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EIGENFUNCTION EXPANSIONS OF A QUADRATIC PENCIL OF DIFFERENTIAL OPERATORS WITH PERIODIC GENERALIZED POTENTIAL

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Abstract. In this article we obtain the eigenfunction expansions of a quadratic pencil of Sturm–Liouville operators with periodic coefficients. The important point to note here is the given potential is a first order generalized function.

1. INTRODUCTION

The idea of expanding an arbitrary function in terms of the solutions of a second-order differential equation goes back to the time of Sturm and Liouville, more than a hundred years ago. The first satisfactory proofs were constructed by various authors early in the twentieth century. The second-order linear differential equation with real periodic coefficients, commonly known as Hill's equation, has been investigated by many mathematicians. An account of much of this theory is given in [9]. Further results relating to spectral theory are given in [4]. A characterization of the spectrum of Hill's operator is studied in [11]. The spectral problems of the quadratic pencil of differential operators with periodic potential are investigated in [7]. In the same place one can find wide bibliography. Differential operators with periodic generalized potential are widely used in applications to quantum and atomic physics to produce exact solvable models of complicated physical phenomena in [1–3, 5, 8, 12, 15].

In this paper, we study the eigenfunction problems of the following quadratic pencil of differential equation with generalized potential

$$\ell_\alpha[y] := -y'' + 2\alpha\lambda \sum_{n=-\infty}^{\infty} \delta(x-n)y + q(x)y = \lambda^2 y, \quad x \in \mathbb{R}, \tag{1.1}$$

where $q(x)$ is a 1-periodic, real, non-negative and piecewise continuous function; $\delta(x)$ is the Dirac's delta function; $\alpha \neq 0$ is a real number and λ is a spectral parameter.

The equation (1.1) is equivalent to the following many-point boundary problem:

$$-y''(x) + q(x)y = \lambda^2 y \tag{1.2}$$

$$\begin{pmatrix} y(n^+) \\ y'(n^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2\alpha\lambda & 1 \end{pmatrix} \begin{pmatrix} y(n^-) \\ y'(n^-) \end{pmatrix} \tag{1.3}$$

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such that $y(x) \in H^{2,2}(\mathbb{R}\backslash\mathbb{Z}) \bigcap H^{2,1}(\mathbb{R})$, where the symbol $H^{m,n}$ denotes the Sobolev space (see [14]).

In order to obtain an eigenfunction expansion of (1.1) we have to know the structure of spectrum and we will expose this in Sections 2–4.

2. Hill's discriminant and Floquet theory

The Hill discriminant is at the heart of the spectral theory of the periodic Sturm–Liouville operator. If $y(x)$ is a solution of (1.1), then so is also $y(x+1)$. But generally, $y(x+1)$ is not the same as $y(x)$ and, indeed, (1.1) does not need to have a non-trivial solution with period 1, (see [4]). From (1.3), we give below that (1.1) has the property that there is a non-zero constant ρ and a non-trivial solution $y(x)$ such that from the properties of the delta function

$$\begin{pmatrix} 1 & 0 \\ -2\alpha\lambda & 1 \end{pmatrix} \begin{pmatrix} y(x+1) \\ y'(x+1) \end{pmatrix} = \rho \begin{pmatrix} y(x) \\ y'(x) \end{pmatrix}. \tag{2.1}$$

Let $\theta(x,\lambda)$ and $\varphi(x,\lambda)$ be linearly independent solutions of (1.2) which satisfy the initial conditions

$$\theta(0,\lambda) = 1, \;\; \theta'(0,\lambda) = 0, \;\; \varphi(0,\lambda) = 0, \;\; \varphi'(0,\lambda) = 1. \tag{2.2}$$

Since $\theta(x+1,\lambda)$ and $\varphi(x+1,\lambda)$ are also linearly independent solutions of (1.2), they can be written as a linear combination of $\theta(x,\lambda)$ and $\varphi(x,\lambda)$. Furthermore, every solution of (1.2) has the form

$$y(x,\lambda) = c_1\theta(x,\lambda) + c_2\varphi(x,\lambda)$$

where c_1 and c_2 are constants. To obtain a non-trivial solution of the system with respect to c_1 and c_2 by using the facts above and condition (2.1), the following equality must be satisfied

$$\rho^2 - [\varphi'(1,\lambda) + \theta(1,\lambda) - 2\alpha\lambda\varphi(1,\lambda)]\rho + 1 = 0.$$

This is a quadratic equation for ρ and it is satisfied by at least one non-zero value of ρ. Suppose that this equation has distinct solutions ρ_1 and ρ_2. Since ρ_1 and ρ_2 are non-zero, we can define μ_1 and μ_2 such that $e^{\mu_k} = \rho_k$, $(k = 1, 2)$. Now define $Y_k(x,\lambda) = e^{-\mu_k x} y_k(x,\lambda)$. Thus the general solution of (1.1) has the Floquet form

$$y(x,\lambda) = c_1 e^{\mu_1 x} Y_1(x,\lambda) + c_2 e^{\mu_2 x} Y_2(x,\lambda).$$

The function $F(\lambda)$ defined by

$$F(\lambda) = \varphi'(1,\lambda) + \theta(1,\lambda) - 2\alpha\lambda\varphi(1,\lambda) \tag{2.3}$$

is called a *discriminant* of (1.1) and we consider five cases as follows (see [4, pp. 6–7]):

(1) $F(\lambda) > 2$. There is a real number $\mu \neq 0$ such that $\rho = e^{\mu}, \rho = e^{-\mu}$. Thus

$$y(x,\lambda) = c_1 e^{\mu x} Y_1(x,\lambda) + c_2 e^{-\mu x} Y_2(x,\lambda). \tag{2.4}$$

(2) $F(\lambda) < -2$. The situation here is the same as in (1). Here μ must now be replaced by $\mu + i\pi$ in (2.4).

(3) $-2 < F(\lambda) < 2$. There is a real number ν with $0 < \nu < \pi$ (or $-\pi < \nu < 0$) such that

$$y(x,\lambda) = c_1 e^{i\nu x} Y_1(x,\lambda) + c_2 e^{-i\nu x} Y_2(x,\lambda).$$

(4) $F(\lambda) = \pm 2$. Then there is only one non-trivial solution $y_1(x,\lambda)$. Let us denote the other solution by $y_2(x,\lambda)$, defined as below, such that $y_1(x,\lambda)$ and $y_2(x,\lambda)$ are linearly independent.

$$\begin{aligned} y_1(x,\lambda) &= e^{\mu x} Y_1(x,\lambda), \\ y_2(x,\lambda) &= e^{\mu x}\left[\frac{d_1}{\rho} x Y_1(x,\lambda) + Y_2(x,\lambda)\right]. \end{aligned}$$

where d_1 is a constant and $d_1 = 0$ if and only if $\theta'(1,\lambda) = 2\alpha\lambda$ and $\varphi(1,\lambda) = 0$.

(5) If λ is not a real number, then the possible alternatives are: If $F(\lambda)$ is real, then one of the above cases is valid. If $F(\lambda)$ is not real, then there is a complex number μ such that $\rho = e^{\mu}, \rho = e^{-\mu}$ and (1.1) has two linearly independent solutions $y_1(x,\lambda) = e^{\mu x} Y_1(x,\lambda)$, $y_2(x,\lambda) = e^{-\mu x} Y_2(x,\lambda)$, where $Y_1(x,\lambda)$ and $Y_2(x,\lambda)$ are periodic with period 1.

Definition 2.1. The equation (1.1) is said to be (a) *unstable* if all non-trivial solutions are unbounded in $(-\infty,\infty)$, (b) *conditionally stable* if there is a non-trivial solution which is bounded in $(-\infty,\infty)$, (c) *stable* if all non-trivial solutions are bounded in $(-\infty,\infty)$.

From this definition and five cases above, we obtain the following theorem.

Theorem 2.2. *For fixed $\lambda \in (-\infty,\infty)$, the equation (1.1) is unstable if $|F(\lambda)| > 2$ and stable if $|F(\lambda)| < 2$ and also stable if $|F(\lambda)| = 2$ and $\theta'(1,\lambda) = 2\alpha\lambda, \varphi(1,\lambda) = 0$. Finally if $|F(\lambda)| = 2$ and $\theta'(1,\lambda) \neq 2\alpha\lambda$ or $\varphi(1,\lambda) \neq 0$ then (1.1) is conditionally stable.*

3. Stability and instability intervals

For $0 \le x \le 1$ the equation

$$y'' + [\lambda^2 - q(x)]y = 0 \tag{3.1}$$

and the boundary conditions

$$y(1) = e^{it} y(0), \quad y'(1) = e^{it}[y'(0) + 2\alpha\lambda y(0)] \tag{3.2}$$

are called a t-quasi-periodic boundary problem, where $t \in [0, 2\pi)$.

First we define the linear operator in the Hilbert space $H = L_2(0,1)$ by L^t as follows:

$$L^t y := -y'' + q(x)y$$

with domain

$$D(L^t) = \{y | y(x) \in H^{2,1}(0,1), \ell_0[y] \in H,\ y(1) = e^{it} y(0),\ y'(1) = e^{it}[y'(0) + 2\alpha\lambda y(0)]\}.$$

Theorem 3.1. *The eigenvalues of the operator L^t are real and the eigenvalues $\lambda_n(t)$ are the values of λ which satisfy the equation $F(\lambda) = 2\cos t$.*

Proof. Suppose that λ is an eigenvalue of the operator L^t and that $y(x)$ is a corresponding eigenfunction such that $(y,y) = 1$. Taking the inner product of both sides of (3.1) with $y(x)$ and using (3.2) we get

$$\lambda^2 + 2\alpha|y(0)|^2\lambda - \int_0^1 \left\{|y'(x)|^2 + q(x)|y(x)|^2\right\} dx = 0.$$

Since α is a real number and $q(x) \geq 0$, the roots of this equation are real numbers. Substituting $\rho = \exp(it)$ into (2.3), we obtain $F(\lambda) = 2\cos t$. ∎

Theorem 3.2. *The eigenvalues of the operator L^t are simple for $t \neq m\pi$ $(m = 0, \pm 1, \pm 2, \ldots)$.*

Proof. We suppose that $t \neq m\pi$ $(m = 0, \pm 1, \pm 2, \ldots)$ and $y_1(x)$ and $y_2(x)$ are linearly independent eigenfunctions corresponding to the eigenvalue λ of the operator L^t. Thus, for all λ, the solutions of the equation (3.1) especially $\theta(x, \lambda)$ and $\varphi(x, \lambda)$ can be written as a linear combination of the functions $y_1(x)$ and $y_2(x)$ and these solutions satisfy the boundary conditions (3.2). It follows that

$$\begin{aligned} F(\lambda) &= \theta(1, \lambda) + \varphi'(1, \lambda) - 2\alpha\lambda\varphi(1, \lambda) \\ &= e^{it}\theta(0, \lambda) + e^{it}[\varphi'(0, \lambda) + 2\alpha\lambda\varphi(0, \lambda)] - 2\alpha\lambda\varphi(0, \lambda) = 2e^{it}. \end{aligned}$$

From Theorem 3.1, we arrive at $\cos t = e^{it}$. But this equality holds only for $t = m\pi$ $(m = 0, \pm 1, \pm 2, \ldots)$ which contradicts the assumption. ∎

The periodic and quasi-periodic problems associated with (3.1) and (3.2) correspond to the cases when m is an even (resp. odd) number and their eigenvalues are the zeros of $F(\lambda) = 2$ (resp. $F(\lambda) = -2$).

Theorem 3.3. *The eigenvalues of the operator L^t are of the second order if and only if*

$$\varphi(1, \lambda) = 0, \quad \theta'(1, \lambda) = 2\alpha\lambda.$$

Proof. The proof of the theorem is immediately obtained from the fact that $\varphi(x, \lambda)$ and $\theta(x, \lambda)$ satisfy the conditions (3.2). ∎

It follows from the formula in [16, p. 292], for large $|\lambda|$

$$F(\lambda) = 2\sqrt{1 + \alpha^2}\sin(\lambda + \beta) + O\left(\frac{e^{|\mathrm{Im}\lambda|}}{|\lambda|}\right), \tag{3.3}$$

where $\tan\beta = -1/\alpha$. From using the asymptotic formula (3.3) and applying Rouché's theorem, we can see that the equation $F(\lambda) = 2\cos t$ has countable many roots: $\lambda_k(t)$ $(k = 0, \pm 1, \pm 2, \ldots)$. Hence from Theorem 3.1, all eigenvalues $\lambda_k(t)$ $(k = 0, \pm 1, \pm 2, \ldots)$ are nonzero real numbers and satisfy the following inequalities

$$\cdots \leq \lambda_{-2}(t) \leq \lambda_{-1}(t) \leq \lambda_0(t) \leq \lambda_1(t) \leq \lambda_2(t) \leq \cdots \tag{3.4}$$

Now we will give existence and certain form of stability and instability intervals of the equation (1.1). For this, we will use the properties of the function $F(\lambda)$.

Lemma 3.4. *For $q(x) \neq 0$, $F(0) > 2$.*

Lemma 3.5. *If $F(\lambda) < 2$ then $\frac{dF(\lambda)}{d\lambda} \neq 0$.*

Lemma 3.6. *Let $|F(\lambda)| = 2$. Then $\frac{dF(\lambda_0)}{d\lambda} = 0$ if and only if*

$$\varphi(1, \lambda_0) = 0, \quad \theta'(1, \lambda_0) = 2\alpha\lambda_0. \tag{3.5}$$

In addition

a) Let $F(\lambda_0) = 2$. If $\frac{dF(\lambda_0)}{d\lambda} = 0$ then $\frac{d^2F(\lambda_0)}{d\lambda^2} < 0$.

b) Let $F(\lambda_0) = -2$. If $\frac{dF(\lambda_0)}{d\lambda} = 0$ then $\frac{d^2F(\lambda_0)}{d\lambda^2} > 0$.

The proofs of these lemmas are seen by using the method in [16, p. 290]. Considering all lemmas above, we can derive the following results:

Corollary 3.7. *The functions $F(\lambda) \mp 2$ do not have a zero of order higher than second.*

Corollary 3.8. *The zeros of the function $F(\lambda) - 2$ are of the second order if and only if $F(\lambda)$ has a maximum value at these zeros. The zeros of the function $F(\lambda)+2$ are of the second order if and only if $F(\lambda)$ has a minimum value at these zeros.*

Theorem 3.9.

1) Let $\alpha_{2k}^{\pm}, \alpha_{2k+1}^{\pm}$ $(k = 0, \pm 1, \pm 2, \ldots)$ be eigenvalues of the periodic and quasi-periodic boundary problem respectively. Then the numbers $\alpha_{2k}^{\pm}$ and $\alpha_{2k+1}^{\pm}$ occur in the order

$$\cdots < \alpha_{-2}^{-} \leq \alpha_{-2}^{+} < \alpha_{-1}^{-} \leq \alpha_{-1}^{+} < \alpha_{0}^{-} \leq \alpha_{0}^{+} < \alpha_{1}^{-} \leq \alpha_{1}^{+} < \alpha_{2}^{-} \leq \alpha_{2}^{+} < \cdots$$

2) In the intervals $[\alpha_{2k}^{+}, \alpha_{2k+1}^{-}]$, $F(\lambda)$ decreases from $+2$ to -2. In the intervals $[\alpha_{2k+1}^{+}, \alpha_{2k+2}^{-}]$, $F(\lambda)$ increases from -2 to $+2$.

3) In the intervals $(\alpha_{2k}^{-}, \alpha_{2k}^{+})$, $F(\lambda) > 2$. In the intervals $(\alpha_{2k+1}^{-}, \alpha_{2k+1}^{+})$, $F(\lambda) < -2$.

Thus the stability intervals of (1.1) are $(\alpha_{k-1}^{+}, \alpha_{k}^{-})$ and that the conditional stability intervals are the closures of these intervals. The instability intervals of (1.1) are $(\alpha_{k}^{-}, \alpha_{k}^{+})$.

4. Nature of the spectrum of the operator $L(\lambda)$

We denote the pencil operator in $L_2(\mathbb{R})$ of the differential expression

$$\frac{d^2}{dx^2} + \lambda^2 - 2\alpha\lambda \sum_{n=-\infty}^{\infty} \delta(x-n) - q(x)$$

by $L(\lambda)$ and D is a maximal domain such that

$$D = \Big\{ y \mid y(x) \in H^{2,2}(\mathbb{R}\backslash\mathbb{Z}) \cap H^{2,1}(\mathbb{R}),\ y(n^+) = y(n^-) = y(n),$$
$$y'(n^+) - y'(n^-) = 2\alpha\lambda y(n),\ -y'' + 2\alpha\lambda \sum_{n=-\infty}^{\infty} \delta(x-n)y + q(x)y \in L_2(\mathbb{R}) \Big\}.$$

We note that the functions y'' and $2\alpha\lambda \sum_{n=-\infty}^{\infty} \delta(x-n)y$ are (delta type) generalized functions of order 1, such that $-y'' + 2\alpha\lambda \sum_{n=-\infty}^{\infty} \delta(x-n)y$ in $L_2(\mathbb{R})$, (see [10]).

Let us denote the set consisting of the conditional stability intervals of (1.1) by S. We prove first in this section that the spectrum of $L(\lambda)$ denoted by σ is continuous, that is, $L(\lambda)$ has no eigenvalues, and then that σ coincides with S.

Theorem 4.1. *The spectrum of $L(\lambda)$ is continuous.*

Proof. If $L(\lambda)$ had an eigenvalue λ_0 with corresponding eigenfunction $\psi(x)$, we would have $L(\lambda)\psi(x) = 0$. Then $\psi(x)$ would be a non-trivial solution of (1.1). But from cases 1–5 of §2, (1.1) has no such non-trivial solution $\psi(x)$ in $L_2(-\infty, \infty)$ for any complex number λ and this finishes the proof. ∎

Theorem 4.2. *The sets σ and S are identical.*

Proof. We show first that $S \subset \sigma$. We suppose then that if λ_0 is any point in S then λ_0 is also in σ. Referring to cases 1–5 of §2, there is, for λ_0 in S, at least one non-trivial solution $\psi(x)$ of (1.1), with $\lambda = \lambda_0$, such that $|\rho| = 1$.

On the other hand, let $g(x)$ be any function with a continuous second derivative in $[0, 1]$ such that

$$g(0) = 0,\ g(1) = 1,\ g'(0) = g''(0) = g'(1) = g''(1) = 0,\ 0 \le g(x) \le 1.$$

Now define a sequence $\{f_n(x)\}$ as follows

$$f_n(x) = b_n \psi(x) h_n(x)$$

in $(-\infty, \infty)$, where

$$h_n(x) = \begin{cases} 1, & |x| \le (n-1) \\ g(n - |x|), & (n-1) \le |x| \le n \\ 0, & |x| \ge n \end{cases}$$

and b_n is the normalization constant making $\|f_n\| = 1$. Since $h_n(x) = 1$ throughout $(-n, n)$ except for any interval of length 1 at each end, we have

$$b_n \sim \left(2n \int_0^1 |\psi(x)|^2 dx\right)^{-\frac{1}{2}}$$

as $n \to \infty$. In particular,

$$b_n \to 0 \tag{4.1}$$

as $n \to \infty$. It is clear that, $f_n(x) \in D$ and

$$L(\lambda_0) f_n(x) = b_n [2\psi'(x) h_n'(x) + \psi(x) h_n''(x)]$$

hence

$$\|L(\lambda_0) f_n(x)\| \le |b_n| [2 \|\psi'(x) h_n'(x)\| + \|\psi(x) h_n''(x)\|] \le K |b_n|,$$

where K is a nonnegative number and does not depend on n. From (4.1), we get that $\|L(\lambda_0) f_n(x)\| \to 0$ as $n \to \infty$. It follows from Theorem 5.2.2 in [4, p. 81] that λ_0 is in σ, and therefore $S \subset \sigma$.

To prove the reverse inclusion $\sigma \subset S$, we suppose now that λ_0 is not in S and prove that λ_0 is not in σ. For λ_0 is not in S, the following three possibilities can occur:

i) $\lambda_0 \in (-\infty, \infty)$ and $|F(\lambda_0)| > 2$;
ii) $\operatorname{Im} \lambda_0 \neq 0$ and $|F(\lambda_0)|$ is real;
iii) $\operatorname{Im} \lambda_0 \neq 0$ and $|F(\lambda_0)|$ is not real.

i) The analysis for $F(\lambda_0) > 2$ and $F(\lambda_0) < -2$ is virtually the same and so we write it out only for $F(\lambda_0) > 2$. Then there are solutions $\psi_1(x)$ and $\psi_2(x)$ of (1.1), with $\lambda = \lambda_0$. Thus, we define the Green's function $G(x, \xi; \lambda_0)$ for the equation $L(\lambda_0) = f(x)$ from using the method in [13] and then we define the linear operator R by

$$y(x, \lambda_0) = Rf(x) = \int_{-\infty}^{\infty} G(x, \xi; \lambda_0) f(\xi) d\xi \tag{4.2}$$

where $f(x) \in L_2(-\infty, \infty)$. It can be seen that R is a bounded operator. This means that λ_0 is in the resolvent set of $L(\lambda_0)$ and so λ_0 is not in σ.

ii) It is enough to show that $|F(\lambda_0)| > 2$, then the same proof works for this case. Indeed, if $|F(\lambda_0)| \leq 2$, then there's at least one $t_0 \in (-\infty, \infty)$ which satisfies the equality $F(\lambda_0) = 2\cos t_0$. This means that λ_0 is the eigenvalue and from Theorem 3.1 this eigenvalue is a real number. This contradicts our assumption $\operatorname{Im} \lambda_0 \neq 0$.

iii) There are two linearly independent solutions of (1.1) for $\lambda = \lambda_0$. Hence the same proof as in i) works for this case, too.

∎

Corollary 4.3. *The operator $L(\lambda)$ has a continuous spectrum consisting of the intervals $[\alpha_{k-1}^+, \alpha_k^-]$ $(k = 0, \pm 1, \pm 2, \ldots)$.*

Definition 4.4. In the sequel, the segments on the real axis $[\alpha_{k-1}^+, \alpha_k^-]$ $(k = 0, \pm 1, \pm 2, \ldots)$ will be said to be the bands of the spectrum of the operator $L(\lambda)$, and the intervals (α_k^-, α_k^+) $(k = 0, \pm 1, \pm 2, \ldots)$ will be called the gaps.

Theorem 4.5. *The number of gaps in the spectrum of $L(\lambda)$ is infinite and the lengths of gaps tend to infinity as $n \to \infty, (q(x) \neq 0)$.*

Proof. If we apply arguments stated in [16, p. 296] for the function $F(\lambda) = \theta(1, \lambda) + \varphi'(1, \lambda) + 2\alpha\lambda\varphi(1, \lambda)$ then we find

$$(\alpha_{2n}^+)^2 - (\alpha_{2n}^-)^2 = 4\alpha\pi n + O(1).$$

∎

5. Eigenfunction expansions

In the present section, we obtain the eigenfunction expansions by using the above results and the methods in [6] and [7]. First, we consider the Green's function of the t-quasi periodic boundary problem (3.1), (3.2). After long processes, we have the Green's function, (see [13]),

$$G_t(x, \xi; \lambda) = \frac{1}{\Delta_t(\lambda)} [\theta(x, \lambda) h_t(\xi, \lambda) - \varphi(x, \lambda) g_t(\xi, \lambda)] + \omega(x, \xi, \lambda), \tag{5.1}$$

where

$$\Delta_t(\lambda) = -[F(\lambda) - 2\cos t], \tag{5.2}$$

$$h_t(\xi, \lambda) = \varphi(1, \lambda)\theta(\xi, \lambda) + [e^{-it} - \theta(1, \lambda)]\varphi(\xi, \lambda), \tag{5.3}$$

$$g_t(\xi,\lambda) = \left[e^{-it} - \varphi'(1,\lambda)\right]\theta(\xi,\lambda) + \theta'(1,\lambda)\varphi(\xi,\lambda)$$
$$-2\alpha\lambda\left[\theta(1,\lambda)\varphi(\xi,\lambda) - \theta(\xi,\lambda)\varphi(1,\lambda)\right], \tag{5.4}$$
$$\omega(x,\xi;\lambda) = \begin{cases} \varphi(x,\lambda)\theta(\xi,\lambda) - \theta(\xi,\lambda)\varphi(x,\lambda), & 0 \le \xi \le x \\ 0, & x \le \xi \le 1. \end{cases}$$

Thus for all $f(x) \in L_2[0,1]$, the solution of the t-quasi periodic boundary problem (3.1), (3.2) can be written as

$$y(x,\lambda) = \int_0^1 G_t(x,\xi;\lambda) f(\xi) d\xi.$$

Theorem 5.1. *The following formula is correct for $t \neq \pi m$, $(m = 0, \pm 1, \pm 2, \ldots)$.*

$$G_t(x,\xi;\lambda) = -\frac{b_k(t)\psi_{k,t}(x)\overline{\psi_{k,t}(\xi)}}{\lambda - \lambda_k(t)} + \omega_{k,t}(x,\xi;\lambda), \tag{5.5}$$

where

$$b_k(t) = \left\{\varphi(1,\lambda_k(t))\frac{dF(\lambda_k(t))}{d\lambda}\right\}^{-1}, \tag{5.6}$$

$$\psi_{k,t}(x) = \varphi(1,\lambda_k(t))\theta(x,\lambda_k(t)) + [e^{it} - \theta(1,\lambda_k(t))]\varphi(x,\lambda_k(t)) \tag{5.7}$$

and $\omega_{k,t}(x,\xi;\lambda)$ is a regular function about a point $\lambda = \lambda_k(t)$.

It is easy to check that $\psi_{k,t}(x)$, determined by (5.7), satisfies the following two equalities

$$\psi_{k,t}(1) = e^{it}\psi_{k,t}(0), \quad \psi'_{k,t}(1) = e^{it}\left[\psi'_{k,t}(0) + 2\alpha\lambda\psi_{k,t}(0)\right]. \tag{5.8}$$

Consequently, these functions are the solutions of the t-quasi periodic boundary problem (3.1), (3.2).

Theorem 5.2. *We suppose that the function $f(x)$ is twice (continuously) differentiable and* $\operatorname{supp} f(x) \subset (0,1)$. *Then as $|\lambda| \to \infty$*

$$\int_0^1 G_t(x,\xi;\lambda) f(\xi) d\xi = \frac{f(x)}{\lambda^2} + O\left(\frac{1}{\lambda^3}\right). \tag{5.9}$$

So by using contour integration method, Parseval's equality, (5.5) and (5.9) we arrive at

$$\sum_{k=-\infty}^{\infty} b_k(t)\left|\int_0^1 f(x)\bar{\psi}_{k,t}(x)dx\right|^2 = 0,$$
$$-\sum_{k=-\infty}^{\infty} \lambda_k(t) b_k(t)\left|\int_0^1 f(x)\bar{\psi}_{k,t}(x)dx\right|^2 = \int_0^1 |f(x)|^2 dx. \tag{5.10}$$

These equalities can also be derived for $f(x) \in L_2[0,1]$. Now we will obtain the eigenfunction expansion on the real axis. Let $f(x)$ be a continuous function and vanish except on a finite interval. Let us consider the following function (see [6])

$$f_t(x) = \sum_{m=-\infty}^{\infty} f(x+m)e^{-imt} \tag{5.11}$$

Since $f(x)$ is a finite function, this sum is also finite. It is easy to check that the function $\psi_{k,t}(x)$ defined for $-\infty < x < \infty$ satisfies the following equality

$$\psi_{k,t}(x+1) = e^{it}\psi_{k,t}(x). \tag{5.12}$$

Thus from (5.11) and (5.12), we get

$$\int_0^1 f_t(x)\,\bar{\psi}_{k,t}(x)\,dx = \int_{-\infty}^{\infty} f(x)\,\bar{\psi}_{k,t}(x)\,dx. \tag{5.13}$$

Replacing $f_t(x)$ by $f(x)$ in equalities (5.10) we obtain

$$\begin{gathered}\sum_{k=-\infty}^{\infty} b_k(t)\left|\int_0^1 f_t(x)\,\bar{\psi}_{k,t}(x)\,dx\right|^2 = 0,\\ -\sum_{k=-\infty}^{\infty}\lambda_k(t)\,b_k(t)\left|\int_0^1 f_t(x)\,\bar{\psi}_{k,t}(x)\,dx\right|^2 = \int_0^1 |f_t(x)|^2\,dx.\end{gathered} \tag{5.14}$$

Furthermore, from (5.7) and (5.13) we have the following equality

$$\int_0^1 f_t(x)\,\bar{\psi}_{k,t}(x)\,dx = \varphi(1,\lambda_k)\,\mathcal{F}_1(\lambda_k) + \left[e^{-it} - \theta(1,\lambda_k)\right]\mathcal{F}_2(\lambda_k),$$

where $\lambda_k = \lambda_k(t)$ and

$$\mathcal{F}_1(\lambda) = \int_{-\infty}^{\infty} f(x)\,\theta(x,\lambda)\,dx, \quad \mathcal{F}_2(\lambda) = \int_{-\infty}^{\infty} f(x)\,\varphi(x,\lambda)\,dx. \tag{5.15}$$

Without loss of generality we can assume that $f(x)$ is a real function. After some operations we have

$$\left|\int_0^1 f_t(x)\,\bar{\psi}(x)\,dx\right|^2 = \varphi(1,\lambda_k(t))\,\Phi(\lambda_k(t)), \tag{5.16}$$

where

$$\begin{aligned}\Phi(\lambda_k(t)) = {}& \varphi(1,\lambda)\,\mathcal{F}_1^2(\lambda) - \left[\theta'(1,\lambda) - 2\alpha\lambda\theta(1,\lambda)\right]\mathcal{F}_2^2(\lambda)\\ &+ \left[\varphi'(1,\lambda) - \theta(1,\lambda) - 2\alpha\lambda\varphi(1,\lambda)\right]\mathcal{F}_1(\lambda)\,\mathcal{F}_2(\lambda).\end{aligned} \tag{5.17}$$

By substituting (5.16) in (5.14) and using (5.6) we arrive at

$$\begin{gathered}\sum_{k=-\infty}^{\infty}\Phi(\lambda_k(t))\,\upsilon(\lambda_k(t))\,\dot{\lambda}_k(t) = 0,\\ \sum_{k=-\infty}^{\infty}\lambda_k(t)\,\Phi(\lambda_k(t))\,\upsilon(\lambda_k(t))\,\dot{\lambda}_k(t) = \int_0^1 |f_t(x)|^2\,dx\end{gathered} \tag{5.18}$$

where the dot denotes the derivative with respect to t and

$$\upsilon(\lambda) = \left\{4 - \left[\theta(1,\lambda) + \varphi'(1,\lambda) - 2\alpha\lambda\varphi(1,\lambda)\right]^2\right\}^{-\frac{1}{2}}. \tag{5.19}$$

We integrate both sides of the equations (5.18) with respect to t over $[0,\pi]$ and take into account the conditional stability set of the equation (1.1). Then substitute λ for $\lambda_k(t)$ in all integrals we arrive at the following expansion formulas

$$
\begin{aligned}
0 &= \left(\sum_{k=-\infty}^{\infty} \int_{\alpha_{2k}^{+}}^{\alpha_{2k+1}^{-}} - \sum_{k=-\infty}^{\infty} \int_{\alpha_{2k+1}^{+}}^{\alpha_{2k+2}^{-}} \right) \Psi(x,\lambda)\, \upsilon(\lambda)\, d\lambda, \\
f(x) &= \frac{1}{\pi} \left(\sum_{k=-\infty}^{\infty} \int_{\alpha_{2k}^{+}}^{\alpha_{2k+1}^{-}} - \sum_{k=-\infty}^{\infty} \int_{\alpha_{2k+1}^{+}}^{\alpha_{2k+2}^{-}} \right) \lambda \Psi(x,\lambda)\, \upsilon(\lambda)\, d\lambda,
\end{aligned}
\tag{5.20}
$$

where

$$
\begin{aligned}
\Psi(x,\lambda) &= \Phi_1(\lambda)\,\theta(x,\lambda) + \Phi_2(\lambda)\,\varphi(x,\lambda), \\
\Phi_1(\lambda) &= \varphi(1,\lambda)\,\mathcal{F}_1(\lambda) + \frac{1}{2}\left[\varphi'(1,\lambda) - \theta(1,\lambda)\right]\mathcal{F}_2(\lambda), \\
\Phi_2(\lambda) &= \frac{1}{2}\left[\varphi'(1,\lambda) - \theta(1,\lambda) - 2\alpha\lambda\varphi(1,\lambda)\right]\mathcal{F}_1(\lambda) \\
&\quad - \left[\theta'(1,\lambda) - 2\alpha\lambda\theta(1,\lambda)\right]\mathcal{F}_2(\lambda),
\end{aligned}
\tag{5.21}
$$

where the functions $\mathcal{F}_1(\lambda)$ and $\mathcal{F}_2(\lambda)$ are obtained from formulas (5.15) and $\upsilon(\lambda)$ is given in (5.19).

We note that these results have been given for differential operators but not for the quadratic pencil with regular potential in [16] and for higher order self-adjoint differential operators in [17].

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