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Spectral characterizations for Hyers–Ulam stability

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Abstract. First we prove that an $n \times n$ complex linear system is Hyers–Ulam stable if and only if it is dichotomic (i.e. its associated matrix has no eigenvalues on the imaginary axis $i\mathbb{R}$). Also we show that the scalar differential equation of order n,

$$x^{(n)}(t) = a_1 x^{(n-1)}(t) + \cdots + a_{n-1} x'(t) + a_n x(t), \quad t \in \mathbb{R}_+ := [0, \infty),$$

is Hyers–Ulam stable if and only if the algebraic equation

$$z^n = a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

has no roots on the imaginary axis.

Keywords: differential equations, dichotomy, Hyers–Ulam stability.

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1 Introduction

In 1940 S. M. Ulam posed some open problems, see [28] and [29]. One of these problems refers to the stability of a certain functional equation. The first answer to this problem was given by D. H. Hyers in 1941, see [10]. After that, this was called the Hyers–Ulam problem and its study became a widely studied subject for many mathematicians. It seems that M. Obłoza [21] was the first author who proved a result concerning Hyers–Ulam stability of differential equations. C. Alsina and R. Ger, [1], investigated Hyers–Ulam stability of first order linear differential equations, and, after that, their results were generalized by S. E. Takahasi, H. Takagi, T. Miura and S. Miyajima in [27], L. Sun and S.-M. Jung, in [11], [12] and [13] and G. Wang, M. Zhou in [30]. For comprehensive information we refer readers to the two recent expository papers by N. Brillouët-Belluot, J. Brzdęk, K. Ciepliński [2] and by Z. Moszner [20]. The Hyers–Ulam problems for second order differential equations were studied by Y. Li, J. Huang in [18], Y. Li,

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Y. Shen [16], Y. Li in [17] and P. Găvruţă, S.-M. Jung, Y. Li in [6]. Also M. N. Qarawani, [25], studied Hyers–Ulam stability for linear and nonlinear second order differential equations.

In [15], Y. Li and Y. Shen characterized the Hyers–Ulam stability of linear differential equation of order two, under the assumption that its associated characteristic equation has two different positive roots.

M. Obłoza, [22], has connected Hyers–Ulam and Lyapunov stability for ordinary differential equations. See also the papers of J. Brzdęk and S.-M. Jung [14], and of D. Popa and I. Raşa [23] and [24] for further interesting details concerning this subject.

Over the past decades, the Hyers–Ulam stability of operator equations has been widely discussed. In [19] the authors describe the results on Hyers–Ulam stability for n-th order linear differential operator $p(D)$, p being a complex valued polynomial of degree n and D a differential operator. They prove that the differential operator equation $p(D)f = 0$ is Hyers–Ulam stable if and only if the algebraic equation $p(z) = 0$ has no pure imaginary solutions.

In the very recent paper [7], the authors investigate a special case of Hyers–Ulam stability for linear differential equations by using the Laplace transform method. Instead of uniform distance between solutions they estimate the pointwise distance.

In this paper we prove that a linear differential systems (driven by a $n \times n$ matrix A) is Hyers–Ulam stable if and only if it is dichotomic, that is spectrum of A does not intersect the imaginary axis. Thus we provide a spectral criteria for Hyers–Ulam stability. Our method uses only elementary settings. Nevertheless, the idea that Hyers–Ulam stability and exponential dichotomy are equivalent seems to be new and it can enlarge the area of investigations on Hyers–Ulam stability. As a special case, we also show that the scalar differential equation of order n,

$$x^{(n)}(t) = a_1 x^{(n-1)}(t) + \cdots + a_{n-1}x'(t) + a_n x(t), \quad t \in \mathbb{R}_+ := [0, \infty),$$

is Hyers–Ulam stable if and only if its associated algebraic equation

$$z^n = a_1 z^{n-1} + \cdots + a_{n-1} z + a_n,$$

has no roots on the imaginary axis.

Now we outline the Hyers–Ulam problem for a matrix A.

Let $\mathbb{R}_+$ be the set of all nonnegative real numbers, and let A be an $n \times n$ complex matrix, n being a positive integer. Consider the system

$$x'(t) = Ax(t), \quad t \in \mathbb{R}_+ := [0, \infty). \tag{A}$$

Let ε be a positive real number. A $\mathbb{C}^n$-valued function y is called an ε-approximate solution for (A) if

$$\|y'(t) - Ay(t)\| \leq \varepsilon, \quad \forall t \in \mathbb{R}_+,$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{C}^n$, i.e. for

$$x = (\xi_1, \ldots, \xi_n)^T \in \mathbb{C}^n, \quad \|x\|^2 = \sum_{k=0}^{n} |\xi_k|^2.$$

Let n and m be two positive integers. The set of all $n \times m$ matrices having complex entries is denoted by $\mathbb{C}^{n\times m}$. The spaces $\mathbb{C}^n$ and $\mathbb{C}^{n\times 1}$ are identified by the usual way. The space $\mathbb{C}^{n\times n}$ becomes a Banach algebra when we endow it with the operatorial norm

$$L \mapsto \|L\| := \sup_{\|x\|\leq 1} \|Lx\| : \mathbb{C}^{n\times n} \to \mathbb{R}_+.$$

In the following we denote by $[M]_{ij}$ the element of the matrix M located at the intersection of the i-th row and the j-th column. The matrix A is said to be Hyers–Ulam stable if there exists a nonnegative absolute constant L such that for every ε-approximate solution ϕ of (A), there exists an exact solution θ of (A) such that

$$\sup_{t\in\mathbb{R}_+} \|\phi(t)-\theta(t)\| \leq L\varepsilon.$$

2 Notations and some results

Throughout the paper, A stands for an $n\times n$ complex matrix while $P_A(z) := \det(zI-A)$ denotes its characteristic polynomial. I denotes the identity matrix of order n. The set $\sigma(A) := \{\lambda_1,\lambda_2,\ldots,\lambda_k\}$, consisting of all roots of P_A, is called the spectrum of A. As is well-known,

$$P_A(z) = (z-\lambda_1)^{m_1}\cdots(z-\lambda_k)^{m_k},$$

where $m_1, m_2, \ldots, m_k$ are the algebraic multiplicities of the eigenvalues $\lambda_1,\ldots,\lambda_k$, respectively. Then, $m_1+\cdots+m_k = n$ and

$$\mathbb{C}^n = \ker(A-\lambda_1 I)^{m_1}\oplus\cdots\oplus\ker(A-\lambda_k I)^{m_k}. \tag{2.1}$$

We also mention that the dimension of $\ker(A-\lambda_j I)^{m_j}$ is m_j. For every integer j with $1\leq j\leq k$ and every $t\in\mathbb{R}$, the subspace $\ker(A-\lambda_j I)^{m_j}$ is e^{tA}-invariant. Indeed, let $F_N(t) := \sum_{j=0}^{N}\frac{(tA)^j}{j!}$, N being a positive integer. As is well-known, the sequence of functions (F_N) converges uniformly on real compact intervals to the map $t\mapsto e^{tA}$. On the other hand,

$$F_N(\cdot)(A-\lambda_j I)^{m_j} = (A-\lambda_j I)^{m_j}F_N(\cdot),$$

and we get the assertion by passing to the limit for $N\to\infty$. As a consequence of (2.1), for each $x\in\mathbb{C}^n$ there exists $x_j\in\ker(A-\lambda_j I)^{m_j}$ such that

$$e^{tA}x = e^{tA}x_0 + e^{tA}x_1+\cdots+e^{tA}x_k, \quad t\in\mathbb{R}_+.$$

Moreover, $e^{tA}x_j$ belongs to $\ker(A-\lambda_j I)^{m_j}$ for all $t\in\mathbb{R}$ and there exists a $\mathbb{C}^n$-valued polynomial $p_{jx}(t)$ of degree at most m_j-1 such that

$$x_j(t) := e^{tA}x_j = e^{\lambda_j t}p_{jx}(t), \quad t\in\mathbb{R}, \quad 1\leq j\leq k. \tag{2.2}$$

This is well-known from properties of the generalized eigenspace. See [8, pp. 104–107] for further details.

The decomposition (2.1) yields

$$\mathbb{C}^n = \mathcal{X}_s(A)\oplus\mathcal{X}_0(A)\oplus\mathcal{X}_u(A),$$

where

$$\mathcal{X}_s(A) = \bigoplus_{j=1,\ \mathrm{Re}\,(\lambda_j)<0}^{k} \ker(A-\lambda_j I)^{m_j},$$

$$\mathcal{X}_0(A) = \bigoplus_{j=1,\ \mathrm{Re}\,(\lambda_j)=0}^{k} \ker(A-\lambda_j I)^{m_j}$$

and

$$\mathcal{X}_u(A) = \bigoplus_{j=1,\ \mathrm{Re}\ (\lambda_j)>0}^{k} \ker(A - \lambda_j I)^{m_j}.$$

The subspaces $\mathcal{X}_s(A)$ and $\mathcal{X}_u(A)$ are called the stable and respectively the unstable subspace of A.

The circle and closed disk of radius r which are centered on the eigenvalue $\lambda_j = \sigma(A)$, are respectively:

$$C_r(\lambda_j) = \{z \in \mathbb{C} : |z - \lambda_j| = r\}$$

and

$$\overline{D}_r(\lambda_j) = \{z \in \mathbb{C} : |z - \lambda_j| \leq r\}$$

where r is a positive real number, small enough such that $\sigma(A) \cap \overline{D}_r(\lambda_j) = \{\lambda_j\}$. Recall that an $n \times n$ complex matrix P, verifying $P^2 = P$, is called a projection. Let $1 \leq j \leq k$. From (2.1) it follows that

$$I = E_{\lambda_1} + E_{\lambda_2} + \cdots + E_{\lambda_k},$$

where $E_{\lambda_j} := E_j : \mathbb{C}^n \to \mathbb{C}^n$ is defined by $E_j x := x_j$. Obviously, E_j $(1 \leq j \leq k)$ are projections which are called spectral projections associated to the matrix A. It is well-known [4, Chapter 7] that

$$E_j = \frac{1}{2\pi i} \oint_{C_r(\lambda_j)} (zI - A)^{-1}\, dz. \tag{2.3}$$

The equation (2.3) will be used in the proof of Lemma 4.4 below.

The first result of this paper reads as follows.

Theorem 2.1. *The matrix A is Hyers–Ulam stable if and only if it is dichotomic.*

For the proof of the Theorem 2.1, we need the following proposition, which contains equivalent characterizations for exponential dichotomy. This result is certainly known but we include it and its proof here for the sake of completeness. Further details about different characterizations of dichotomy can be found in the book of W. A. Coppel, see [3, Chapter 3].

Proposition 2.2. *The following three statements concerning the matrix A are equivalent.*

(α) *A is dichotomic.*

(β) *There exists a projection P, commuting with A, and there exist positive constants N_1, N_2, ν_1, ν_2 such that*

(β_1) $\|e^{tA}Px\| \leq N_1 e^{-\nu_1 t}\|Px\|$, *for all $x \in \mathbb{C}^n$, for every $t \geq 0$,*

(β_2) $\|e^{tA}(I-P)x\| \leq N_2 e^{\nu_2 t}\|(I-P)x\|$, *for all $x \in \mathbb{C}^n$ and for all $t \leq 0$.*

(γ) *For each continuous and bounded function $f\colon \mathbb{R}_+ \to \mathbb{C}^n$, there exists a unique bounded solution, starting from the unstable subspace of A (i.e. with the initial conditions belonging to $\mathcal{X}_u(A)$), of the equation*

$$y'(t) = Ay(t) + f(t), \quad t \geq 0. \tag{A,f}$$

Proof. $(\alpha) \Rightarrow (\beta)$. A is dichotomic, so $\mathcal{X}_0(A) = \{0\}$ and then $\mathbb{C}^n = \mathcal{X}_s(A) \oplus \mathcal{X}_u(A)$. Every $x \in \mathbb{C}^n$ can be written as $x = x_s + x_u$ with $x_s \in \mathcal{X}_s(A)$ and $x_u \in \mathcal{X}_u(A)$. Let $P := \mathbb{C}^n \to \mathbb{C}^n$ defined by $Px := x_s$. It is obvious that the matrix P is a projection. Moreover, using (2.2) it can be seen that (β_1) and (β_2) are fulfilled for certain positive constants N_1, N_2, ν_1, ν_2.

$(\beta) \Rightarrow (\alpha)$. Suppose that there exists $\lambda \in \sigma(A)$ with $\mathrm{Re}(\lambda) = 0$. Then there is $x_0 \neq 0$, $x_0 \in \mathbb{C}^n$ such that $Ax_0 = \lambda x_0$ and thus $e^{tA}Px_0 = e^{t\lambda}Px_0$ for all $t \in \mathbb{R}$, where the fact that P commutes with A (and then with e^{tA}) was used. If $Px_0 \neq 0$, then (β_1) yields

$$\left\|e^{tA}Px_0\right\| = \left\|e^{t\lambda}Px_0\right\| = \|Px_0\| \leq N_1 e^{-\nu_1 t}\|Px_0\|, \quad \forall t \geq 0,$$

which is a contradiction. If $Px_0 = 0$, then $(I-P)x_0 \neq 0$ and (β_2) gives

$$\left\|e^{tA}(I-P)x_0\right\| = \left\|e^{t\lambda}(I-P)x_0\right\| = \|(I-P)x_0\| \leq N_2 e^{\nu_2 t}\|(I-P)x_0\|, \quad \forall t \leq 0,$$

which is also a contradiction.

$(\alpha) \Rightarrow (\gamma)$. Since the matrix A is dichotomic, the map

$$t \mapsto y(t) := \int_0^t e^{(t-s)A}Pf(s)\,ds - \int_t^\infty e^{(t-s)A}(I-P)f(s)\,ds,$$

is a solution of (A, f). See [3, Chapter 3] for more details. Indeed, the second integral is well defined because, from (β_2), we have

$$\begin{aligned}\int_t^\infty \left\|e^{(t-s)A}(I-P)f(s)\right\|\,ds &\leq \int_t^\infty N_2 e^{\nu_2(t-s)}\|I-P\|\|f\|_\infty\,ds\\ &= \frac{N_2}{\nu_2}\|I-P\|\|f\|_\infty.\end{aligned}$$

Also from (β), the solution $y(\cdot)$ is bounded on $\mathbb{R}_+$, since

$$\sup_{t\geq 0}|y(t)| \leq \left(\frac{N_1}{\nu_1}\|P\| + \frac{N_2}{\nu_2}\|I-P\|\right)\sup_{t\geq 0}|f(t)|.$$

Moreover, $y(0) = -\int_0^\infty e^{-sA}(I-P)f(s)\,ds \in \mathcal{X}_u(A)$ because $\mathcal{X}_u(A)$ is a closed subspace and it is invariant under any exponential of A.

It remains to show that we have uniqueness. Suppose that there exist two bounded solutions on $\mathbb{R}_+$ of (A, f), denoted by $y_1(\cdot)$ and $y_2(\cdot)$. Then

$$y_1(t) = e^{tA}z_1 + \int_0^t e^{(t-s)A}f(s)\,ds, \quad t \geq 0$$

and

$$y_2(t) = e^{tA}z_2 + \int_0^t e^{(t-s)A}f(s)\,ds, \quad t \geq 0,$$

with $z_1, z_2 \in \mathcal{X}_u(A)$.

Since $y_1(t) - y_2(t) = e^{tA}(z_1 - z_2)$, $y_1(\cdot) - y_2(\cdot)$ is bounded on $\mathbb{R}_+$ and because A is dichotomic it follows that $z_1 - z_2 \in \mathcal{X}_s(A)$. On the other hand, by the assumption, we have that $z_1, z_2 \in \mathcal{X}_u(A)$. This yields $z_1 - z_2 \in \mathcal{X}_u(A)$. But $\mathcal{X}_s(A) \cap \mathcal{X}_u(A) = \{0\}$ and therefore $z_1 = z_2$.

$(\gamma) \Rightarrow (\alpha)$. Suppose that there exists $\lambda \in \sigma(A)$, with $\mathrm{Re}(\lambda) = 0$. Then there exists $x_0 \neq 0$ such that $Ax_0 = \lambda x_0$, and therefore $e^{tA}x_0 = e^{\lambda t}x_0$, for all $t \in \mathbb{R}$.

Let $f(t) := e^{\lambda t}x_0$ for $t \geq 0$. Obviously, f is a bounded and continuous function and from the hypothesis, there exists a unique $z_0 \in \mathcal{X}_u(A)$ such that the map

$$t \mapsto e^{tA}z_0 + \int_0^t e^{(t-s)A}e^{\lambda s}x_0\, ds$$

is bounded on $\mathbb{R}_+$. But

$$\begin{aligned} e^{tA}z_0 + \int_0^t e^{(t-s)A}e^{\lambda s}x_0\, ds &= e^{tA}z_0 + \int_0^t e^{(t-s)\lambda}e^{\lambda s}x_0\, ds \\ &= e^{tA}z_0 + \int_0^t e^{\lambda t}x_0\, ds \\ &= e^{tA}z_0 + te^{\lambda t}x_0. \end{aligned}$$

If $z_0 = 0$, obviously we arrive at a contradiction, since the map $t \mapsto te^{\lambda t}x_0$ is unbounded. If $z_0 \neq 0$, from the spectral decomposition theorem there are two positive constants N and ν such that $\|e^{tA}z_0\| \geq Ne^{\nu t}$ for all $t \geq 0$, and a contradiction arises again. □

3 Hyers–Ulam stability and exponential dichotomy for linear differential systems

We can see an ε-approximate solution of (A) as an exact solution of (A, ρ) corresponding to a forced term $\rho(\cdot)$ which is bounded by ε.

Remark 3.1. Let ε be an arbitrary positive number. The matrix A (or the system (A)) is Hyers–Ulam stable if and only if there exists a nonnegative constant L such that for every $\mathbb{C}^n$-valued continuous map $\rho = \rho(t)$ bounded by ε on $\mathbb{R}_+$, and every $x \in \mathbb{C}^n$, there exists $x_0 \in \mathbb{C}^n$ such that

$$\sup_{t \geq 0} \left\| e^{tA}(x - x_0) + \int_0^t e^{(t-s)A}\rho(s)\, ds \right\| \leq L\varepsilon.$$

Proof. Let $\varepsilon > 0$. Assume that the system (A) is Hyers–Ulam stable. Let $\rho(\cdot)$ be a $\mathbb{C}^n$-valued continuous function on $\mathbb{R}_+$ and let $x \in \mathbb{C}^n$. We prove that the map

$$t \mapsto \phi(t) := e^{tA}x + \int_0^t e^{(t-s)A}\rho(s)\, ds : \mathbb{R}_+ \to \mathbb{C}^n \tag{3.1}$$

is an ε-approximative solution for (A). Indeed, the derivative of ϕ is given by

$$\begin{aligned} \phi'(t) &= Ae^{tA}x + \left(e^{tA} \int_0^t e^{-sA}\rho(s)\, ds \right)' \\ &= Ae^{tA}x + Ae^{tA} \int_0^t e^{-sA}\rho(s)\, ds + e^{tA}e^{-tA}\rho(t) \\ &= A\phi(t) + \rho(t), \quad \forall t \geq 0. \end{aligned}$$

Therefore, $\|\phi'(t) - A\phi(t)\| = \|\rho(t)\| \leq \varepsilon$. Let now L be as in the definition of Hyers–Ulam stability and $\theta(\cdot)$ an exact solution of (A) such that $\|\phi - \theta\|_\infty \leq L\varepsilon$. This inequality yields

$$\sup_{t \geq 0} \left\| e^{tA}(x - x_0) + \int_0^t e^{(t-s)A}\rho(s)\, ds \right\| \leq L\varepsilon, \tag{3.2}$$

where $x_0 := \theta(0)$.

To prove the converse statement, let $\varepsilon > 0$ and ϕ be an ε-approximative solution of (A). Then the map $t \mapsto \rho(t) := \phi'(t) - A\phi(t)$ is continuous on $\mathbb{R}_+$ and $\|\rho\|_\infty \leq \varepsilon$. Let $L \geq 0$ as in the assumption and for $x = \phi(0)$ let choose $x_0 \in \mathbb{C}^n$ such that (3.2) holds. Set $\theta(t) := e^{tA}x_0$. To finish the proof it is enough to show that

$$e^{-tA}\phi(t) = x + \int_0^t e^{-sA}\rho(s)\,ds.$$

This is an elementary fact and the details are omitted. □

Proof of Theorem 2.1.

Necessity. Suppose that A is not dichotomic, i.e. $\mathcal{X}_0(A) \neq \{0\}$. Then, there exists λ_j in $\sigma(A)$, with $\lambda_j = i\mu_j$, $\mu_j \in \mathbb{R}$. Let $\varepsilon > 0$ be fixed and set $\rho(t) := e^{i\mu_j t}u_0$, with $\|u_0\| \leq \varepsilon$. Obviously, the function ρ is continuous and bounded by ε. By assumption, the matrix A is Hyers–Ulam stable. Hence, the solution

$$y(t) = e^{tA}(x - x_0) + \int_0^t e^{(t-s)A}\rho(s)\,ds, \quad x, x_0 \in \mathbb{C}^n,$$

of the Cauchy problem

$$\begin{cases} y'(t) = Ay(t) + \rho(t), & t \geq 0 \\ y(0) = x - x_0, \end{cases} \tag{A,ρ}$$

is bounded by $L\varepsilon$. By using the spectral decomposition theorem, (see also Lemma 4.3 below, [9, Theorem 2], [5, p. 510] or [26, p. 308]), there exists an $n \times n$ matrix-valued polynomial $P_j(t)$ having the degree at most $m_j - 1$, such that

$$E_j e^{tA} = e^{i\mu_j t}P_j(t), \qquad \forall t \geq 0. \tag{3.3}$$

Then the map

$$t \mapsto E_j\left[e^{tA}(x - x_0) + \int_0^t e^{(t-s)A}\rho(s)\,ds\right], \quad x, x_0 \in \mathbb{C}^n,$$

should also be bounded by $L\varepsilon$.

On the other hand,

$$E_j\left[e^{tA}(x - x_0) + \int_0^t e^{(t-s)A}\rho(s)\,ds\right] = e^{i\mu_j t}P_j(t)(x - x_0) + \int_0^t E_j(e^{(t-s)A}\rho(s))\,ds,$$

and

$$\begin{aligned} \int_0^t E_j e^{(t-s)A}\rho(s)\,ds &= \int_0^t E_j e^{(t-s)A}e^{i\mu_j s}u_0\,ds \\ &= \int_0^t e^{i\mu_j s}e^{(t-s)i\mu_j}P_j(t-s)u_0\,ds \\ &= e^{i\mu_j t}\int_0^t P_j(t-s)u_0\,ds = e^{i\mu_j t}q_j(t), \end{aligned}$$

where

$$q_j(t) = \int_0^t P_j(t-s)u_0\,ds,$$

is a polynomial, as well. Now by choosing an appropriate vector $u_0 \neq 0$,

$$\deg[P_j(t)(x-x_0)] \leq \deg[P_j(t)] = \deg[P_j(t)u_0] < 1 + \deg[P_j(t)] = \deg[q_j(t)].$$

Therefore, the solution $y(t) = e^{i\mu_j t}\left[P_j(t)(x-x_0) + q_j(t)\right]$ is unbounded and we have a contradiction.

Sufficiency. The absolute constant L will be chosen later.

Let $\rho : \mathbb{R}_+ \to \mathbb{C}^n$ be a continuous function, with $\|\rho\|_\infty \leq \varepsilon$ and let $x \in \mathbb{C}^n$. By Proposition 2.2, there exists a unique bounded solution $y(\cdot)$ of (A, ρ) starting from the subspace $\mathcal{X}_u(A)$. Set $u_0 := y(0) \in \mathcal{X}_u(A)$. Since A is dichotomic, the map

$$t \mapsto \int_0^t e^{(t-s)A} P\rho(s)\, ds - \int_t^\infty e^{(t-s)A}(I-P)\rho(s)\, ds$$

is a bounded solution on $\mathbb{R}_+$ of (A, f). Then,

$$\begin{aligned} \|y(t)\| &= \left\| e^{tA}u_0 + \int_0^t e^{(t-s)A}\rho(s)\, ds \right\| \\ &= \left\| \int_0^t e^{(t-s)A} P\rho(s)\, ds - \int_t^\infty e^{(t-s)A}(I-P)\rho(s)\, ds \right\| \\ &\leq \left(\frac{N_1}{\nu_1}\|P\| + \frac{N_2}{\nu_2}\|I-P\| \right) \varepsilon, \end{aligned}$$

The desired assertion follows by choosing $L = \left(\frac{N_1}{\nu_1}\|P\| + \frac{N_2}{\nu_2}\|I-P\|\right)$ and setting $x_0 = x - u_0$. □

4 Hyers–Ulam stability and exponential dichotomy for scalar differential equations of higher order

Let us consider the following differential equations for $t \in \mathbb{R}_+$

$$x^{(n)}(t) = a_1 x^{(n-1)}(t) + \cdots + a_{n-1}x'(t) + a_n x(t) \tag{4.1}$$

and

$$x^{(n)}(t) = a_1 x^{(n-1)}(t) + \cdots + a_n x(t) + \theta(t), \tag{4.2}$$

where $\theta : \mathbb{R}_+ \to \mathbb{C}$ is a continuous function and $a_j \in \mathbb{C}, 1 \leq j \leq n$.

To the differential equation (4.2) we associate the system

$$X'(t) = AX(t) + \Theta(t), \quad X(t), \Theta(t) \in \mathbb{C}^n,$$

where

$$X(t) = \left(x(t), x'(t), \ldots, x^{(n-1)}(t)\right)^T,$$

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_1 \end{pmatrix}$$

is an $n \times n$ matrix and

$$\Theta(t) = (0, \ldots, 0, \theta(t))^T.$$

Remark 4.1. Let ε be an arbitrary positive number. The differential equation (4.1) is Hyers–Ulam stable if and only if there exists a nonnegative constant L such that for every $\mathbb{C}$-valued continuous map $\theta = \theta(t)$ bounded by ε on $\mathbb{R}_+$, and every $x \in \mathbb{C}^n$, there exists $x_0 \in \mathbb{C}^n$ such that

$$\sup_{t\geq 0}\left|\left[e^{tA}(x-x_0)+\int_0^t e^{(t-s)A}\Theta(s)\,ds\right]_{11}\right| \leq L\varepsilon.$$

For every $z \in \mathbb{C}$, consider the $n \times n$ matrix

$$zI - A = \begin{pmatrix} z & -1 & 0 & \cdots & 0 \\ 0 & z & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & z-a_1 \end{pmatrix}.$$

If $z \in \rho(A) := \mathbb{C} \setminus \sigma(A)$, this matrix is invertible and it is obvious to see that the n-th column of its inverse is given by

$$\mathrm{col}_n[(zI-A)^{-1}] = \frac{1}{P_A(z)} \cdot \begin{pmatrix} 1 \\ z \\ \vdots \\ z^{n-1} \end{pmatrix}.$$

Theorem 4.2. *The following statements are equivalent:*

(α) *The differential equation (4.1) is Hyers–Ulam stable.*

(β) *The matrix A is dichotomic.*

(γ) *The characteristic equation*

$$\lambda^n - a_1\lambda^{n-1} - a_{n-2}\lambda^{n-2} - \cdots - a_n = 0 \tag{4.3}$$

has no roots on the imaginary axis.

Proof. The statements (β) and (γ) are equivalent since the spectrum of A is equal to the set of all roots of (4.3).

(α) $\Rightarrow$ (β). Suppose that A is not dichotomic. Then, there exists λ_j in $\sigma(A)$, with $\lambda_j = i\mu_j$, $\mu_j \in \mathbb{R}$. Let $\varepsilon > 0$ and set $\Theta(t) := e^{i\mu_j t}u_0$, where

$$u_0 = (0,\ldots,0,v_0)^T \in \mathbb{C}^n \tag{4.4}$$

and v_0 is a nonzero complex scalar satisfying $|v_0| \leq \varepsilon$. Clearly, the function Θ is continuous and bounded by ε. The differential equation (4.1) is Hyers–Ulam stable, so

$$\sup_{t\geq 0}\left|\left[e^{tA}(x-x_0)+\int_0^t e^{(t-s)A}\Theta(s)\,ds\right]_{11}\right| \leq L\varepsilon.$$

Then the map $t \mapsto \left[E_j(e^{tA}(x-x_0)+\int_0^t e^{(t-s)A}\Theta(s)\,ds)\right]_{11}$ is bounded on $\mathbb{R}_+$ by $L\varepsilon$, as well. On the other hand, in view of (3.3), one has

$$\left[E_j\left(e^{tA}(x-x_0)+\int_0^t e^{(t-s)A}\Theta(s)\,ds\right)\right]_{11} = \left[e^{i\mu_j t}P_j(t)(x-x_0)\right]_{11} + \left[\int_0^t E_j e^{(t-s)A}\Theta(s)\,ds\right]_{11}.$$

We already know from the proof of Theorem 2.1 that the degree of the scalar-valued polynomial in t, $[[P_j(t)(x-x_0)]]_{11}$, is less than or equal to m_j-1. In the following we prove that

$$\rho_j(t) := e^{-i\mu_j t}\left[\int_0^t E_j e^{(t-s)A}\Theta(s)\,ds\right]_{11}$$

is a polynomial in t of degree m_j. More exactly, we show that $\rho_j(t) = c_j t^{m_j}$ where c_j is a certain nonzero constant which will be settled later.

We need two lemmas.

Lemma 4.3. *With the above notations, we have*

$$e^{-i\mu_j t}e^{tA}E_j = \sum_{k=0}^{m_j-1}\frac{(A-i\mu_k I_n)^k}{k!}E_j t^k =: Q_{jA}(t). \tag{4.5}$$

Proof. For every $x \in \ker(A-i\mu_j I)^{m_j}$ and any integer $p \geq m_j$ one has $(A-i\mu_j I)^p x = 0$. Therefore,

$$(A-i\mu_j I)^p E_j = 0, \quad \text{for all } p \geq m_j.$$

Thus,

$$e^{-i\mu_j t}e^{tA}E_j = e^{(A-i\mu_j I)t}E_j = \sum_{r=0}^{\infty}\frac{(A-i\mu_j I)^r}{r!}E_j t^r = \sum_{r=0}^{m_j-1}\frac{(A-i\mu_j I)^r}{r!}E_j t^r.$$

□

Lemma 4.4. *The degree of the scalar polynomial $[Q_{jA}(t)]_{1n}$, given in (4.5), is equal to m_j-1.*

Proof. Let us consider the scalar polynomial $q_j(z) := \frac{P_A(z)}{(z-\lambda_j)^{m_j}}$. Clearly, the map $z \mapsto \frac{1}{q_j(z)}$ is analytic on $\overline{D}_r(\lambda_j)$. By (2.3) and (4.5) it is enough to prove that

$$a_{1n}^{(m_j-1)} := \frac{1}{2\pi i}\oint_{C_r(\lambda_j)}\left[\frac{(A-\lambda_j I)^{m_j-1}}{(m_j-1)!}R(z,A)\right]_{1n} dz$$

is a nonzero scalar.

We analyse two particular cases and then the general case arises naturally.

For $m_j = 1$, $[Q_{jA}(t)]_{1n} = [E_j]_{1n}$ and therefore

$$a_{1n}^{(0)} = \frac{1}{2\pi i}\oint_{C_r(\lambda_j)}\frac{1}{P_A(z)}\,dz = \frac{1}{2\pi i}\oint_{C_r(\lambda_j)}\frac{\frac{1}{q_j(z)}}{z-\lambda_j}\,dz = \frac{1}{q_j(\lambda_j)} \neq 0,$$

where the Cauchy integral formula was used.

For $m_j = 2$, we have

$$\left[\frac{A-\lambda_j I}{1!}R(z,A)\right]_{1n} = \frac{1}{P_A(z)}\begin{pmatrix} -\lambda_j & 1 & 0 & \cdots & 0 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ z \\ \vdots \\ z^{n-1} \end{pmatrix}$$

$$= \frac{z-\lambda_j}{P_A(z)} = \frac{\frac{1}{q_j(z)}}{z-\lambda_j}$$

which yields

$$a_{1n}^{(1)} = \frac{1}{2\pi i}\oint_{C_r(\lambda_j)} \left[\frac{A-\lambda_j I}{1!}R(z,A)\right]_{1n} dz = \frac{1}{2\pi i}\oint_{C_r(\lambda_j)} \frac{\frac{1}{q_j(z)}}{z-\lambda_j}\,dz = \frac{1}{q_j(\lambda_j)} \neq 0.$$

By continuing in this way, we obtain:

$$\begin{aligned}
&\left[\frac{(A-\lambda_j I)^{(m_j-1)}}{(m_j-1)!}R(z,A)\right]_{11} \\
&= \frac{1}{P_A(z)(m_j-1)!}\cdot\begin{pmatrix} C^0_{m_j-1}(-\lambda_j)^{m_j-1} & \cdots & C^{m_j-1}_{m_j-1}(-\lambda_j)^0 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ z \\ \vdots \\ z^{n-1}\end{pmatrix} \\
&= \frac{\sum_{k=o}^{m_j-1} C^k_{m_j-1} z^k (-\lambda_j)^{m_j-1-k}}{(m_j-1)!P_A(z)} \\
&= \frac{(z-\lambda_j)^{m_j-1}}{(m_j-1)!P_A(z)} = \frac{\frac{1}{q_j(z)}}{(m_j-1)!(z-\lambda_j)}
\end{aligned}$$

and by applying again the Cauchy theorem, we get

$$\begin{aligned}
a_{1n}^{(m_j-1)} &= \frac{1}{2\pi i}\oint_{C_r(\lambda_j)} \left[\frac{(A-\lambda_j I)^{(m_j-1)}}{(m_j-1)!}R(z,A)\right]_{11} dz \\
&= \frac{1}{(m_j-1)!q_j(\lambda_j)}
\end{aligned}$$

which is a nonzero scalar and we get the desired assertion.

□

Remark 4.5. A similar argument allows us to state that

$$a_{1n}^{(k)} := \left[\frac{1}{2\pi i}\oint_{C_r(\lambda_j)} \frac{(A-\lambda_j I)^k}{k!}R(z,A)\,dz\right]_{1n} = 0$$

whenever $m_j > 1$ and $k < m_j - 1$.

Returning to the proof of the theorem, note that in view of (4.5):

$$\begin{aligned}
\left[\int_0^t E_j e^{(t-s)A}\Theta(s)\,ds\right]_{11} &= \int_0^t e^{i\mu_j(t-s)}\left[e^{-i\mu_j(t-s)}E_j e^{(t-s)A}e^{i\mu_j s}u_0\right]_{11} ds \\
&= e^{i\mu_j t}\int_0^t [Q_{jA}]_{1n}(t-s)v_0\,ds \\
&= e^{i\mu_j t}\int_0^t \left[E_j\sum_{k=0}^{m_j-1}\frac{(A-\lambda_j I)^k}{k!}(t-s)^k\right]_{1n} v_0\,ds \\
&= e^{i\mu_j t}\int_0^t \frac{1}{(m_j-1)!q_j(\lambda_j)}(t-s)^{m_j-1}v_0\,ds \\
&= e^{i\mu_j t}\frac{1}{m_j!q_j(\lambda_j)}t^{m_j}v_0,
\end{aligned}$$

v_0 being the scalar defined in (4.4).

Then

$$e^{-i\mu_j t}\left[E_j\left(e^{tA}(x-x_0)+\int_0^t e^{(t-s)A}\Theta(s)\,ds\right)\right]_{11}=[P_j(t)(x-x_0)]_{11}+\rho_j(t)$$

is a polynomial in t of degree $m_j \geq 1$, since it is the sum of a polynomial of degree m_j with a polynomial of degree at most $m_j - 1$. This contradicts the fact that the map

$$t \mapsto \left[E_j\left(e^{tA}(x-x_0)+\int_0^t e^{(t-s)A}\Theta(s)\,ds\right)\right]_{11}$$

is bounded on $\mathbb{R}_+$.

$(\beta) \Rightarrow (\alpha)$. The assertion follows via the proof of the second part of the Theorem 2.1.

□

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