamma(n) \\ &= f(n, u(n-\tau(n)) + \gamma(n-\tau(n))) + r(n) \\ &= f(n, y(n-\tau(n))) + r(n), \quad n \in \mathbb{Z}. \end{aligned}$$

Thus, $y(n)$ is a positive periodic solution of (1.1) with the required properties.

From (H4), there exists $k_0 \in \mathbb{N}$ such that

$$R > \frac{1}{k_0} + \delta_2 \omega g(\delta R)\left(1 + \frac{h(R)}{g(R)}\right)\sum_{i=1}^{\omega}\phi(i). \tag{3.2}$$

Let $\mathbb{K}_0 = \{k_0, k_0+1, \dots\}$. For any fixed $k \in \mathbb{K}_0$, consider the family of equations

$$u(n) = \frac{1}{k} + \lambda \sum_{i=1}^{\omega}\sum_{j=1}^{\omega} G(i,j) f_k(n_{ij}, u(n_{ij}^{\tau}) + \gamma(n_{ij}^{\tau})) = \frac{1}{k} + \lambda T_k u(n), \quad n \in \mathbb{Z}, \tag{3.3}$$

where $\lambda \in (0,1)$, $f_k(n,x) = f(n, \max\{x, 1/k\})$ for $(n,x) \in \mathbb{Z} \times \mathbb{R}$, $n_{ij}^{\tau} = n_{ij} - \tau(n_{ij})$, and

$$T_k u(n) = \sum_{i=1}^{\omega}\sum_{j=1}^{\omega} G(i,j) f_k(n_{ij}, u(n_{ij}^{\tau}) + \gamma(n_{ij}^{\tau})).$$

We now prove two claims.

Claim 1. For any $\lambda \in (0,1)$, any possible solution $u(n)$ of (3.3) satisfies $u(n) \geq \delta ||u||$ for $n \in \mathbb{Z}$, where $\delta = \delta_1/\delta_2 \in (0,1]$ is given in (H4).

In fact, from (2.4), we have

$$u(n) \leq \frac{1}{k} + \delta_2 \sum_{i=1}^{\omega}\sum_{j=1}^{\omega} f(n_{ij}, u(n_{ij}^{\tau}) + \gamma(n_{ij}^{\tau}))$$

and

$$\begin{aligned} u(n) &\geq \frac{1}{k} + \delta_1 \sum_{i=1}^{\omega}\sum_{j=1}^{\omega} f(n_{ij}, u(n_{ij}^{\tau}) + \gamma(n_{ij}^{\tau})) \\ &\geq \delta \left(\frac{1}{k} + \delta_2 \sum_{i=1}^{\omega}\sum_{j=1}^{\omega} f(n_{ij}, u(n_{ij}^{\tau}) + \gamma(n_{ij}^{\tau})) \right) \end{aligned}$$

for $n \in \mathbb{Z}$. Thus, $u(n) \geq \delta ||u||$ on $\mathbb{Z}$, i.e., Claim 1 holds.

Claim 2. For any $\lambda \in (0,1)$, any possible solution of (3.3) satisfies $\|u\| \neq R$.

Suppose the claim is not true and assume that $u(n)$ is a solution of (3.3) for some $\lambda \in (0,1)$ with $\|u\| = R$. Since $\lambda T_k u(n) \geq 0$ on $\mathbb{Z}$, we have $u(n) \geq 1/k$, which implies that $u(n) + \gamma(n) \geq u(n) \geq 1/k$ on $\mathbb{Z}$. Then, (3.3) becomes

$$u(n) = \frac{1}{k} + \lambda \sum_{i=1}^{\omega}\sum_{j=1}^{\omega} G(i,j) f(n_{ij}, u(n_{ij}^{\tau}) + \gamma(n_{ij}^{\tau})). \tag{3.4}$$

From (2.4), (H2), and Claim 1, it follows that

$$\begin{aligned} R = \|u\| &\leq \frac{1}{k_0} + \lambda \sum_{i=1}^{\omega}\sum_{j=1}^{\omega} G(i,j) f(n_{ij}, u(n_{ij}^{\tau}) + \gamma(n_{ij}^{\tau})) \\ &\leq \frac{1}{k_0} + \delta_2 \omega \sum_{i=1}^{\omega} f(i, u(i-\tau(i)) + \gamma(i-\tau(i))) \\ &\leq \frac{1}{k_0} + \delta_2 \omega \sum_{i=1}^{\omega} \phi(i) g(u(i-\tau(i))) \left(1 + \frac{h(u(i-\tau(i)))}{g(u(i-\tau(i)))}\right) \\ &\leq \frac{1}{k_0} + \delta_2 \omega g(\delta\|u\|) \left(1 + \frac{h(\|u\|)}{g(\|u\|)}\right) \sum_{i=1}^{\omega} \phi(i) \\ &= \frac{1}{k_0} + \delta_2 \omega g(\delta R) \left(1 + \frac{h(R)}{g(R)}\right) \sum_{i=1}^{\omega} \phi(i), \end{aligned}$$

which contradicts (3.2). Hence, Claim 2 holds.

Note that $1/k \le 1/k_0 < R$ and (3.3) can be rewritten as

$$u(n) = (1-\lambda)\frac{1}{k} + \lambda N_k u(n),$$

where $N_k u(n) = T_k u(n) + 1/k$. Clearly, N_k is compact, and as in Claim 1, we have

$$N_k u(n) \ge \delta ||N_k u|| \quad \text{for } n \in \mathbb{Z}.$$

Then, N_k maps $\overline{\Omega}$ into K, where $K = \{u \in X \ : \ u(n) \ge \delta\|u\| \text{ on } \mathbb{Z}\}$. Therefore, by Lemma 3.2, N_k has at least one fixed point $u_k \in \overline{\Omega}$. Thus, for any $k \in \mathbb{K}_0$, we have proven that the equation

$$u_k(n) = \frac{1}{k} + \sum_{i=1}^{\omega}\sum_{j=1}^{\omega} G(i,j) f_k(n_{ij}, u(n_{ij}^{\tau}) + \gamma(n_{ij}^{\tau})), \quad n \in \mathbb{Z},$$

has a solution $u_k(n)$ with $\|u_k\| \le R$. Since $u_k(n) + \gamma(n) \ge u_k(n) \ge 1/k$, we see that $u_k(n)$ is actually a solution of the equation

$$u_k(n) = \frac{1}{k} + \sum_{i=1}^{\omega}\sum_{j=1}^{\omega} G(i,j) f(n_{ij}, u(n_{ij}^{\tau}) + \gamma(n_{ij}^{\tau})), \quad n \in \mathbb{Z}. \tag{3.5}$$

This, together with (H3), implies that

$$\begin{aligned} u_k(n) &> \sum_{i=1}^{\omega}\sum_{j=1}^{\omega} G(i,j) f(n_{ij}, u(n_{ij}^{\tau}) + \gamma(n_{ij}^{\tau})) \\ &\ge \omega\delta_1 \sum_{i=1}^{\omega} \psi_R(i) > 0 \quad \text{on } \mathbb{Z}. \end{aligned} \tag{3.6}$$

Since $\|u_k\| \le R$ for all $k \in \mathbb{K}_0$, we know that the sequence $\{u_k(n)\}_{k\in\mathbb{K}_0}$ has a subsequence, which converges uniformly to a function $u \in X$. For simplicity, we still denote this subsequence by $\{u_k(n)\}_{k\in\mathbb{K}_0}$. If we let $k \to \infty$ in (3.5) and (3.6), we obtain

$$u(n) = \sum_{i=1}^{\omega}\sum_{j=1}^{\omega} G(i,j) f(n_{ij}, u(n_{ij}^{\tau}) + \gamma(n_{ij}^{\tau})) \tag{3.7}$$

and

$$u(n) \ge \omega\delta_1 \sum_{i=1}^{\omega} \psi_R(i) > 0$$

for $n \in \mathbb{Z}$. Thus, by Lemma 3.1, $u(n)$ is a positive solution of (3.1). Since $\|u(n)\| \le R$ and $u = \lim_{k\to\infty} u_k$, we have $\|u\| \le R$. By an argument similar to the one used to show Claim 2, we have $\|u\| < R$. Hence, $0 < \|u\| < R$. This completes the proof of the theorem. □

Proof of Corollary 2.2. We will apply Theorem 2.1. To this end, let $f(n,x) = c(n)x^{-\alpha} + \mu d(n)x^{\beta}$, $g(x) = x^{-\alpha}$, $h(x) = \mu x^{\beta}$, and $\phi(n) = \max\{c(n),\ d(n)\}$. Then, (H1), (H2), and (H3) with $\psi_q(n) = q^{-\alpha}c(n)$ hold. Let $A = \delta_2\omega\sum_{i=1}^{\omega}\phi(i)$. Then, $0 < A < \infty$ and (H4) becomes

$$\mu < \frac{\delta^{\alpha}R^{\alpha+1} - A}{AR^{\alpha+\beta}} \quad \text{for some } R > 0.$$

Hence, equation (1.2) has at least one positive solution for

$$0 < \mu < \bar{\mu} := \sup_{R>0} \frac{\delta^{\alpha}R^{\alpha+1} - A}{AR^{\alpha+\beta}}.$$

Note that $\bar{\mu} = \infty$ if $\beta < 1$ and $\bar{\mu} < \infty$ if $\beta \ge 1$. This completes the proof of the corollary. □

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