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On the continuity of the state constrained minimal time function

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Abstract. We obtain results on the propagation of the (Lipschitz) continuity of the minimal time function associated with a finite dimensional autonomous differential inclusion with state constraints and a closed target. To this end, we first obtain new regularity results of the solution map with respect to initial data.

Keywords: regularity of solutions, minimal time function, state constraints.

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1 Introduction

Let S be a nonempty subset of $\mathbb{R}^p$, F a multifunction mapping S to nonempty subsets of $\mathbb{R}^p$ and consider the state constrained differential inclusion

$$y'(t) \in F(y(t)). \tag{1.1}$$

A solution of (1.1) on $[0, T]$ is an absolutely continuous function $y\colon [0, T] \to S$ that satisfies $y'(t) \in F(y(t))$ for a.e. $t \in [0, T]$. A solution of (1.1) on a semi-open interval $[0, T)$ is defined similarly.

The S-constrained minimal time problem associated to a nonempty subset Σ of S (called the *target set*) is the problem in which the goal is to steer an initial point $x \in S$ to Σ along a solution of (1.1) in minimal time. The minimal time value is denoted by $T(x)$, which is defined to be $+\infty$ if no solution of (1.1) from x can reach Σ. The function T is called the S-constrained minimal time function. When $S = \mathbb{R}^p$, T coincides with the well known (unconstrained) minimal time function associated with the target Σ. In this paper we study continuity properties of the S-constrained minimal time function.

The regularity properties of the minimal time function, being strongly connected to controllability properties of the system, have been the object of an extensive literature. For more

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details on controllability see, e.g., [3,12]. The Lipschitz continuity of the unconstrained minimal time function associated to a point target was first studied in [29]. In that paper, Petrov introduced a necessary and sufficient condition, called Petrov condition, for the Lipschitz continuity of the minimal time function in a neighborhood of the origin. That result was extended later to more general target sets in [4,32]. In [33], Veliov obtained a necessary and sufficient condition for the local Lipschitz continuity of the unconstrained minimal time function for closed target sets, when the multifunction F is nonautonomous and depends measurably on time. In [35], in the absence of constraints, Wolenski and Zhuang showed that the Lipschitz continuity of the minimal time function near the target Σ is equivalent to the boundedness of the proximal subgradient of the minimal time function on Σ.

For the state constrained case we mention the paper [28], where the authors generalize the results obtained for the unconstrained minimal time function in [35]. They gave necessary and sufficient conditions for the proto-Lipschitzness of T (the definition is given in Section 3), imposing some geometric assumptions for the pair (Σ, S) (the admissibility of Σ for S and conditions involving points near Σ which are exterior to S). Moreover, under further geometric assumptions on S, in [28] there are given necessary and sufficient conditions for T to be Lipschitz on a neighborhood of Σ in S.

In [27], a Petrov type condition is provided for the state constrained minimal time function T to be proto-Lipschitz. More exactly, the following result is proved.

Theorem 1.1. *Let $F\colon S \to \mathbb{R}^p$ be an upper semi-continuous multifunction with nonempty compact convex values, S a nonempty closed subset of $\mathbb{R}^p$ and Σ a closed subset of S. Suppose that there exist $\rho > 0$ and $\gamma > 0$ such that*

$$\inf_{s\in\pi_\Sigma(x)} \inf_{u\in F(x)\cap\mathcal{T}_S(x)} \langle x-s, u\rangle \le -\gamma d_\Sigma(x) \tag{1.2}$$

for all $x \in S \cap (\Sigma + \rho B)$. Then the S-constrained minimal time function T is proto-Lipschitz.

We denoted by $\pi_\Sigma(x)$ the set of projections of x on Σ and $\mathcal{T}_S(x)$ is the Bouligand tangent cone to S at x. Moreover, in [27] there are given examples where the hypothesis of Theorem 1.1 holds, but the geometric conditions from [28] are not satisfied.

This paper is a continuation of [27] and its goal is to get the propagation of the continuity of the state constrained minimal time function T around the target to the whole reachable set, without imposing explicitly the geometric assumptions from [28]. Instead, we use some regularity properties of the multifunction

$$S \ni x \rightsquigarrow F(x) \cap \mathcal{T}_S(x). \tag{1.3}$$

The propagation of the continuity properties of the S-constrained minimal time function was previously discussed in [9] and [17]. In [9] the authors considered the control system $y' \in f(t, y, U)$ with state constraints and proved the Lipschitz continuity of T under Lipschitz hypotheses on f and some regularity assumptions on the set of constraints. In that paper, the set of constraints is the closure of an open set $\Omega \subset \mathbb{R}^p$ and the target Σ is a subset of Ω. In our paper we require only that $\Sigma \subset S$ with S a closed subset of $\mathbb{R}^p$ and we do not assume Lipschitz continuity of F. In [17] we imposed that $F(x) \subset \mathcal{T}_S(x)$ for any $x \in S$, which, in fact, implies the invariance of S with respect to the solutions of the differential inclusion $y' \in F(y)$. For other results on the propagation of continuity properties of the minimal time function, in the absence of constraints, see, [10,13,15,35]. A key role in obtaining these results is played by the dependence of the solutions on the initial conditions. In this paper, in order to obtain the propagation results, we first prove a Filippov type result for our state constrained differential inclusion (1.1), which is a main result of the paper.

In the absence of state constraints, we recall the celebrated Filippov theorem and various extensions of it, under different frames and assumptions on F (see, e.g., [1,2,14,16,21,22,24,36]).

There are also many papers on Filippov type results, in the state constrained case. We recall the paper of Frankowska and Rampazzo [25], where there are given Filippov and Filippov–Wazewski theorems in the case when the state variable is constrained to the closure of an open subset of $\mathbb{R}^n$. Nour and Stern [28], while investigating the Lipschitz continuity of the minimal time function, established the Lipschitz dependence of the solutions of (1.1) on the initial data, under Lipschitz hypothesis on F and certain assumptions on S. In [8], Bressan and Facchi established a result of this type, assuming that S is compact and convex, F is Lipschitz and satisfies a strict inward pointing condition at every boundary point $x \in \partial S$, that is

$$\operatorname{co} F(x) \cap \operatorname{int} \mathcal{T}_S(x) \neq \emptyset. \tag{1.4}$$

Filippov type results were also obtained in [5,7]. We want to remark that in all the papers above, the Filippov type results were obtained under the Lipschitz hypothesis on F.

In this paper, we prove a Filippov type result for our state constrained differential inclusion (1.1), avoiding explicit geometric assumptions on S or Σ and using regularity properties of the multifunction defined by (1.3). We give examples that do not satisfy the conditions imposed in [28] and/or [8], but satisfy our hypotheses. It is important to remark that the technique for obtaining our result, by viability, was used for the first time in [14], for a semilinear system, with F Lipschitz. This technique was also used in [16,17,31]. It requires the convexity of the values of F, as it was remarked also in [31]. From this point of view, the Filippov type results of this paper are new compared to the previous ones, because this technique of the proof allows us to weaken the Lipschitz conditions; moreover, they are new and important even in the absence of state constraints. However, by these results we relax the Lipschitz hypothesis, but we impose F to have convex values.

2 Preliminaries

For any subset $K \subseteq \mathbb{R}^p$ we denote by $\operatorname{int} K$ the interior of K, $\overline{K}$ the closure of K, $\pi_K(x)$ the set of projections of $x \in \mathbb{R}^p$ in K and by $d_K(x)$ the Euclidean distance from x to the set K. The open unit ball is denoted by B.

A vector $\eta \in \mathbb{R}^p$ is tangent to the set K at a point $\xi \in K$ if

$$\liminf_{h \downarrow 0} \frac{1}{h} d_K(\xi + h\eta) = 0.$$

We denote by $\mathcal{T}_K(\xi)$ the set of all tangent vectors to K at $\xi \in K$. For each $\xi \in K$, the set $\mathcal{T}_K(\xi)$ is a closed cone. A well-known characterization by sequences is the following: $\eta \in \mathcal{T}_K(\xi)$ if and only if there exist two sequences $(h_n)_n$ in $\mathbb{R}_+$ with $h_n \downarrow 0$ and $(q_n)_n$ in $\mathbb{R}^p$ with $\lim_{n\to\infty} q_n = \eta$ such that $\xi + h_n q_n \in K$ for each $n \in \mathbb{N}$.

We recall that a closed set K is called *sleek* if the multifunction

$$K \ni x \rightsquigarrow \mathcal{T}_K(x)$$

is lower semicontinuous. For more details on tangent cones we refer for instance to [2].

Let $\mathcal{F}\colon \mathcal{K} \rightsquigarrow \mathbb{R}^p$ be a given multifunction and consider the differential inclusion

$$w'(t) \in \mathcal{F}(w(t)). \tag{2.1}$$

The set $\mathcal{K}$ is viable with respect to $\mathcal{F}$ if for each $\xi \in \mathcal{K}$ there exists $\theta > 0$ such that (2.1) has at least one solution $w\colon [0,\theta] \to \mathcal{K}$ with $w(0) = \xi$.

The following viability theorem can be found, for instance, in [1,2,18].

Theorem 2.1. *Let $\mathcal{K}$ be a nonempty, locally closed subset in $\mathbb{R}^p$ and let $\mathcal{F}\colon \mathcal{K} \rightsquigarrow \mathbb{R}^p$ be an upper semicontinuous multifunction with nonempty, compact and convex values. A necessary and sufficient condition in order that $\mathcal{K}$ be viable with respect to $\mathcal{F}$ is the following tangency condition:*

$$\mathcal{F}(\xi) \cap \mathcal{T}_{\mathcal{K}}(\xi) \neq \emptyset \tag{2.2}$$

for each $\xi \in \mathcal{K}$.

The following conditions on a multifunction, weaker than the Lipschitz continuity, introduced in [20,23], will be used in the next sections of the paper.

Definition 2.2. A multifunction $G\colon K \rightsquigarrow \mathbb{R}^p$ is said to be
1) *one-sided Lipschitz* of constant L if for any $x, y \in K$, any $v \in F(x)$ there exists $w \in F(y)$ such that

$$\langle x - y, v - w \rangle \leq L \|x - y\|^2 .$$

2) *one-sided Perron* if for any $x, y \in K$, any $v \in F(x)$ there exists $w \in F(y)$ such that

$$\langle x - y, v - w \rangle \leq \vartheta(\|x - y\|) \|x - y\| ,$$

where $\vartheta\colon [0,\infty) \to [0,\infty)$ is a Perron function.

By a *Perron* function we mean a continuous function $\vartheta\colon [0,\infty) \to [0,\infty)$ with $\vartheta(0) = 0$ such that the differential equation $z' = \vartheta(z)$ has the null function as the unique solution with $z(0) = 0$. This function was introduced by Perron in [30]. It is clear that the class of one-sided Perron multifunctions is larger than the class of one-sided Lipschitz ones.

3 Lipschitz continuity of the state constrained minimal time function

Let $S \subset \mathbb{R}^p$ be a closed nonempty set and let $\Sigma \subset S$ be a closed subset. The S-constrained minimal time function $T\colon S \to [0, +\infty]$ is defined by

$$T(x) = \inf\{\tau \geq 0;\ \text{there exists a solution } \ y \text{ of (1.1) with } y(0) = x,\ y(\tau) \in \Sigma\}.$$

If no solution from x can reach Σ then $T(x) = +\infty$. We denote by $\mathcal{R}$ the set of all points $x \in S$ such that $T(x) < +\infty$.

Following [28], the minimal time function T is said to be *proto-Lipschitz* if there exist $\rho > 0$ and $M > 0$ such that

$$T(x) \leq M d_\Sigma(x)$$

for all $x \in (\Sigma + \rho B) \cap S$.

In the same spirit, we say that T is *proto-continuous* if there exist $\rho > 0$ and $\omega\colon [0,\rho] \to [0,+\infty)$ such that $\lim_{s\to 0^+} \omega(s) = 0$ and

$$T(x) \le \omega(d_\Sigma(x))$$

for all $x \in (\Sigma + \rho B) \cap S$. As it is proved in [3, p. 229], when Σ is closed with compact boundary, T is proto-continuous iff it is continuous in each point of Σ.

In the case when ω is of type $\omega(s) = Ms^\alpha$, with $0 < \alpha < 1$, $M > 0$, we say that T is *proto-Hölder continuous*.

We define the multifunction $G\colon S \rightsquigarrow \mathbb{R}^p$ by

$$G(x) = F(x) \cap \mathcal{T}_S(x) \tag{3.1}$$

and we impose some regularity properties for G in order to obtain the Lipschitz/continuous dependence of the solutions of (1.1) on the initial data, that is the key for the propagation Theorems 3.4 and 4.5.

First, we give an extension of the Filippov theorem, on the Lipschitz dependence of the solutions of (1.1) on the initial data, in the state constraints case. The proof is based on the viability Theorem 2.1 with $\mathcal{F}$ and $\mathcal{K}$ appropriately chosen as in [17, Theorem 2.1].

Theorem 3.1. *Let $F\colon S \rightsquigarrow \mathbb{R}^p$ be an upper semicontinuous multifunction, with convex and compact values. Assume that G, defined by (3.1), has nonempty convex values, is lower semicontinuous and one-sided Lipschitz of constant L. Then, for any $x_1, x_2 \in S$, any solution $y_1\colon [0,\sigma] \to S$ of (1.1) with $y_1(0) = x_1$, there exists a solution $y_2\colon [0,\sigma] \to S$ of (1.1) with $y_2(0) = x_2$ such that*

$$\|y_1(t) - y_2(t)\| \le e^{Lt}\,\|x_1 - x_2\| \tag{3.2}$$

for all $t \in [0,\sigma]$.

Proof. Let $x_1, x_2 \in S$ and let $y_1\colon [0,\sigma] \to S$ be a solution of (1.1) with $y_1(0) = x_1$. Since G has nonempty values, we can apply Theorem 2.1 to conclude that the solution y_1 can be continued up to a noncontinuable one, denoted also $y_1\colon [0,\sigma_1) \to S$, $\sigma_1 > \sigma$. Consider the space $\mathcal{X} = \mathbb{R}^{p+2}$, the set

$$\mathcal{K} = \{(\tau, x, \lambda) \in [0,\sigma_1) \times S \times \mathbb{R};\ \|y_1(\tau) - x\| \le \lambda\}$$

and the multifunction $\mathcal{F} : \mathcal{K} \to \mathcal{X}$ defined by

$$\mathcal{F}(\tau, x, \lambda) = \{1\} \times F(x) \times \{L\lambda\}\,.$$

We shall prove that the tangency condition

$$\mathcal{T}_\mathcal{K}(\tau, x, \lambda) \cap \mathcal{F}(\tau, x, \lambda) \ne \emptyset \tag{3.3}$$

holds for any $(\tau, x, \lambda) \in \mathcal{K}$. To this end, we show that there exists $w \in F(x)$ such that $(1, w, L\lambda) \in \mathcal{T}_\mathcal{K}(\tau, x, \lambda)$. Indeed, let $(\tau, x, \lambda) \in \mathcal{K}$, hence $\|y_1(\tau) - x\| \le \lambda$. By a result of Wazewski [34, p. 866] (see also [19, Proposition 1]), there exists $v \in F(y_1(\tau))$ and a sequence $(h_n)_n \subset [0,\sigma_1)$, $h_n \downarrow 0$, such that the sequence $v_n := (y_1(\tau + h_n) - y_1(\tau))/h_n$ converges to v. Moreover, we have that $y_1(\tau) + h_n v_n \in S$ for n sufficiently large, i.e., $v \in \mathcal{T}_S(y_1(\tau))$. Hence $v \in G(y_1(\tau))$. Using now the one-sided Lipschitz property of G, we get $w \in G(x)$ such that

$$\langle y_1(\tau) - x, v - w\rangle \le L\,\|y_1(\tau) - x\|^2\,.$$

As G is lower semicontinuous and has closed convex values, by Michael's selections theorem and Peano's existence result, there exists an $y(\cdot)$ solution of (1.1) with $y(0) = x$ such that $w_n := (y(h_n) - x)/h_n$ converges to w and $x + h_n w_n \in S$, for every $n \in \mathbb{N}$. We have that

$$\|y_1(\tau + h_n) - (x + h_n w_n)\| \leq \lambda + h_n L\lambda + h_n r_n,$$

where

$$r_n = \left\| \frac{y_1(\tau + h_n) - y_1(\tau)}{h_n} - v \right\| + \frac{\|y_1(\tau) - x + h_n(v - w)\| - \|y_1(\tau) - x\|}{h_n} \\ + \|w_n - w\| - \frac{\langle y_1(\tau) - x, v - w\rangle}{\|y_1(\tau) - x\|},$$

and $(r_n)_n$ converges to 0. So, we obtained that

$$(\tau + h_n, x + h_n w_n, \lambda + h_n L\lambda + h_n r_n) \in \mathcal{K}$$

for every $n \in \mathbb{N}$, hence the tangency condition (3.3) holds. Then, by Theorem 2.1, the set $\mathcal{K}$ is viable with respect to $\mathcal{F}$. Since $(0, x_2, \|x_1 - x_2\|) \in \mathcal{K}$, there exist $\theta > 0$ and a solution $w = (t, y, z)$ of the problem $w' \in \mathcal{F}(w)$, on $[0, \theta]$, with $w(0) = (0, x_2, \|x_1 - x_2\|)$, such that $(t(s), y(s), z(s)) \in \mathcal{K}$ for all $s \in [0, \theta]$. It is easy to see that $t(s) = s$, y is a solution of (1.1) with $y(0) = x_2$ and $z(s) = e^{Ls}\|x_1 - x_2\|$. Hence, on $[0, \theta]$, we have that

$$\|y_1(s) - y(s)\| \leq e^{Ls}\|x_1 - x_2\|.$$

By usual continuation arguments, there exists a solution $\overline{y}\colon [0, c) \to S$ of (1.1) with $\overline{y}(0) = x_2$ such that

$$\|y_1(s) - \overline{y}(s)\| \leq e^{Ls}\|x_1 - x_2\| \tag{3.4}$$

for all $s \in [0, c)$, noncontinuable with this property. Finally, we shall prove that $c = \sigma_1$. Assume by contradiction that $c < \sigma_1$. By (3.4) we have that $\overline{y}$ is bounded on $[0, c)$ and, since F is compact valued, we have that there exists $y^* := \lim_{s\uparrow c} \overline{y}(s)$, which belongs to the closed set S. Moreover, by (3.4) we get that $\|y_1(c) - y^*\| \leq e^{Lc}\|x_1 - x_2\|$. Applying now Theorem 2.1 for $(c, y^*, e^{Lc}\|x_1 - x_2\|) \in \mathcal{K}$ we obtain that $\overline{y}$ can be continued to the right of c with property (3.4), which contradicts the maximality of $\overline{y}$. Hence $c = \sigma_1$. In conclusion, there exists a noncontinuable solution $y_2\colon [0, \sigma_2) \to S$, $\sigma_2 \geq \sigma_1$, of (1.1) with $y_2(0) = x_2$ such that (3.2) holds for all $t \in [0, \sigma_1)$. □

Remark 3.2. The lower semicontinuity and convexity hypotheses on G are satisfied, for instance, if S is sleek, F is lower semicontinuous and

$$F(x) \cap \operatorname{int} \mathcal{T}_S(x) \neq \emptyset \tag{3.5}$$

for any $x \in S$. Indeed, if the set S is sleek it is known that $\mathcal{T}_S(x)$ is a convex cone (see, e.g., [2]), hence the multifunction G has convex values. If, in addition, F is lower semicontinuous and (3.5) is satisfied, then the multifunction G is lower semicontinuous (see [6, Lemma 3.1]). However, condition (3.5) is not necessary for the lower semicontinuity of G (see the Example below). It should be interesting to find general conditions on S and F to ensure that the multifunction G is one-sided Lipschitz. An interesting case when this happens is when $F(x) \subset \mathcal{T}_S(x)$ for any $x \in S$ (which, in fact, assures invariance) and F is one-sided Lipschitz.

Example 3.3. Consider the set $S = \{(x_1, x_2);\ x_2 \geq 0\}$ and the multifunction $F(x_1, x_2) = \overline{B} \cap \{(y_1, y_2);\ y_2 \leq 0\}$ for all $(x_1, x_2) \in S$. We have that $\mathcal{T}_S(x_1, x_2) = S$ for $(x_1, x_2) \in \partial S$, so condition (3.5) is not satisfied. However, it is easy to see that the multifunction G, given by

$$G(x_1, x_2) = \begin{cases} \overline{B} \cap \{(y_1, y_2);\ y_2 \leq 0\} & \text{if } x_2 > 0 \\ [-1, 1] \times \{0\} & \text{if } x_2 = 0 \end{cases} \tag{3.6}$$

is lower semicontinuous.

For this system, Lemma 1 from [8] can not be applied because S does not satisfy the following assumption required there, that there exist a non-zero vector $a \in F = F(x_1, x_2)$ and $\rho > 0$ such that

$$S + \Gamma_{a,\rho} = S, \tag{3.7}$$

where $\Gamma_{a,\rho} := \{\lambda y; \lambda \geq 0, \|y - a\| \leq \rho\}$. Indeed, for any $a = (a_1, a_2) \in F$ and $\rho > 0$ take $s = 0 \in S$ and $y = a \in \Gamma_{a,\rho}$ if $a_2 < 0$ or $y = (a_1, -\rho) \in \Gamma_{a,\rho}$ if $a_2 = 0$. It is easy to see that $s + y \notin S$, so (3.7) does not hold. Neither [28, Lemma 14] can be used because one of the conditions required is not fulfilled, that is

$$\min_{v \in F(x)} \langle \eta, v \rangle < 0 \quad \text{for all } \eta \in N_S^C(x),\ x \in \partial S,$$

where $N_S^C(x)$ denotes the Clarke normal cone to S at x. Take, for instance, $x = (0, 0)$ and $\eta = (0, -1)$, then $\min_{v \in F(x)} \langle \eta, v \rangle = 0$.

However, it is easy to see that our hypotheses from Theorem 3.1 hold. We shall only prove that G is one-sided Lipschitz. Take $(x_1, x_2) \in \operatorname{int} S$, $(y_1, y_2) \in \partial S$ and $(v_1, v_2) \in G(x_1, x_2)$, hence $x_2 > 0$, $y_2 = 0$, $|v_1| \leq 1$ and $v_2 \leq 0$. Then there exists $(v_1, 0) \in G(y_1, y_2)$ such that

$$\langle (x_1, x_2) - (y_1, y_2), (v_1, v_2) - (v_1, 0) \rangle = \langle (x_1 - y_1, x_2), (0, v_2) \rangle = x_2 v_2 \leq 0.$$

The other cases can be solved similarly. In conclusion, by Theorem 3.1, we get the Lipschitz dependence of solutions on initial states.

Now we are ready to prove the propagation of the Lipschitz continuity of the state constrained minimal time function associated to (1.1).

Theorem 3.4. *Assume the hypotheses of Theorem 3.1. Suppose that T is proto-Lipschitz. Then $\mathcal{R}$ is open in S and T is locally Lipschitz on $\mathcal{R}$, i.e., for every $x \in \mathcal{R}$ there exists a neighborhood $\mathcal{U}$ of x and a constant $k > 0$ such that*

$$|T(z_1) - T(z_2)| \leq k \|z_1 - z_2\|$$

for every $z_1, z_2 \in \mathcal{U} \cap S$.

Proof. Let $\rho > 0$ and $M > 0$ be from the definition of the proto-Lipschitzness of T and L be from the one-sided Lipschitzness of G.

Let $x \in \mathcal{R}$. We prove that if $z \in S$ with $\|z - x\| < \rho e^{-L(T(x)+1)}$ then $z \in \mathcal{R}$ and

$$T(z) \leq T(x) + M e^{L(T(x)+1)} \|z - x\|. \tag{3.8}$$

To this end, fix $\varepsilon \in (0, 1)$ and consider $\tau < T(x) + \varepsilon$ and a solution $y \colon [0, \tau] \to S$ of (1.1) with $y(0) = x$ such that $y(\tau) \in \Sigma$. Let $z \in S$ be such that $\|z - x\| < \rho e^{-L(T(x)+1)}$. By Theorem 3.1 there exists $y_z \colon [0, \tau] \to S$ a solution of (1.1) with $y_z(0) = z$ such that

$$\|y_z(t) - y(t)\| \leq e^{Lt} \|z - x\|$$

for each $t \in [0, \tau]$. Therefore,

$$d_\Sigma(y_z(\tau)) \le e^{L\tau} \|z - x\| < \rho.$$

Since T is the proto-Lipschitz, we get

$$T(y_z(\tau)) \le M d_\Sigma(y_z(\tau)) \le M e^{L(T(x)+1)} \|z - x\| .$$

This implies that $T(z) \le \tau + M e^{L(T(x)+1)} \|z - x\|$. Further, $T(z) \le T(x) + \varepsilon + M e^{L(T(x)+1)} \|z - x\|$. Finally, since $\varepsilon \in (0, 1)$ is arbitrary, we get (3.8).

Now, let $x_0 \in \mathcal{R}$ and let $z_1, z_2 \in S$ be such that

$$\|z_i - x_0\| < \frac{\rho}{2} e^{-L(T(x_0)+M\rho+1)},$$

for $i = 1, 2$. We show that

$$\|T(z_1) - T(z_2)\| \le M e^{L(T(x_0)+M\rho+1)} \|z_1 - z_2\| . \tag{3.9}$$

To this end, we observe, by the first part of the proof, that $z_i \in \mathcal{R}$ and $T(z_i) \le T(x_0) + M e^{L(T(x_0)+1)} \|z_i - x_0\| \le T(x_0) + M\rho$, for $i = 1, 2$. Moreover,

$$\|z_1 - z_2\| \le \rho e^{-L(T(x_0)+M\rho+1)} \le \rho e^{-L(T(z_i)+1)}$$

for $i = 1, 2$. Therefore, by the first part of the proof,

$$\begin{aligned} T(z_1) &\le T(z_2) + M e^{L(T(z_2)+1)} \|z_1 - z_2\| \\ &\le T(z_2) + M e^{L(T(x_0)+M\rho+1)} \|z_1 - z_2\| . \end{aligned}$$

By symmetry, we get (3.9), as claimed. □

In [9, Theorem 3.8] the Lipschitz continuity of the minimal time function is proved under some regularity assumptions on the set of constraints. We remind that in [9] $S = \overline{\Omega}$ with Ω open and $\Sigma \subset \Omega$. Moreover, the following condition on the boundary of Ω is imposed: there exist $\alpha > 0$ and $\mathcal{I}$ a multifunction with some properties (called there uniformly hypertangent conical field) such that for any $x \in \partial\Omega$

$$F(x) \cap \mathcal{I}(x) \cap \{v \in \mathbb{R}^p;\ \ \|v\| \ge \alpha\} \ne \emptyset. \tag{3.10}$$

In the following example we present a system with $\Sigma \subset \Omega$ that does not satisfy (3.10) because $F(x) \cap \{v \in \mathbb{R}^p;\ \ \|v\| \ge \alpha\} = \emptyset$ for some $x \in \partial\Omega$ and any $\alpha > 0$, but satisfies our hypotheses.

Example 3.5. Let $S = \left\{(x_1, x_2) \in \mathbb{R}^2;\ x_2 \ge 0\right\}$, the target set

$$\Sigma = \left\{(x_1, x_2) \in \mathbb{R}^2;\ x_2 \ge 1\right\}$$

and the multifunction $F\colon S \rightsquigarrow \mathbb{R}^2$ given by

$$F(x_1, x_2) = \begin{cases} \{0\} \times [-x_2, x_2] & \text{if } x_2 > 0 \\ \{0\} \times [-|x_1|, 0] & \text{if } x_1 \ne 0, x_2 = 0 \\ \{(0,0)\} & \text{if } (x_1, x_2) = (0,0). \end{cases}$$

It is easy to see that for any $\alpha > 0$,

$$F(0,0) \cap \{v \in \mathbb{R}^2;\ \|v\| \geq \alpha\} = \emptyset,$$

so Theorem 3.8 from [9] can not be applied for this system. However, the conditions of Theorem 3.4 hold. Indeed, F is upper semicontinuous, with convex compact values. The multifunction G is given by

$$G(x_1, x_2) = \begin{cases} \{0\} \times [-x_2, x_2] & \text{if } x_2 > 0 \\ \{(0,0)\} & \text{if } x_2 = 0 \end{cases}$$

and it is easy to see that G is convex valued and Lipschitz continuous. We only have to show that T is proto-Lipschitz. To this end, we shall prove that condition (1.2) is satisfied. Let $\rho \in (0,1)$, $(x_1, x_2) \in S$, with $1 \geq x_2 \geq \rho$. Then $G(x_1, x_2) = \{0\} \times [-x_2, x_2]$, $\pi_\Sigma(x_1, x_2) = (x_1, 1)$ and $d_\Sigma(x_1, x_2) = 1 - x_2$. Then

$$\begin{aligned} \min_{u \in G(x_1,x_2)} \langle (x_1, x_2) - (x_1, 1), u \rangle &= \min_{u_2 \in [-x_2, x_2]} (x_2 - 1) u_2 \\ &= -x_2(1 - x_2) \leq -\rho d_\Sigma(x_1, x_2). \end{aligned}$$

Then, by Theorem 1.1, T is proto-Lipschitz. Applying now Theorem 3.4, we get that T is locally Lipschitz on $\mathcal{R}$.

In the following example, we consider a system with F not Lipschitz continuous, which can not be framed in the settings of [9] or [28], but satisfies the conditions of Theorem 3.4, therefore we get the Lipschitz continuity of the associated minimal time function on the reachable set.

Example 3.6. Consider $S = \{(x_1, x_2);\ x_2 \geq 0\}$, $\Sigma = \{(0,0)\}$ and $F \colon S \rightsquigarrow \mathbb{R}^2$ defined by

$$F(x_1, x_2) = \begin{cases} \overline{B} \cap \{(x, y);\ y \leq 0\} & \text{if } x_2 = 0 \\ \{(x, y);\ x^2 + 9y^2 \leq 1,\ y \leq 0\} & \text{if } x_2 > 0. \end{cases}$$

Clearly, F is upper semicontinuous with convex compact values. Since $\mathcal{T}_S(x) = S$ for $x \in \partial S$, we have that G is given by

$$G(x_1, x_2) = \begin{cases} [-1, 1] \times \{0\} & \text{if } x_2 = 0, \\ \{(x, y);\ x^2 + 9y^2 \leq 1,\ y \leq 0\} & \text{if } x_2 > 0, \end{cases}$$

and it is easy to see that G is convex valued, lower semicontinuous and one-sided Lipschitz. Moreover, (1.2) holds. Indeed, for $(x_1, x_2) \in S$, $x_2 > 0$, we have that

$$\langle (x_1, x_2), u \rangle \leq -\frac{1}{3} \|(x_1, x_2)\| \tag{3.11}$$

for $u = -(1/\|(x_1, x_2)\|)(x_1, \frac{1}{3}x_2)$ which obviously belongs to $G(x_1, x_2)$. For $(x_1, 0)$ take $u = (-\operatorname{sgn}(x_1), 0)$ and (3.11) holds. Therefore, by Theorem 1.1, T is proto-Lipschitz. Finally, by Theorem 3.4, we get that T is locally Lipschitz on $\mathcal{R}$. To get this final result we can not apply [9, Theorem 3.8], because $\Sigma \subset \partial S$, or [28, Theorem 15] because the condition that $\min_{v \in F(x)} \langle \eta, v \rangle < 0$ for all $\eta \in N_S^C(x)$ and $x \in \partial S$ is not satisfied. Moreover, the multifunction F is not locally Lipschitz continuous.

By Theorems 1.1 and 3.4 we get the following corollary.

Corollary 3.7. *Assume the hypotheses of Theorem 3.1. Moreover, assume that there exist $\rho > 0$ and $\gamma > 0$ such that*

$$\inf_{s\in\pi_\Sigma(x)} \inf_{u\in G(x)} \langle x-s,u\rangle \leq -\gamma d_\Sigma(x)$$

for all $x \in S \cap (\Sigma+\rho B)$. Then $\mathcal{R}$ is open in S and T is locally Lipschitz on $\mathcal{R}$.

4 Small time controllability and continuity of the state constrained minimal time function

In the previous section we assumed that the multifunction G is one-sided Lipschitz and we obtained the Lipschitz continuity of the S-constrained minimal time function. In this section we study the propagation of the regularity of the S-constrained minimal time function when the proto-Lipschitz condition is replaced by a weaker one (proto-continuous, proto-Hölder continuous), related to small time controllability on Σ, studied in [3, Chapter IV].

First, we give a Petrov-type condition that assures that the S-constrained minimal time function is proto-continuous and then we present a propagation result of this continuity property. In order to get the propagation result we consider a weaker condition on G than one-sided Lipschitz, used in the previous section, that assures the continuity of the solution map of (1.1) in the sense of Hausdorff metric.

Theorem 4.1. *Let $F\colon S \rightsquigarrow \mathbb{R}^p$ be an upper semicontinuous multifunction, with convex and compact values. Suppose that G, defined by (3.1), is nonempty valued and there exist $\rho > 0$ and $\mu\colon [0,\rho] \to [0,\infty)$ an integrable function with $\int_0^\rho \frac{1}{\mu(s)}\,ds < +\infty$ such that*

$$\inf_{s\in\pi_\Sigma(x)} \inf_{u\in G(x)} \langle x-s,u\rangle \leq -\mu(d_\Sigma(x))d_\Sigma(x) \tag{4.1}$$

for all $x \in S \cap (\Sigma+\rho B)$. Then $S \cap (\Sigma+\rho B) \subseteq \mathcal{R}$ and we have that

$$T(x) \leq \int_0^{d_\Sigma(x)} \frac{1}{\mu(s)}\,ds$$

for any $x \in S \cap (\Sigma+\rho B)$, therefore T is proto-continuous.

Proof. Take $x \in (S \cap (\Sigma+\rho B))\setminus\Sigma$.

Step 1. We first prove that there exists an $y\colon [0,\tau) \to (S \cap (\Sigma+\rho B))\setminus\Sigma$ a noncontinuable solution of

$$y'(t) \in F(y(t)), \quad y(0) = x, \tag{4.2}$$

and $z\colon [0,\tau) \to \mathbb{R}$ a solution of

$$z'(t) = -\mu(z(t)), \quad z(0) = d_\Sigma(x), \tag{4.3}$$

such that

$$d_\Sigma(y(t)) \leq z(t) \tag{4.4}$$

for all $t \in [0,\tau)$.

To this aim, we consider the set

$$\mathcal{K} = \{(y,z);\ y \in (S \cap (\Sigma+\rho B))\setminus\Sigma,\ d_\Sigma(y) \leq z,\}$$

the multifunction $\mathcal{F}\colon \mathcal{K} \rightsquigarrow \mathbb{R}^{p+1}$ defined by

$$\mathcal{F}(y,z) = F(y) \times \{-\mu(z)\}$$

for all $(y,z) \in \mathcal{K}$ and we apply Theorem 2.1. To this end, we use (4.1) to prove the tangency condition (2.2). For details, see the proof of Theorem 1.1 developed in [27], where $\mu(z) = -\gamma$.

Step 2. We prove that x can be transferred to the target Σ in time $\tau \le \int_0^{d_\Sigma(x)} (1/\mu(s))\, ds$. To this aim, let us first observe that the solution z (obtained in Step 1) is continuous, nonincreasing and $0 \le z(t) \le z(0) = d_\Sigma(x) < \delta$, for all $t \in [0,\tau)$. We have that

$$\int_0^t \frac{z'(s)}{\mu(z(s))}\, ds = \int_{z(0)}^{z(t)} \frac{ds}{\mu(s)},$$

hence

$$t = \int_{z(t)}^{d_\Sigma(x)} \frac{ds}{\mu(s)}. \tag{4.5}$$

Passing to the limit for $t \uparrow \tau$ in (4.5), we get that

$$\tau = \int_{z(\tau)}^{d_\Sigma(x)} \frac{ds}{\mu(s)} \le \int_0^\delta \frac{ds}{\mu(s)} < \infty. \tag{4.6}$$

By (4.4), y is bounded on $[0,\tau)$ and, as F maps bounded sets into bounded sets, we have that $F(y)$ is bounded on $[0,\tau)$. Then there exists $y^* := \lim_{t\uparrow\tau} y(t)$ and $y^* \in S$. Passing to the limit for $t \uparrow \tau$ in (4.4) we get that

$$d_\Sigma(y^*) \le z(\tau) \le z(0) = d_\Sigma(x) < \rho,$$

so $y^* \in S \cap (\Sigma + \rho B)$. Moreover, since $y(\cdot)$ is noncontinuable, it follows that $y^* \in \Sigma$, hence x can be transferred to the target in time τ. Therefore, $T(x) \le \tau$, and, using (4.6), we get

$$T(x) \le \int_0^{d_\Sigma(x)} \frac{ds}{\mu(s)},$$

as claimed. □

Example 4.2. Let S be the closed ball of center 0 and radius $1/2$ from $\mathbb{R}^2$ and $\Sigma = \{(0,0)\}$. Define the function $f\colon S \to \mathbb{R}^2$ by

$$f(x_1,x_2) = \begin{cases} \left(\frac{x_1}{\sqrt[4]{x_1^2+x_2^2}\ln(x_1^2+x_2^2)}, \frac{x_2}{\sqrt[4]{x_1^2+x_2^2}\ln(x_1^2+x_2^2)} \right), & \text{if } (x_1,x_2) \ne (0,0), \\ (0,0), & \text{if } (x_1,x_2) = (0,0), \end{cases}$$

and consider the multifunction $F\colon S \rightsquigarrow \mathbb{R}^2$ given by

$$F(x_1,x_2) = \{u f(x_1,x_2);\ u \in [0,1]\}.$$

It is clear that F has compact convex values and is continuous, since f is continuous on S. Moreover, we have that $F(x_1,x_2) \subset \mathcal{T}_S(x_1,x_2)$ for any $(x_1,x_2) \in S$, hence $G(x_1,x_2) = F(x_1,x_2)$. Take $(x_1,x_2) \in S\backslash\{(0,0)\}$. We have that

$$\begin{aligned} \inf_{v\in G(x_1,x_2)} \langle (x_1,x_2), v\rangle &= \inf_{u\in[0,1]} u \frac{x_1^2+x_2^2}{\sqrt[4]{x_1^2+x_2^2}\ln(x_1^2+x_2^2)} \\ &= \frac{x_1^2+x_2^2}{\sqrt[4]{x_1^2+x_2^2}\ln(x_1^2+x_2^2)} \\ &= -\mu(d_\Sigma(x_1,x_2))d_\Sigma(x_1,x_2), \end{aligned}$$

where $\mu\colon [0,1/2] \to [0,+\infty)$ is defined by $\mu(s) = -\sqrt{s}/2 \ln s$ for $s \neq 0$ and $\mu(0) = 0$. It is easy to see that μ is continuous on $[0,1/2]$ and $\int_0^{1/2}(1/\mu(s))ds < +\infty$. Hence, by Theorem 4.1, T satisfies the following estimate

$$T(x_1,x_2) \leq 2\int_0^{\sqrt{x_1^2+x_2^2}} \frac{|\ln s|}{\sqrt{s}}\,ds$$

for any $(x_1,x_2) \in S$, therefore T is proto-continuous.

In the next example, inspired by [11], T is proto-Hölder continuous of exponent $1/2$.

Example 4.3. Let $S = \left\{(x_1,x_2) \in \mathbb{R}^2;\ x_2 \geq 0\right\}$, the target set

$$\Sigma = ([0,\infty) \times [1,\infty)) \cup \Delta,$$

where $\Delta = \{(x_1,x_2);\, x_1 \geq 1 - (-x_2^2 + 2x_2)^{1/2},\, x_2 \in [0,1]\}$, and the multifunction $F\colon S \rightsquigarrow \mathbb{R}^2$ given by

$$F(x_1,x_2) = \begin{cases} [0,1] \times [-1,0] & \text{if } x_2 = 0 \\ [0,1] \times \{0\} & \text{if } x_2 \in (0,1] \\ [0,1] \times [0,x_2-1] & \text{if } x_2 > 1. \end{cases}$$

It is easy to see that F is upper semicontinuous, is not lower semicontinuous at $(x_1,0)$ (so F is not Lipschitz), has convex compact values and G is given by

$$G(x_1,x_2) = \begin{cases} [0,1] \times \{0\} & \text{if } 0 \leq x_2 \leq 1 \\ [0,1] \times [0,x_2-1] & \text{if } x_2 > 1. \end{cases} \tag{4.7}$$

Let $(x_0,y_0) \in S\backslash\Sigma$ with $y_0 \in [0,1]$. Then $(\overline{x},\overline{y}) := \pi_\Sigma(x_0,y_0) \in \partial\Delta$ is given by

$$(\overline{x},\overline{y}) = \left(1 - \frac{1-x_0}{\sqrt{(1-x_0)^2+(1-y_0)^2}}, 1 - \frac{1-y_0}{\sqrt{(1-x_0)^2+(1-y_0)^2}}\right)$$

and

$$d_\Sigma(x_0,y_0) = \sqrt{(1-x_0)^2+(1-y_0)^2} - 1.$$

We have that

$$\begin{aligned} \min_{u\in G(x_0,y_0)} \langle (x_0,y_0)-(\overline{x},\overline{y}),u\rangle &= \min_{u_1\in[0,1]} (x_0-\overline{x})u_1 \\ &= (x_0-1)\left(1 - \frac{1}{\sqrt{(1-x_0)^2+(1-y_0)^2}}\right) \\ &\leq -\frac{d_\Sigma(x_0,y_0)}{d_\Sigma(x_0,y_0)+1}\sqrt{d_\Sigma(x_0,y_0)^2 + 2d_\Sigma(x_0,y_0)}. \end{aligned}$$

Moreover, for any $(x_0,y_0) \in S\backslash\Sigma$ with $y_0 = 0$ we have that

$$\min_{u\in G(x_0,y_0)} \langle (x_0,y_0)-(\overline{x},\overline{y}),u\rangle = -\frac{d_\Sigma(x_0,y_0)}{d_\Sigma(x_0,y_0)+1}\sqrt{d_\Sigma(x_0,y_0)^2 + 2d_\Sigma(x_0,y_0)}.$$

Now, let $(x_0,y_0) \in S\backslash\Sigma$ with $y_0 > 1$. Then $(\overline{x},\overline{y}) = (0,y_0)$, $d_\Sigma(x_0,y_0) = -x_0$ and

$$\min_{u\in G(x_0,y_0)} \langle (x_0,y_0)-(\overline{x},\overline{y}),u\rangle = -d_\Sigma(x_0,y_0).$$

In conclusion, for any $(x_0, y_0) \in S \cap (\Sigma + \frac{1}{2}B)$ we have that

$$\min_{u \in G(x_0,y_0)} \langle (x_0, y_0) - (\overline{x}, \overline{y}), u \rangle \leq -\sqrt{d_\Sigma(x_0, y_0)} d_\Sigma(x_0, y_0),$$

so, (4.1) holds with $\mu(s) = \sqrt{s}$, for $s \in [0, 1/2]$. By Theorem 4.1, we get that T is proto-continuous. More precisely, we have that

$$T(x_0, y_0) \leq 2\sqrt{d_\Sigma(x_0, y_0)} \tag{4.8}$$

for any $(x_0, y_0) \in S \cap (\Sigma + \frac{1}{2}B)$.

Now, relaxing the one-sided Lipschitz condition on G (assumed in Theorem 3.1) to one-sided Perron, we obtain the continuity of the solution map of (1.1) in the sense of Hausdorff metric. The continuity of the solution map was also proved in [17] but under a stronger assumption, that is, $F(x) \subseteq \mathcal{T}_S(x)$ for all $x \in S$.

Theorem 4.4. *Let $F \colon S \rightsquigarrow \mathbb{R}^p$ be an upper semicontinuous multifunction, with convex and compact values. Assume that G, defined by (3.1), has nonempty convex values, is lower semicontinuous and one-sided Perron. Then, for any $\varepsilon > 0$ there exists $\delta > 0$ such that, for any $x_1, x_2 \in S$ with $\|x_1 - x_2\| < \delta$ and for any solution $y_1 \colon [0, \sigma] \to S$ of (1.1) with $y_1(0) = x_1$, there exists a solution $y_2 \colon [0, \sigma] \to S$ of (1.1) with $y_2(0) = x_2$ such that*

$$\|y_1(t) - y_2(t)\| \leq \varepsilon \tag{4.9}$$

for all $t \in [0, \sigma]$.

Proof. The technique of the proof is similar to the one of Theorem 3.1, this time defining the multifunction $\mathcal{F}$ by

$$\mathcal{F}(\tau, x, \lambda) = \{1\} \times F(x) \times \{\vartheta(\lambda)\}.$$

A key role in the proof is played by the result from [26, p. 24] on the upper semicontinuity of the solution map for the differential equation $z'(t) = \vartheta(z(t))$. See also the proof of [17, Theorem 2.4]. □

By Theorem 4.4 we obtain the propagation of the continuity of the state constrained minimal time function associated to (1.1).

Theorem 4.5. *Assume the hypotheses of Theorem 4.4. Suppose that T is proto-continuous. Then $\mathcal{R}$ is open in S and T is locally uniformly continuous on $\mathcal{R}$.*

Proof. The proof is similar to the one of [17, Theorem 3.1], where the target is zero. □

Remark 4.6. Under the assumptions of Theorem 4.4, with G one-sided Lipschitz, if T is proto-continuous with $\omega(s) = Ms^\alpha$, $M > 0$, $\alpha \in (0, 1)$, we get that T is locally Hölder continuous of exponent α on $\mathcal{R}$.

Example 4.7. Consider again the system from Example 4.3. We have proved that T satisfies (4.8) for any $(x_0, y_0) \in S \cap (\Sigma + \frac{1}{2}B)$. It is easy to verify that G, given by (4.7), is Lipschitz continuous and convex valued. Then, by Remark 4.6, T is locally Hölder continuous of exponent $1/2$ on $\mathcal{R}$.

By Theorems 4.1 and 4.5 we get the following corollary.

Corollary 4.8. *Assume the hypotheses of Theorem 4.1. Moreover, assume that G is convex valued, lower semicontinuous and one-sided Perron. Then $\mathcal{R}$ is open in S and T is locally uniformly continuous on $\mathcal{R}$.*

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