# Global exponential stability for coupled systems of neutral delay differential equations

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**Abstract.** In this paper, a novel class of neutral delay differential equations (NDDEs) is presented. By using the Razumikhin method and Kirchhoff's matrix tree theorem in graph theory, the global exponential stability for such NDDEs is investigated. By constructing an appropriate Lyapunov function, two different kinds of sufficient criteria which ensure the global exponential stability of NDDEs are derived in the form of Lyapunov functions and coefficients of NDDEs, respectively. A numerical example is provided to demonstrate the effectiveness of the theoretical results.

**Keywords:** coupled systems, neutral delay differential equations, exponential stability.

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## 1   Introduction

It is well known that neutral differential equations (NDEs) are extensively used to model many of the phenomena arising in areas such as mechanics, physics, biology, medicine, economics, ecological systems, and engineering systems [1, 2, 4, 8, 9, 16, 17, 25, 28, 31, 32]. In recent years, the properties of NDEs have been a very active area of research, and a lot of interesting results have been obtained [10, 30]. Stability is one of the most important concepts concerning the properties of NDEs. Hence, it is taken for granted that the stability analysis of NDEs has attracted considerable attention of an increasing number of scientists [11, 24, 27, 29]. In reality, time delays exist in many physical systems and population ecology, because the future states depend not only on the present state but also on the past states. It is widely known that time delays often lead to the failure of stability for a stable system. Therefore, the study of neutral delay differential equations (NDDEs) has become the subject of many investigations.

   As is known to all, the Lyapunov functional method and the Lyapunov function method are two basic methods in studying the stability of delay differential equations. It is from the

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authors' point of view that the Razumikhin–Lyapunov function method allows us to use simple functions rather than functionals. And compared with the Lyapunov functional method, the imposed conditions of Razumikhin–Lyapunov function method are less restricted. More recently, the Razumikhin-type stability theorems for different kinds of dynamical systems were established in [3, 18, 19, 21, 26]. By the previous literatures, it is not difficult to see that the Razumikhin method provides a powerful tool to study the stability of delay differential equations.

In the study of coupled systems, the direct Lyapunov method is one of the most powerful and effective techniques, and is an indispensable tool in the theory of stability. It plays an important role in the establishment and development of the theory of stability for coupled systems. However, an unpleasant fact in this approach is that it is very difficult to straightly construct an appropriate Lyapunov function for specific coupled systems, because the dynamics of coupled systems depend not only on the individual vertex dynamics but also on the coupling topology. Obviously, it is the key point to construct an appropriate Lyapunov function for specific coupled systems in the study of stability. Over the past few years, based on graph theory, Li et al. advanced a new approach to construct Lyapunov functions for differential equations in [5, 12]. In [6, 13, 14, 15, 20, 22, 23], the global stability for several classes of coupled systems was effectively investigated by the method.

Motivated by the above discussion, it is feasible to investigate the global exponential stability theory for coupled systems of NDDEs by this effective approach. In this paper, a novel class of NDDEs are presented. Based on Razumikhin technique and Kirchhoff's matrix tree theorem in graph theory, the global exponential stability for these coupled systems of NDDEs was investigated. By constructing the appropriate Lyapunov function, two different kinds of sufficient criteria which ensure the global exponential stability for coupled systems of NDDEs are derived in the form of Lyapunov functions and coefficients of NDDEs, respectively. And it is worth mentioning that we get the sufficient stability conditions that could be verified more easily than by using the usual methods of Lyapunov functions.

The organization of this paper is as follows. The problem formulation and some basic preliminaries are given in Section 2. In Section 3, the main results, which guarantee that the coupled system of NDDEs is globally exponentially stable, are provided. In Section 4, we discuss a numerical example to illustrate the advantages of our results.

## 2  Preliminaries

The following basic concepts on the graph theory can be found in [13]. A digraph $\mathcal{G} = (V, E)$ contains a set $V = \{1, 2, \ldots, n\}$ of vertices and a set $E$ of arcs $(j, i)$ leading from initial vertex $i$ to terminal vertex $j$. A subgraph $\mathcal{H}$ of a graph $\mathcal{G}$ is a graph whose set of vertices and set of edges are all subsets of $\mathcal{G}$. A subgraph $\mathcal{H}$ of $\mathcal{G}$ is said to be spanning if $\mathcal{H}$ and $\mathcal{G}$ have the same vertex set. A digraph $\mathcal{G}$ is weighted if each arc $(j, i)$ is assigned a positive weight $a_{ij}$. Here $a_{ij} > 0$ if and only if there exists an arc from vertex $j$ to vertex $i$ in $\mathcal{G}$. The weight $W(\mathcal{G})$ of $\mathcal{G}$ is the product of the weights on all its arcs. A directed path $\mathcal{P}$ in $\mathcal{G}$ is a subgraph with distinct vertices $\{i_1, i_2, \ldots, i_m\}$ such that its set of arcs is $\{(i_k, i_{k+1}) : k = 1, 2, \ldots, m - 1\}$. If $i_m = i_1$, we call $\mathcal{P}$ a directed cycle. A connected subgraph $\mathcal{T}$ is a tree if it contains no cycles. A tree $\mathcal{T}$ is rooted at vertices $i$, called the root, if $i$ is not a terminal vertex of any arcs, and each of the remaining vertices is a terminal vertex of exactly one arc. A digraph $\mathcal{G}$ is strongly connected if, for any pair of distinct vertices, there exists a directed path from one to the other. Given a weighted digraph $\mathcal{G}$ with $n$ vertices, define the weighted matrix $A = (a_{ij})_{n \times n}$

whose entry $a_{ij}$ equals the weight of arc $(j, i)$ if it exists, and 0 otherwise. Denote the directed graph with weight matrix $A$ as $(\mathcal{G}, A)$. A weighted digraph $(\mathcal{G}, A)$ is said to be balanced if $W(\mathcal{C}) = W(-\mathcal{C})$ for all directed cycles $\mathcal{C}$. Here, $-\mathcal{C}$ denotes the reverse of $\mathcal{C}$ and is constructed by reversing the direction of all arcs in $\mathcal{C}$. A subgraph $\mathcal{Q}$ is unicyclic if it is a disjoint union of rooted trees whose roots form a directed cycle. A spanning unicyclic subgraph of $\mathcal{G}$ is a spanning directed subgraph consisting of a collection of disjoint rooted directed trees whose roots are connected by a directed cycle. For a unicyclic graph $\mathcal{G}$ with cycle $\mathcal{C}_\mathcal{Q}$, let $\tilde{\mathcal{Q}}$ be the unicyclic graph obtained by replacing $\mathcal{C}_\mathcal{Q}$ with $-\mathcal{C}_\mathcal{Q}$. Suppose that $(\mathcal{G}, A)$ is balanced, then $W(\mathcal{Q}) = W(\tilde{\mathcal{Q}})$. The Laplacian matrix of $(\mathcal{G}, A)$ is defined as

$$
L = \begin{pmatrix}
\sum_{k \neq 1} a_{1k} & -a_{12} & \cdots & -a_{1n} \\
-a_{21} & \sum_{k \neq 2} a_{2k} & \cdots & -a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
-a_{n1} & -a_{n2} & \cdots & \sum_{k \neq n} a_{nk}
\end{pmatrix}.
$$

To prove our results, the following lemma is necessary, which can be found in [12].

**Lemma 2.1.** *Assume $l \geq 2$. Let $c_i$ denote the cofactor of the i-th diagonal element of L. Then the following identity holds:*

$$
\sum_{i,j=1}^{l} c_i a_{ij} F_{ij}(x_i, x_j) = \sum_{\mathcal{Q} \in \mathbb{Q}} W(\mathcal{Q}) \sum_{(s,r) \in E(C_\mathcal{Q})} F_{rs}(x_r, x_s).
$$

*Here $F_{ij}(x_i, x_j)$, $1 \leq i, j \leq l$, are arbitrary functions, $\mathbb{Q}$ is the set of all spanning unicyclic graph of $(\mathcal{G}, A)$, $W(\mathcal{Q})$ is the weight of $\mathcal{Q}$, and $C_\mathcal{Q}$ denotes the directed cycle of $\mathcal{Q}$. In particular, if $(\mathcal{G}, A)$ is strongly connected, then $c_i > 0$ for $i \in \mathbb{L}$.*

Throughout this paper, the following notations will be used.

| |
|---|
| $\mathbb{R}^n$: $n$-dimensional Euclidean space |
| $\mathbb{R}_+^1$: $[0, +\infty)$ |
| $\mathbb{Z}^+$: $\{1, 2, \dots\}$ |
| $\mathbb{L}$: $\{1, 2, \dots, l\}$, $l \in \mathbb{Z}^+$ |
| $m = \sum_{i=1}^{l} m_i$ for $m_i \in \mathbb{Z}^+$ |
| $\tau = \max\{\tau_1, \dots, \tau_l\}$ for $\tau_i \in \mathbb{R}_+^1$ |
| $\|x\|$: the Euclidean norm for vectors $x$ |
| $I_A$: indicator function of a set $A$ |
| $C([-\tau, 0]; \mathbb{R}^n)$: space of continuous functions |
|     $x: [-\tau, 0] \to \mathbb{R}^n$ with norm $\|x\| = \sup_{-\tau \leq u \leq 0} |x(u)|$ |
| $C^1(\mathbb{R}^n; \mathbb{R}_+^1)$: the family of all nonnegative functions |
|     $V(x)$ on $\mathbb{R}^n$ that are continuously differentiable in $x$ |

Considering the coupled systems of NDDEs as follows:

$$
\frac{\mathrm{d}\left[x_k(t) - \gamma_k x_k(t - \tau_k)\right]}{\mathrm{d}t}
$$
$$
= f_k(x_k(t), x_k(t - \tau_k), t) + \sum_{h=1}^{l} H_{kh}(x_h(t) - \gamma_h x_h(t - \tau_h)), \quad t \geq 0, \quad k \in \mathbb{L}, \tag{2.1}
$$

where $\tau_k \geq 0$, $\gamma_k \geq 0$ are constants, functions $f_k: \mathbb{R}^{m_k} \times \mathbb{R}^{m_k} \times \mathbb{R}_+^1 \to \mathbb{R}^{m_k}$, $H_{kh}: \mathbb{R}^{m_h} \to \mathbb{R}^{m_k}$ are continuous. Throughout this section we assume that functions $f_k$ and $H_{kh}$, $k, h \in \mathbb{L}$ satisfy

Lipschitz condition, by Theorems 12.2.1–12.2.3 in [7], system (2.1) has unique solution for each initial state $x_0 = \Psi \in C([-\tau, 0]; \mathbb{R}^m)$. We denote by $x(t, \Psi) = (x_1^T(t, \Psi), \ldots, x_l^T(t, \Psi))^T$ the unique solution of the system (2.1). Before proceeding with the main result of this work, the following assumption is made.

**Assumption 2.2.** *Functions $f_k$ and $H_{kh}$ satisfy $f_k(0, 0, t) = 0$, $H_{kh}(0) = 0$.*

We note that Assumption 2.2 implies that system (2.1) has a trivial solution $x(t, 0) = 0$. The definition on the exponential stability of the trivial solution is given as follows.

**Definition 2.3.** The trivial solution to system (2.1) is said to be exponentially stable if there exist positive constants $C$ and $\gamma$ such that

$$\sum_{k=1}^{l} |x_k(t, \Psi)|^p \leq Ce^{-\gamma t}, \quad t \geq 0,$$

for some $p > 0$ and all $\Psi \in C([-\tau, 0]; \mathbb{R}^m)$.

## 3   Main results

In this section, we investigate the global exponential stability for system (2.1). In order to facilitate the following proof, the definition of vertex Lyapunov functions set is given as follows.

**Definition 3.1.** The set $\{V_k(x_k) \in C^1(\mathbb{R}^{m_k}; \mathbb{R}_+^1), k \in \mathbb{L}\}$ is called a vertex Lyapunov functions set for (2.1) if the following conditions hold:

**V1.** There exist positive constants $p$, $\alpha_k$, $\beta_k$, such that

$$\alpha_k |x_k|^p \leq V_k(x_k) \leq \beta_k |x_k|^p. \tag{3.1}$$

**V2.** There exist constants $a_{kh} \geq 0$, $k, h \in \mathbb{L}$, positive constants $q > 1$, $\sigma_k$ and functions $F_{kh} : \mathbb{R}^{m_k} \times \mathbb{R}^{m_h} \to \mathbb{R}^1$, such that

$$
\begin{aligned}
V_k'(x_k) &\triangleq (V_k)_{x_k}' \left[ f_k(x_k(t), x_k(t - \tau_k), t) + \sum_{h=1}^{l} H_{kh}(x_h(t) - \gamma_h x_h(t - \tau_h)) \right] \\
&\leq -\sigma_k V_k(x_k(t) - \gamma_k x_k(t - \tau_k)) \\
&\quad + \sum_{h=1}^{l} a_{kh} F_{kh}(x_k(t) - \gamma_k x_k(t - \tau_k), x_h(t) - \gamma_h x_h(t - \tau_h)),
\end{aligned}
\tag{3.2}
$$

for all $t \geq 0$ and those $x_k(t) \in \mathbb{R}^{m_k}$, $k \in \mathbb{L}$, satisfying

$$V_k(x_k(t - \theta) - \gamma_k x_k(t - \theta - \tau_k)) < qV_k(x_k(t) - \gamma_k x_k(t - \tau_k)), \quad -\tau \leq \theta \leq 0.$$

**V3.** Along each directed cycle $\mathcal{C}_Q$ of weighted digraph $(\mathcal{G}, A)$, in which $A = (a_{kh})_{l \times l}$, there is

$$\sum_{(h,k) \in E(\mathcal{C}_Q)} F_{kh}(x_k, x_h) \leq 0, \quad \text{for all} \quad x_k \in \mathbb{R}^{m_k}, x_h \in \mathbb{R}^{m_h}. \tag{3.3}$$

For simplicity, fix any $\Psi \in C([-\tau, 0]; \mathbb{R}^m)$ and write $x_k(t) = x_k(t, \Psi)$. For $V_k(x_k) \in C^1(\mathbb{R}^{m_k}; \mathbb{R}_+^1)$ and positive constant $\gamma$, define

$$\Phi_k(t) = \max_{-\tau \leq \theta \leq 0} \{e^{\gamma(t+\theta)} V_k(x_k(t+\theta) - \gamma_k x_k(t+\theta-\tau_k))\}, \quad t \geq \tau,$$

and

$$D^+ \left( \sum_{k=1}^{l} c_k \Phi_k(t) \right) = \sum_{k=1}^{l} c_k \left( \limsup_{\Delta_0 \to 0^+} \frac{\Phi_k(t+\Delta_0) - \Phi_k(t)}{\Delta_0} \right). \tag{3.4}$$

In order to obtain our results, we establish the following lemma.

**Lemma 3.2.** *Suppose that system* (2.1) *admits a vertex Lyapunov functions set* $\{V_k(x_k), k \in \mathbb{L}\}$, *and that the digraph* $(\mathcal{G}, A)$ *is strongly connected. Then*

$$D^+ \left( \sum_{k=1}^{l} c_k \Phi_k(t) \right) \leq 0, \tag{3.5}$$

*where* $c_k$ *is the cofactor of the i-th diagonal element of the Laplacian matrix of* $(\mathcal{G}, A)$ *and* $\gamma < \min\{\sigma_1, \ldots, \sigma_l, \ln(q)/\tau\}$.

*Proof.* We fix $t \geq \tau$, define

$$\bar{\theta}_k = \max\{\theta \in [-\tau, 0] : e^{\gamma(t+\theta)} V_k(x_k(t+\theta) - \gamma_k x_k(t+\theta-\tau_k)) = \Phi_k(t)\}. \tag{3.6}$$

It follows easily that $\bar{\theta}_k \in [-\tau, 0]$ and

$$\Phi_k(t) = e^{\gamma(t+\bar{\theta}_k)} V_k(x_k(t+\bar{\theta}_k) - \gamma_k x_k(t+\bar{\theta}_k - \tau_k)).$$

Set $\Omega_1(k) = \{k \in \mathbb{L} : -\tau < \bar{\theta}_k < 0\}$, $\Omega_2(k) = \{k \in \mathbb{L} : \bar{\theta}_k = -\tau\}$, $\Omega_3(k) = \{k \in \mathbb{L} : \bar{\theta}_k = 0\}$. Then, we discuss inequality (3.5) as follows:

**Case 1.** If $k \in \mathbb{L}$ and $-\tau < \bar{\theta}_k < 0$, then $I_{\Omega_1(k)} = 1$. We observe from (3.6) that

$$e^{\gamma t} V_k(x_k(t) - \gamma_k x_k(t-\tau_k)) < e^{\gamma(t+\bar{\theta}_k)} V_k(x_k(t+\bar{\theta}_k) - \gamma_k x_k(t+\bar{\theta}_k - \tau_k)).$$

This implies immediately that by the continuity of $V_k(x_k(t) - \gamma_k x_k(t-\tau_k))$,

$$\lim_{\Delta \to 0^+} e^{\gamma(t+\Delta)} V_k(x_k(t+\Delta) - \gamma_k x_k(t+\Delta-\tau_k))$$
$$< e^{\gamma(t+\bar{\theta}_k)} V_k(x_k(t+\bar{\theta}_k) - \gamma_k x_k(t+\bar{\theta}_k - \tau_k))$$
$$+ \lim_{\Delta \to 0^+} \left[ \int_t^{t+\Delta} \left( e^{\gamma r} \sum_{h=1}^{l} a_{kh} F_{kh}(x_k(r) - \gamma_k x_k(r-\tau_k), x_h(r) - \gamma_h x_h(r-\tau_h)) \right) dr \right].$$

Consequently, there exists sufficiently small $\Delta_1 > 0$, such that

$$e^{\gamma(t+\Delta_1)} V_k(x_k(t+\Delta_1) - \gamma_k x_k(t+\Delta_1-\tau_k))$$
$$< e^{\gamma(t+\bar{\theta}_k)} V_k(x_k(t+\bar{\theta}_k) - \gamma_k x_k(t+\bar{\theta}_k - \tau_k))$$
$$+ \int_t^{t+\Delta_1} \left[ e^{\gamma r} \sum_{h=1}^{l} a_{kh} F_{kh}(x_k(r) - \gamma_k x_k(r-\tau_k), x_h(r) - \gamma_h x_h(r-\tau_h)) \right] dr. \tag{3.7}$$

**Case 2.** If $k \in \mathbb{L}$ and $\bar{\theta}_k = -\tau$, then $I_{\Omega_2(k)} = 1$ and

$$e^{\gamma t} V_k(x_k(t) - \gamma_k x_k(t - \tau_k)) < e^{\gamma(t-\tau)} V_k(x_k(t - \tau) - \gamma_k x_k(t - \tau - \tau_k)).$$

So,

$$\lim_{\Delta \to 0^+} e^{\gamma(t+\Delta)} V_k(x_k(t + \Delta) - \gamma_k x_k(t + \Delta - \tau_k))$$
$$< e^{\gamma(t-\tau)} V_k(x_k(t - \tau) - \gamma_k x_k(t - \tau - \tau_k))$$
$$+ \lim_{\Delta \to 0^+} \left[ \int_t^{t+\Delta} \left( e^{\gamma r} \sum_{h=1}^l a_{kh} F_{kh}(x_k(r) - \gamma_k x_k(r - \tau_k), x_h(r) - \gamma_h x_h(r - \tau_h)) \right) dr \right].$$

For sufficiently small $\Delta_2 > 0$, we then have

$$e^{\gamma(t+\Delta_2)} V_k(x_k(t + \Delta_2) - \gamma_k x_k(t + \Delta_2 - \tau_k))$$
$$< e^{\gamma(t-\tau)} V_k(x_k(t - \tau) - \gamma_k x_k(t - \tau - \tau_k)) \tag{3.8}$$
$$+ \int_t^{t+\Delta_2} \left[ e^{\gamma r} \sum_{h=1}^l a_{kh} F_{kh}(x_k(r) - \gamma_k x_k(r - \tau_k), x_h(r) - \gamma_h x_h(r - \tau_h)) \right] dr.$$

**Case 3.** If $k \in \mathbb{L}$ and $\bar{\theta}_k = 0$, then $I_{\Omega_3(k)} = 1$ and

$$e^{\gamma(t+\theta)} V_k(x_k(t + \theta) - \gamma_k x_k(t + \theta - \tau_k)) \le e^{\gamma t} V_k(x_k(t) - \gamma_k x_k(t - \tau_k)), \quad -\tau \le \theta \le 0.$$

But this means that

$$V_k(x_k(t + \theta) - \gamma_k x_k(t + \theta - \tau_k))$$
$$\le e^{\gamma \tau} V_k(x_k(t) - \gamma_k x_k(t - \tau_k))$$
$$< q V_k(x_k(t) - \gamma_k x_k(t - \tau_k)), \quad -\tau \le \theta \le 0.$$

Using condition V2, we obtain that

$$V_k'(x_k(t) - \gamma_k x_k(t - \tau_k))$$
$$\le -\sigma_k V_k(x_k(t) - \gamma_k x_k(t - \tau_k))$$
$$+ \sum_{h=1}^l a_{kh} F_{kh}(x_k(t) - \gamma_k x_k(t - \tau_k), x_h(t) - \gamma_h x_h(t - \tau_h)).$$

Integrate

$$\frac{d}{dr} e^{\gamma r} [V_k(x_k(r) - \gamma_k x_k(r - \tau_k))]$$

with respect to $r$ on $[t, t + \Delta_3]$ and use (3.2) to show that for $\Delta_3 > 0$,

$$e^{\gamma(t+\Delta_3)} [V_k(x_k(t + \Delta_3) - \gamma_k x_k(t + \Delta_3 - \tau_k))]$$
$$= e^{\gamma t} V_k(x_k(t) - \gamma_k x_k(t - \tau_k))$$
$$+ \int_t^{t+\Delta_3} e^{\gamma r} [V_k'(x_k(r) - \gamma_k x_k(r - \tau_k)) + \gamma V_k(x_k(r) - \gamma_k x_k(r - \tau_k))] dr$$
$$\le e^{\gamma t} V_k(x_k(t) - \gamma_k x_k(t - \tau_k)) + \int_t^{t+\Delta_3} e^{\gamma r} \left[ -(\sigma_k - \gamma) V_k(x_k(r) - \gamma_k x_k(r - \tau_k)) \right.$$

$$+ \sum_{h=1}^{l} a_{kh} F_{kh} \big( x_k(r) - \gamma_k x_k(r - \tau_k), x_h(r) - \gamma_h x_h(r - \tau_h) \big) \Big] dr$$

$$\leq e^{\gamma t} V_k(x_k(t) - \gamma_k x_k(t - \tau_k))$$

$$+ \int_t^{t+\Delta_3} e^{\gamma r} \sum_{h=1}^{l} a_{kh} F_{kh} \big( x_k(r) - \gamma_k x_k(r - \tau_k), x_h(r) - \gamma_h x_h(r - \tau_h) \big) dr.$$

By condition V3, (3.7) and (3.8), this yields that for sufficiently small $0 < \Delta_4 < \min\{\Delta_1, \Delta_2, \Delta_3\}$,

$$\sum_{k=1}^{l} c_k e^{\gamma(t+\Delta_4)} \big[ V_k(x_k(t+\Delta_4) - \gamma_k x_k(t+\Delta_4 - \tau_k)) \big]$$

$$= \sum_{k=1}^{l} c_k e^{\gamma(t+\Delta_4)} \big[ V_k(x_k(t+\Delta_4) - \gamma_k x_k(t+\Delta_4 - \tau_k)) \big] (I_{\Omega_1(k)} + I_{\Omega_2(k)} + I_{\Omega_3(k)})$$

$$\leq \sum_{k=1}^{l} c_k \Big[ e^{\gamma(t+\bar{\theta}_k)} V_k(x_k(t+\bar{\theta}_k) - \gamma_k x_k(t+\bar{\theta}_k - \tau_k)) I_{\Omega_1(k)}$$

$$+ e^{\gamma(t-\tau)} V_k(x_k(t-\tau) - \gamma_k x_k(t-\tau-\tau_k)) I_{\Omega_2(k)} + e^{\gamma t} V_k(x_k(t) - \gamma_k x_k(t-\tau_k)) I_{\Omega_3(k)} \Big]$$

$$+ \sum_{k=1}^{l} c_k \int_t^{t+\Delta_4} e^{\gamma r} \sum_{h=1}^{l} a_{kh} F_{kh} \big( x_k(r) - \gamma_k x_k(r-\tau_k), x_h(r) - \gamma_h x_h(r-\tau_h) \big) dr$$

$$= \sum_{k=1}^{l} c_k \Big[ e^{\gamma(t+\bar{\theta}_k)} V_k(x_k(t+\bar{\theta}_k) - \gamma_k x_k(t+\bar{\theta}_k - \tau_k)) I_{\Omega_1(k)}$$

$$+ e^{\gamma(t-\tau)} V_k(x_k(t-\tau) - \gamma_k x_k(t-\tau-\tau_k)) I_{\Omega_2(k)} + e^{\gamma t} V_k(x_k(t) - \gamma_k x_k(t-\tau_k)) I_{\Omega_3(k)} \Big]$$

$$+ \int_t^{t+\Delta_4} e^{\gamma r} \sum_{Q \in \mathbb{Q}} W(Q) \sum_{(h,k) \in E(C_Q)} F_{kh} \big( x_k(r) - \gamma_k x_k(r-\tau_k), x_h(r) - \gamma_h x_h(r-\tau_h) \big) dr$$

$$\leq \sum_{k=1}^{l} c_k \Big[ e^{\gamma(t+\bar{\theta}_k)} V_k(x_k(t+\bar{\theta}_k) - \gamma_k x_k(t+\bar{\theta}_k - \tau_k)) I_{\Omega_1(k)}$$

$$+ e^{\gamma(t-\tau)} V_k(x_k(t-\tau) - \gamma_k x_k(t-\tau-\tau_k)) I_{\Omega_2(k)} + e^{\gamma t} V_k(x_k(t) - \gamma_k x_k(t-\tau_k)) I_{\Omega_3(k)} \Big]$$

$$\leq \sum_{k=1}^{l} c_k \Psi_k(t),$$

where $\mathbb{Q}$ is the set of all spanning unicyclic graphs of $(\mathcal{G}, A)$, $C_Q$ denotes the directed cycle of $Q$. We therefore must have

$$\sum_{k=1}^{l} c_k \Phi_k(t + \Delta) \leq \sum_{k=1}^{l} c_k \Phi_k(t)$$

for $\triangle > 0$ sufficiently small. We then have (3.5) holds. $\square$

**Theorem 3.3.** *Let conditions of Lemma 3.2 hold. If*

$$\gamma_k^p < 2^{1-p}, \ k \in \mathbb{L}, \tag{3.9}$$

*then the trivial solution of system (2.1) is globally exponentially stable.*

*Proof.* By using Lemma 3.2, we can obtain

$$\sum_{k=1}^{l} c_k \left( \limsup_{\Delta_0 \to 0^+} \frac{\Phi_k(t + \Delta_0) - \Phi_k(t)}{\Delta_0} \right) \leq 0.$$

This, together with condition V1, implies

$$\sum_{k=1}^{l} c_k \alpha_k e^{\gamma t} |x_k(t) - \gamma_k x_k(t - \tau_k)|^p$$

$$\leq \sum_{k=1}^{l} c_k e^{\gamma t} V_k(x_k(t) - \gamma_k x_k(t - \tau_k))$$

$$\leq \sum_{k=1}^{l} c_k \max_{-\tau \leq \theta \leq 0} \left\{ e^{\gamma(t+\theta)} V_k(x_k(t + \theta) - \gamma_k x_k(t + \theta - \tau_k)) \right\}$$

$$= \sum_{k=1}^{l} c_k \Phi_k(t)$$

$$\leq \sum_{k=1}^{l} c_k \Phi_k(\tau)$$

$$= \sum_{k=1}^{l} c_k \max_{-\tau \leq \theta \leq 0} \left\{ e^{\gamma(\theta+\tau)} V_k(x_k(\theta + \tau) - \gamma_k x_k(\theta + \tau - \tau_k)) \right\}$$

$$\leq \sum_{k=1}^{l} c_k \beta_k e^{\gamma \tau} \max_{-\tau \leq \theta \leq 0} |x_k(\theta + \tau) - \gamma_k x_k(\theta + \tau - \tau_k)|^p.$$

By using the inequality

$$|a + b|^p \leq 2^{p-1}(|a|^p + |b|^p),$$

we compute that

$$c_k \alpha_k e^{\gamma t} |x_k(t)|^p \leq 2^{p-1} c_k \alpha_k e^{\gamma t} |x_k(t) - \gamma_k x_k(t - \tau_k)|^p + 2^{p-1} c_k \alpha_k e^{\gamma t} |\gamma_k x_k(t - \tau_k)|^p. \qquad (3.10)$$

By (3.10), we can easily show that

$$\sum_{k=1}^{l} c_k \alpha_k e^{\gamma t} |x_k(t)|^p$$

$$\leq 2^{p-1} \sum_{k=1}^{l} c_k \alpha_k e^{\gamma t} |x_k(t) - \gamma_k x_k(t - \tau_k)|^p + 2^{p-1} \sum_{k=1}^{l} c_k \alpha_k e^{\gamma t} |\gamma_k x_k(t - \tau_k)|^p$$

$$\leq 2^{p-1} \sum_{k=1}^{l} \left( c_k \beta_k e^{\gamma \tau} \sup_{-\tau \leq s \leq 0} |x_k(s + \tau) - \gamma_k x_k(s + \tau - \tau_k)|^p + c_k \alpha_k \gamma^p e^{\gamma t} \sup_{-\tau \leq s \leq t} |x_k(s)|^p \right),$$

where $\gamma = \max\{\gamma_1, \gamma_2, \ldots, \gamma_l\}$. It follows from the fact that the function

$$m(t) = 2^{p-1} \sum_{k=1}^{l} \left( c_k \beta_k e^{\gamma \tau} \sup_{-\tau \leq s \leq 0} |x_k(s + \tau) - \gamma_k x_k(s + \tau - \tau_k)|^p + c_k \alpha_k \gamma^p e^{\gamma t} \sup_{-\tau \leq s \leq t} |x_k(s)|^p \right),$$

is increasing that

$$\sum_{k=1}^{l} c_k \alpha_k e^{\gamma t} \sup_{-\tau \le s \le t} |x_k(s)|^p$$

$$\le 2^{p-1} \sum_{k=1}^{l} c_k \beta_k e^{\gamma \tau} \sup_{-\tau \le s \le 0} |x_k(s+\tau) - \gamma_k x_k(s+\tau-\tau_k)|^p$$

$$+ 2^{p-1} \gamma^p \sum_{k=1}^{l} c_k \alpha_k e^{\gamma t} \sup_{-\tau \le s \le t} |x_k(s)|^p.$$

Consequently, by (3.9), we obtain that

$$\sum_{k=1}^{l} c_k \alpha_k e^{\gamma t} \sup_{-\tau \le s \le t} |x_k(s)|^p \le \frac{2^{p-1} \sum_{k=1}^{l} c_k \beta_k e^{\gamma \tau}}{1 - 2^{p-1} \rho^p} \sup_{-\tau \le s \le 0} |x_k(s+\tau) - \gamma_k x_k(s+\tau-\tau_k)|^p.$$

We therefore must have

$$\sum_{k=1}^{l} c_k \alpha_k |x_k(t)|^p \le \frac{2^{p-1} \sum_{k=1}^{l} c_k \beta_k e^{\gamma \tau}}{1 - 2^{p-1} \rho^p} \sup_{-\tau \le s \le 0} |x_k(s+\tau) - \gamma_k x_k(s+\tau-\tau_k)|^p e^{-\gamma t}.$$

As the digraph $(\mathcal{G}, A)$ is strongly connected, we obtain that $c_k > 0$. Therefore,

$$c_k \alpha_k > 0.$$

Consequently,

$$\sum_{k=1}^{l} c_k \alpha_k |x_k(t)|^p \ge \min_{1 \le k \le l} \{c_k \alpha_k\} \sum_{k=1}^{l} |x_k(t)|^p.$$

Therefore, we must have

$$\sum_{k=1}^{l} |x_k(t)|^p \le \frac{2^{p-1} \sum_{k=1}^{l} c_k \beta_k e^{\gamma \tau}}{\min_{1 \le k \le l} \{c_k \alpha_k\}(1 - 2^{p-1} \rho^p)} \sup_{-\tau \le s \le 0} |x_k(s+\tau) - \gamma_k x_k(s+\tau-\tau_k)|^p e^{-\gamma t}.$$

The proof is complete. □

**Remark 3.4.** Theorem 3.3 proves that the Lyapunov function $V(x)$ for system (3.3) is obtained by weighted sum of $V_k(x_k)$, and hence finding the vertex Lyapunov functions set for system (3.3) is a key point in the study of stability for system (3.3). In practice, coupled systems are very complex. To make progress, different fields have suppressed certain complications. For example, in nonlinear dynamics the simple and nearly identical dynamical systems are coupled together in simple, regular ways. These simplifications make that any issue of structural complexity is avoided and the system's potentially formidable dynamics could be studied intensively. In many application fields, the Lyapunov functions for specific system have been obtained by other researchers. Hence, in this paper the Lyapunov functions for specific system can be chosen as the $V_k(x_k)$.

In fact, we can obtain some better results, if anther condition on topology property of the coupled systems is added. Note that if $(\mathcal{G}, A)$ is balanced, then

$$\sum_{k,h=1}^{l} c_k a_{kh} F_{kh}(x_k, x_h) = \frac{1}{2} \sum_{\mathcal{Q} \in \mathbb{Q}} W(\mathcal{Q}) \sum_{(h,k) \in E(C_{\mathcal{Q}})} [F_{kh}(x_k, x_h) + F_{hk}(x_h, x_k)].$$

In this case, condition V3 is replaced by the following:

$$\sum_{(h,k)\in E(C_Q)} [F_{kh}(x_k, x_h) + F_{hk}(x_h, x_k)] \leq 0. \tag{3.11}$$

Consequently, we obtain the following corollary:

**Corollary 3.5.** *Suppose that $(\mathcal{G}, A)$ is balanced. Then the conclusion of Theorem 3.3 holds if (3.3) is replaced by (3.11).*

Since the previous results are based on vertex Lyapunov functions set for system (2.1), the stability criteria are not very convenient to be verified for a given system. We then establish some sufficient conditions for stability of system (2.1) by using coefficients of system (2.1).

**Theorem 3.6.** *Suppose that the following conditions hold.*

*A1. The digraph $(\mathcal{G}, A)$ is strongly connected, and there are positive constants $\xi_k$, $k \in \mathbb{L}$, such that*

$$(x_k(t) - \gamma_k(x_k(t - \tau_k)))^T f_k(x_k(t), x_k(t - \tau_k), t) \leq -\xi_k |x_k(t) - \gamma_k x_k(t - \tau_k)|^2. \tag{3.12}$$

*A2. There are constants $A_{kh}(k, h \in \mathbb{L})$, such that*

$$|H_{kh}(x)| \leq A_{kh}|x|. \tag{3.13}$$

*A3. It holds that*

$$\xi_k - \sum_{h=1}^{l} A_{kh} > 0, \quad \gamma_k^p < 2^{1-p}, \quad k \in \mathbb{L}. \tag{3.14}$$

*Then the trivial solution of (2.1) is globally exponentially stable.*

*Proof.* Define functions $V_k(x_k) = |x_k|^2$, write $f_k = f_k(x_k(t), x_k(t - \tau_k), t)$, $H_{kh} = H_{kh}(x_h(t) - \gamma_h x_h(t - \tau_h))$. By conditions A1 and A2, we calculate $\dot{V}_k$ as follows:

$$\dot{V}_k(x_k) = 2[x_k(t) - \gamma_k x_k(t - \tau_k)]^T \left( f_k + \sum_{h=1}^{l} H_{kh} \right)$$

$$\leq 2[x_k(t) - \gamma_k x_k(t - \tau_k)]^T f_k + 2[x_k(t) - \gamma_k x_k(t - \tau_k)]^T \sum_{h=1}^{l} |H_{kh}|$$

$$\leq -2\xi_k |x_k(t) - \gamma_k x_k(t - \tau_k)|^2 + 2|x_k(t) - \gamma_k x_k(t - \tau_k)| \sum_{h=1}^{l} |H_{kh}|.$$

We have

$$2|x_k(t) - \gamma_k x_k(t - \tau_k)| \sum_{h=1}^{l} |H_{kh}|$$

$$\leq 2|x_k(t) - \gamma_k x_k(t - \tau_k)| \sum_{h=1}^{l} A_{kh}|x_h(t) - \gamma_h x_h(t - \tau_h)|$$

$$= 2 \sum_{h=1}^{l} A_{kh}|x_k(t) - \gamma_k x_k(t - \tau_k)||x_h(t) - \gamma_h x_h(t - \tau_h)| \tag{3.15}$$

$$\leq \sum_{h=1}^{l} A_{kh}|x_k(t) - \gamma_k x_k(t - \tau_k)|^2 + \sum_{h=1}^{l} A_{kh}|x_h(t) - \gamma_h x_h(t - \tau_h)|^2.$$

By (3.15), it implies that

$$
\begin{aligned}
\dot{V}_k(x_k) \leq{}& \left(-2\xi_k + \sum_{h=1}^{l} A_{kh}\right) |x_k(t) - \gamma_k x_k(t - \tau_k)|^2 + \sum_{h=1}^{l} A_{kh} |x_h(t) - \gamma_h x_h(t - \tau_h)|^2 \\
\leq{}& -2\left(\xi_k - \sum_{h=1}^{l} A_{kh}\right) |x_k(t) - \gamma_k x_k(t - \tau_k)|^2 \\
& + \sum_{h=1}^{l} A_{kh} \left(|x_h(t) - \gamma_h x_h(t - \tau_h)|^2 - |x_k(t) - \gamma_k x_k(t - \tau_k)|^2\right) \\
={}& -\sigma_k |x_k(t) - \gamma_k x_k(t - \tau_k)|^2 \\
& + \sum_{h=1}^{l} a_{kh} F_{kh} \left(x_k(t) - \gamma_k x_k(t - \tau_k), x_h(t) - \gamma_h x_h(t - \tau_h)\right),
\end{aligned}
\tag{3.16}
$$

where $\sigma_k = 2\xi_k - 2\sum_{h=1}^{l} A_{kh}$, $a_{kh} = A_{kh}$, $F_{kh}(x_k, x_h) = |x_h|^2 - |x_k|^2$.

Hence, V1, V2 and V3 in Definition 3.1 hold and $\{V_k(x_k),\ k \in \mathbb{L}\}$ is a vertex Lyapunov functions set for system (2.1). Clearly, (3.16) holds and all conditions of Corollary 3.5 are satisfied. The proof is complete. $\qquad\square$

## 4 Numerical test

In this section, we provide a numerical example to illustrate the effectiveness of the proposed criteria in this paper.

Give a digraph $\mathcal{G}$ with 16 vertices. Consider the following coupled retarded dynamical system:

$$
\frac{d\left[x_k(t) - \gamma_k x_k(t - \tau_k)\right]}{dt} = f_k(x_k(t), x_k(t - \tau_k), t) + \sum_{h=1}^{16} H_{kh}(x_h(t) - \gamma_h x_h(t - \tau_h)),
$$
$$
t > 0, \quad k = 1, 2, \dots, 16,
$$

where $\tau_k \geq 0, x_k(t) \in \mathbb{R}, \gamma_k \geq 0, H_{kh}(y) = \mu_{kh} y$.

Now, let

$$
f_k(x_k(t), x_k(t - \tau_k), t) = F(y) = \begin{cases} -\left(y - \frac{y^3}{p_k}\right), & y \leq 1 \\ -q_k y^3, & y > 1, \end{cases}
$$

where $y = x_k(t) - \gamma_k x_k(t - \tau_k)$, and $p_k > 0$, $q_k > 0$, then we consider the following coupled differential delay system (see Fig. 4.1):

$$
\frac{d\left[x_k(t) - \gamma_k x_k(t - \tau_k)\right]}{dt} = \left[F(y) + \sum_{h=1}^{16} H_{kh}(x_h(t) - \gamma_h x_h(t - \tau_h))\right],
\tag{4.1}
$$
$$
k = 1, 2, \dots, 16.
$$

For simulation, we assume that the parameters in (4.1) are given as follows:

$$
\begin{array}{c|c}
p_1 = p_2 = \cdots = p_6 = 2 & q_1 = q_2 = \cdots = q_6 = \frac{1}{2} \\
p_7 = p_8 = \cdots = p_{11} = 3 & q_7 = q_8 = \cdots = q_{11} = \frac{2}{3} \\
p_{12} = p_{13} = \cdots = p_{16} = 6 & q_{12} = q_{13} = \cdots = q_{16} = \frac{5}{6}
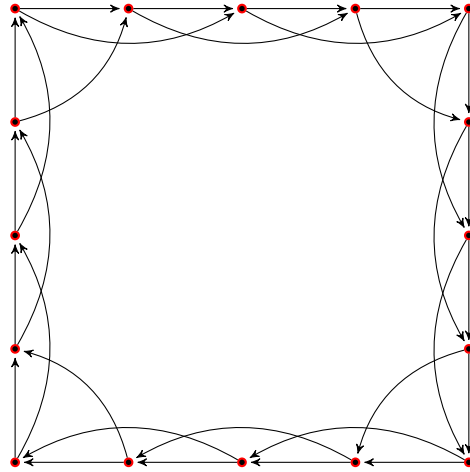\end{array},
$$

Figure 4.1: A complex network.

$$\Xi = (\mu_{kh})_{16 \times 16} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0.1 & 0.1 \\ 0.1 & 0 & 0 & \cdots & 0 & 0 & 0.1 \\ 0.1 & 0.1 & 0 & \cdots & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & \cdots & 0.1 & 0.1 & 0 \end{bmatrix}.$$

   With the above parameters, we can choose $\xi_1 = \xi_2 = \cdots = \xi_6 = 0.5$, $\xi_7 = \xi_8 = \cdots = \xi_{11} = 0.45$, $\xi_{12} = \xi_{13} = \cdots = \xi_{16} = 0.3$. Hence, we have the condition that $\xi_k - \sum_{h=1}^{16} A_{kh} > 0$ holds. Based on the theorem above, we could check that all conditions in Theorem 3.6 are satisfied, which can guarantee the exponential stability for system (4.1). In the end, the simulation results of stable solutions to (4.1) corresponding to weighted digraphs $(\mathcal{G}, \Xi)$ are shown in Fig. 4.2.
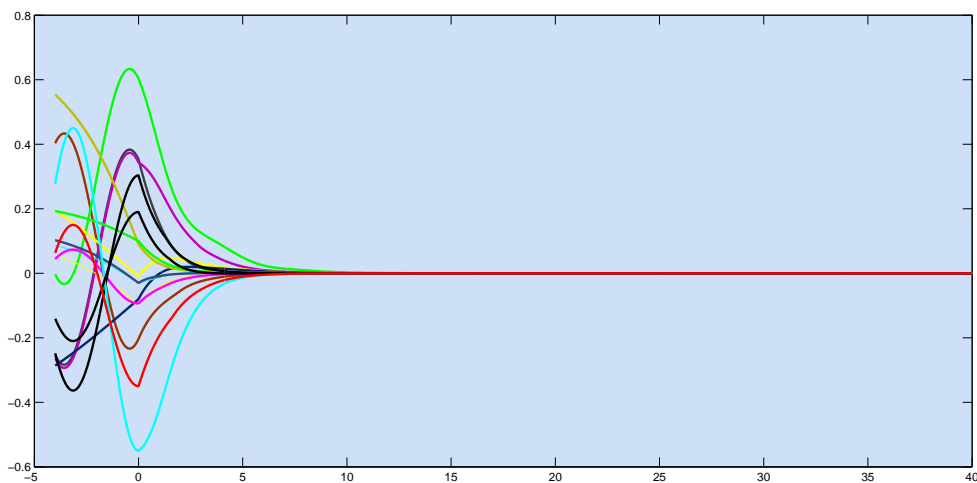


Figure 4.2: Numeric simulation of global exponential stability of (4.1) with $\Xi$.

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