Electronic Journal of Qualitative Theory of Differential Equations
2013, No. **35**, 1-16; **http://www.math.u-szeged.hu/ejqtde/**

Stabilization and robustness of constrained linear systems

M. Ouzahra
Department of Mathematics & Informatics, ENS
University of Sidi Mohamed Ben Abdellah
P.O. Box 5206, Fès, Morocco.
m.ouzahra@yahoo.fr.

Abstract: In this paper, we consider the feedback stabilization of linear systems in a Hilbert state space. The paper proposes a class of nonlinear controls that guarantee exponential stability for linear systems. Applications to stabilization with saturating controls are provided. Also the robustness of constrained stabilizing controls is analyzed.

Mathematics Subject Classification: 93D15, 93D21

Keywords: Distributed linear system, exponential stabilization, robustness, bounded controls.

1 *Introduction*

In this paper, we consider the following linear system :

$$\frac{dz(t)}{dt} = Az(t) + Bu(t), \;\; z(0) = z_0, \tag{1}$$

where the state space is a Hilbert H with inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $\|.\|$, the Hilbert space U with norm $\| \cdot \|_U$ is the space of control and $u(t) \in U$ is a control subject to the constraint $\|u(t)\|_U \leq u_{\max}, \; u_{\max} > 0$. The operator $B : U \to H$ is linear and bounded, and the unbounded operator $A : \mathcal{D}(A) \subset H \to H$ is an infinitesimal of a semigroup of contractions $S(t)$ on H. The radial projection onto the unit ball enables us to define the following bounded control :

$$u_1(t) = \frac{-B^* z(t)}{\sup\left(1, \|B^* z(t)\|_U\right)}.$$

This control guarantees weak and strong stabilization for a class of linear systems under the approximate controllability assumption : $B^* S(t) y = 0, \; \forall t \geq 0 \Rightarrow y = 0$ (see [13, 14]). Furthermore, under the following exact controllability assumption :

$$\int_0^T \|B^* S(t) y\|_U^2 dt \geq \alpha \|y\|^2, \; \forall y \in H, \; (T, \alpha > 0),$$

strong and exponential stabilization results have been established by [3], using the feedback $u_1(t)$ and the following smooth control :

$$u_2(t) = -\frac{B^* z(t)}{1 + \|B^* z(t)\|_U}.$$

The purpose of this paper is to give necessary and sufficient conditions for exponential stabilization of an autonomous nonlinear systems. Then we give applications to problems of local and global exponential stabilization and robustness for constrained control systems. The plan of the paper is as follows : in the second section, we give necessary and sufficient conditions for exponential stability of an autonomous nonlinear system. The third section is devoted to problems of stabilization of the linear system (1) using bounded controls. The robustness problem is considered in the fourth section. Finally, an illustrating example is given in the fifth section.

2 Exponential stability

In this section, we discuss the stabilization question of the following autonomous system :

$$\frac{dz(t)}{dt} = Az(t) + Nz(t), \; z(0) = z_0, \tag{2}$$

where the state space is a Hilbert H with inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $\|.\|$, the dynamic A is an unbounded operator with domain $\mathcal{D}(A) \subset H$ and generates a semigroup of contractions $S(t)$ on H, and N is a nonlinear operator from H into H such that $N(0) = 0$, so that 0 is an equilibrium for (2).

2.1 Definitions and notations

Let us give the following definition regarding the stability of system (2).

Definition 1 *We say the origin is exponentially stable on a set $Y \subset H$ if, for all initial states z_0 in Y, there exist $M, \sigma > 0$ (depending on z_0) such that the mild solution $z(t)$ starting at z_0 satisfies*

$$\|z(t)\| \leq Me^{-\sigma t}\|z_0\|, \ \forall t \geq 0\cdot \tag{3}$$

The origin is said to be uniformly exponentially stable on Y if (3) holds for some σ and M, which are independent of z_0. It is said to be globally exponentially (resp. globally uniformly exponentially) stable if it is exponentially (resp. uniformly exponentially) stable on $Y = H$.

To state stabilization results for (2) we consider, for $\rho > 0$, the assumption :

$$\int_0^T |\langle NS(t)y, S(t)y\rangle| dt \geq \delta_\rho \|y\|^2, \ \forall y \in \mathcal{B}_\rho, \tag{4}$$

where $T, \delta_\rho > 0$ and $\mathcal{B}_\rho = \{y \in H / \ \|y\| \leq \rho\}$. In this case we set

$$\delta_\rho(N) = \inf_{0<\|y\|\leq\rho} \frac{\langle NS(\cdot)y, S(\cdot)y\rangle_{L^1(0,T)}}{\|y\|^2}.$$

We also consider the following strong controllability assumption :

$$\int_0^T |\langle NS(t)y, S(t)y\rangle| dt \geq \delta \|y\|^2, \ \forall y \in H, \tag{5}$$

where $T, \delta > 0$, and let us set : $\delta(N) = \inf_{y\in H-\{0\}} \dfrac{\langle NS(\cdot)y, S(\cdot)y\rangle_{L^1(0,T)}}{\|y\|^2}$.
On the other hand if N is Lipschitz on $\mathcal{B}_\rho$, then there exists $L_\rho > 0$ such that

$$\|N(z) - N(y)\| \leq L_\rho \|z - y\|, \ \ \forall (z, y) \in \mathcal{B}_\rho^2.$$

In this case, we can set : $L_\rho(N) = \sup_{(y,z)\in\mathcal{B}_\rho^2; y\neq z} \dfrac{\|N(z) - N(y)\|}{\|z - y\|}$ so that :

$$\|N(z) - N(y)\| \leq L_\rho(N)\|z - y\|, \ \ \forall (z, y) \in \mathcal{B}_\rho^2, \tag{6}$$

and when N is Lipschitz we set $L(N) = \sup_{z\neq y} \dfrac{\|N(z) - N(y)\|}{\|z - y\|}$.

2.2 Sufficient conditions for exponential stability

Our first result concerns the local exponential stability and is stated as follows :

Theorem 1 *Let (i) A generate a semigroup $S(t)$ of contractions on H, (ii) N be dissipative (i.e., $\langle Ny, y\rangle \leq 0, \ \forall y \in H$) and Lipschitz on any bounded set, and let (iii) (4) hold. Then*

1) for all $z_0 \in \mathcal{B}_\rho$ such that $L_{\|z_0\|}(N) < \dfrac{1}{T}(\dfrac{\delta_\rho(N)}{2})^{\frac{1}{2}}$ we have $z(t) \to 0$, exponentially, as $t \to +\infty$.

2) if $L_\rho(N) < \dfrac{1}{T}(\dfrac{\delta_\rho(N)}{2})^{\frac{1}{2}}$, then the system (2) is uniformly exponentially stable on $\mathcal{B}_\rho$.

Proof. 1) Since N is locally Lipschitz, the system (2) has a unique local mild solution $z(t)$, and since N is dissipative, then $z(t)$ is bounded in time and hence it is defined for all $t \geq 0$. Furthermore, $z(t)$ is given by the variation of constants formula :

$$z(t) = S(t)z_0 + \int_0^t S(t-s)Nz(s)ds. \tag{7}$$

Since $S(t)$ is a semigroup of contractions (so that A is dissipative), then by using approximation techniques and proceeding as in [1], we obtain the following inequality :

$$\|z(t)\|^2 - \|z(s)\|^2 \leq 2\int_s^t \langle Nz(\tau), z(\tau)\rangle \, d\tau, \ \forall t, s \geq 0; \ s \leq t. \tag{8}$$

It follows that

$$\|z(t)\| \leq \|z_0\|, \ \forall t \geq 0. \tag{9}$$

For all $z_0 \in \mathcal{B}_\rho$ and $t \geq 0$, we have the relation

$$\langle NS(t)z_0, S(t)z_0\rangle = \langle NS(t)z_0 - Nz(t), S(t)z_0\rangle + \langle Nz(t), y(t)\rangle - \langle Nz(t), z(t)\rangle \,,$$

where $y(t) = \displaystyle\int_0^t S(t-s)Nz(s)ds.$

Then, using (6) and (9) and the fact that the semigroup $S(t)$ is of contractions, we deduce that

$$|\langle NS(t)z_0, S(t)z_0\rangle| \leq L_{\|z_0\|}(N)\|y(t)\| \left(\|S(t)z_0\| + \|z(t)\|\right) - \langle Nz(t), z(t)\rangle \,, \ \forall t \in [0, T]\cdot$$

It follows that

$$|\langle NS(t)z_0, S(t)z_0\rangle| \leq 2TL^2_{\|z_0\|}\|z_0\|^2 - \langle Nz(t), z(t)\rangle \,, \ \forall t \in [0, T]\cdot \tag{10}$$

By virtue of (9), the inequality (4) also holds for $y = z(t)$. Then, integrating (10), yields

$$(\delta_\rho - 2T^2L^2_{\|z_0\|})\|z(t)\|^2 \leq -\int_t^{t+T} \langle Nz(s), z(s)\rangle \, ds\cdot \tag{11}$$

It follows from the inequality (8) that for all $k \in I\!N$, we have

$$\|z(kT)\|^2 - \|z((k+1)T)\|^2 \ \ \geq \ \ -2\int_{kT}^{(k+1)T} \langle Nz(s), z(s)\rangle \, ds\cdot$$

Then using (11), we get

$$\|z(kT)\|^2 - \|z((k+1)T)\|^2 \ \ \geq \ \ 2(\delta_\rho - 2T^2L^2_{\|z_0\|})\|z(kT)\|^2\cdot$$

This implies

$$\|z((k+1)T)\|^2 \leq C_{\|z_0\|}\|z(kT)\|^2, \tag{12}$$

where $C_{\|z_0\|} = 1 - 2(\delta_\rho(N) - 2T^2L^2_{\|z_0\|}(N))$ which is, by virtue of (12), positive and from the assumption on $L^2_{\|z_0\|}(N)$ we have $C_{\|z_0\|} \leq 1$.

Hence

$$\|z(kT)\|^2 \leq (C_{\|z_0\|})^k \|z_0\|^2,$$

which gives (since $\|z(t)\|$ decreases) the following exponential decay $\|z(t)\| \leq M\|z_0\|e^{-\sigma t}$, where $M = (C_{\|z_0\|})^{-\frac{1}{2}}$ and $\sigma = \frac{-\ln(C_{\|z_0\|})}{2T}$.

2) Under the assumption $L_\rho(N) < \frac{1}{T}(\frac{\delta_\rho(N)}{2})^{\frac{1}{2}}$, we obtain from the above development the estimate : $\|z(t)\| \leq M\|z_0\|e^{-\sigma t}$ with $M = (C_\rho)^{-\frac{1}{2}}$, $\sigma = \frac{-\ln(C_\rho)}{2T}$ and $C_\rho = 1 - 2\left(\delta_\rho(N) - 2T^2L_\rho^2(N)\right)$, so the parameters M and σ are independent of z_0, which gives the uniform stability.■

The following result concerns the global stabilization.

Corollary 1 *Let (i) A generate a semigroup $S(t)$ of contractions on H, (ii) N be dissipative and Lipschitz and let (iii) (5) holds.*

If $L(N) < \frac{1}{T}(\frac{\delta(N)}{2})^{\frac{1}{2}}$, then (2) is uniformly globally exponentially stable.

Proof. From the proof of the above theorem, we have the estimate : $\|z(t)\| \leq M\|z_0\|e^{-\sigma t}$, $\forall z_0 \in H$, where the positive constants $M = \left(1 - 2(\delta(N) - 2T^2L^2(N))\right)^{-\frac{1}{2}}$ and $\sigma = \frac{-\ln\left(1 - 2(\delta(N) - 2T^2L^2(N))\right)}{2T}$ are independent of z_0, which means that the stability is global and uniform.■

Remark 1 *Note that (5) implies that (4) holds for all $\rho > 0$, but the converse is not true as we can see taking $Az = 0$ and $Nz = \frac{-z}{z^2+1}$, $\forall z \in H := R$.*

2.3 Necessary conditions for exponential stability

The next result gives necessary conditions for exponential stability of (2), and will be useful in the next section. For this end, we define, for $\rho > 0$, the following sets : $\Lambda_\rho = \{y \in \mathcal{B}_\rho / S(t)y \to 0$, exponentially, as $t \to +\infty\}$ and $\Lambda = \{y \in H/S(t)y \to 0$, exponentially, as $t \to +\infty\} = \bigcup_{\rho>0} \Lambda_\rho$.

Theorem 2 *1) If the system (2) is exponentially stable on $\mathcal{B}_\rho$, then :*

$$\forall y \in \mathcal{B}_\rho,\ S(t-s)NS(s)y = 0,\ \forall t \geq 0, \forall s \in [0,t] \Rightarrow y \in \Lambda_\rho. \tag{13}$$

2) If the system (2) is globally exponentially stable, then :

$$\forall y \in H,\ S(t-s)NS(s)y = 0,\ \forall t \geq 0, \forall s \in [0,t] \Rightarrow y \in \Lambda. \tag{14}$$

Proof. 1) Let $y \in \mathcal{B}_\rho$ be such that $S(t-s)NS(s)y = 0, \forall t \geq 0,\ \forall s \in [0,t]$. It follows that $z(t) = S(t)y$ satisfies the variation of constants formula (7), and hence it is the unique solution of (2), corresponding to the initial state $z(0) = y$. Then the exponential stability of (2) implies that $z(t) \to 0$, exponentially, and so $y \in \Lambda_\rho$.

2) Let $y \in H$ such that $S(t-s)NS(s)y = 0, \forall t \geq 0,\ \forall s \in [0,t]$, and let $\rho > \|y\|$. Since (2) is globally exponentially stable, then it is also exponentially stable on $\mathcal{B}_\rho$. Then (13) implies $y \in \Lambda_\rho \subset \Lambda$, and hence (14) holds.■

Remarks 1 *1. If the semigroup $S(t)$ is of isometries i.e, $\|S(t)y\| = \|y\|,\ \forall t \geq 0,\ y \in H$, then for all $\rho > 0$, we have $\Lambda_\rho = \{0\}$, and hence $\Lambda = \{0\}$, so (13) and (14) become respectively :*

$$\forall y \in \mathcal{B}_\rho,\ S(t-s)NS(s)y = 0,\ \forall t \geq 0,\ \forall s \in [0,t] \Rightarrow y = 0, \tag{15}$$

and

$$\forall y \in H,\ S(t-s)NS(s)y = 0,\ \forall t \geq 0, \forall s \in [0,t] \Rightarrow y = 0\cdot \tag{16}$$

2. If the semigroup $S(t)$ is not supposed of isometries, then (15) (resp. (16)) is not a necessary condition for exponential stability on $\mathcal{B}_\rho$ (resp. on H), as we can see taking $A = \dfrac{\partial^2}{\partial x^2}$, $\mathcal{D}(A) = H^2(0,1) \cap H_0^1(0,1)$ and $N = 0$. Indeed, it is well known that A generates an exponentially stable semigroup $S(t)$ given by $S(t)y = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} < z_0, \sin(n\pi x) > \sin(n\pi x)$. But for $N = 0$, we have $\Lambda_\rho = H = \Lambda$.

3. Note that (16) $\Rightarrow$ (15), but the converse is not true. Indeed, for $\rho = 1,\ H = I\!R,\ A = 0$ and $N(y) = y\,\mathbf{1}_{\{y;\,|y| \leq 1\}}$, we have $S(t) = I$ (the identity of H) and hence for all $y \in B_1$, and for all $0 \leq s \leq t$, we have $S(t-s)NS(s)y = Ny = y$. Thus the assumption (15) holds. But for $y \notin \mathcal{B}_1 = [-1,1]$, we have $S(t-s)NS(t)y = Ny = 0, \forall t \geq 0$, so (16) does not hold.

4. If N is linear, then we have (15) $\Leftrightarrow$ (16). Indeed, let (15) hold, and let $y \in H$ such that $S(t-s)NS(s)y = 0,\ \forall 0 \leq s \leq t$. If $y \neq 0$, then $y_\rho := \rho\dfrac{y}{\|y\|} \neq 0$ and we have $S(t-s)NS(s)y_\rho = 0$ with $y_\rho \in B_\rho$, which is in contradiction with (15). We conclude that (16) holds.

5. The assumption (16) does not guarantee the exponential stability of (2), as we can see for $A = 0$ and $Nz = -z^3,\ \forall z \in H := I\!R$. Indeed, for all $0 \leq s \leq t$, we have $S(t-s)NS(s)y = Ny = y^3$ and hence (16) holds. However, for all initial state $z_0 \neq 0$, the solution is given by $z(t) = \dfrac{1}{2t + \frac{1}{z_0}}$, which does not converge exponentially to 0, as $t \to +\infty$.

3 Exponential stabilization of linear systems

In this section, we will study the problem of exponential stabilization and robustness of the system (1). For this end, we consider (for some $T, \alpha > 0$) the following exact controllability assumption :

$$\int_0^T \|B^*S(t)y\|_U^2 dt \geq \alpha \|y\|^2, \ \forall y \in H, \tag{17}$$

and let us set $\alpha(B) = \inf\limits_{\|y\|=1} \|B^*S(\cdot)y\|^2_{L^2(0,T;U)}$, (so that $\alpha \leq \alpha(B)$).

3.1 Nonlinear controls

In order to study various kinds of control saturation, it would be more appropriate to consider the general feedback :

$$u(t) = -c\frac{B^*z(t)}{r(z(t))}, \tag{18}$$

where $r : H \to R^{*+}$ is an appropriate function and c is positive constant.

Remark 2 *If $r(y) \geq \nu \|B^*y\|_U$, for all $y \in H$ (for some $\nu > 0$), then we have : $|u(t)| \leq \frac{c}{\nu}$, for all $t \geq 0$.*

The following result gives sufficient conditions for the control (18) to guarantee local and global stabilization of (1).

Theorem 3 *Let (i) A generate a semigroup $S(t)$ of contractions on H, (ii) $B \in \mathcal{L}(U, H)$ such that (17) holds and let (iii) r be Lipschitz on any bounded set.*

1) Let $\rho > 0$ be such that : $0 < m(\rho) \leq r(z) \leq M(\rho)$, for all $z \in \mathcal{B}_\rho$. Then for all c such that

$$0 < c < \frac{\alpha(B)m^4(\rho)}{2T^2M(\rho)\left(M(\rho) + \rho L_\rho(r)\right)^2 \|BB^*\|^2}, \tag{19}$$

the control (18) uniformly exponentially stabilizes (1) on $\mathcal{B}_\rho$.

2) If r is Lipschitz and $0 < m \leq r(y) \leq M$, for all $y \in H$, (for some $m, M > 0$), then there exists $c > 0$ for which the control (18) exponentially globally stabilizes the system (1).

Proof. 1) To study the stabilizability of (1) using the control (18), we introduce the operator $Nz = -c\frac{BB^*z}{r(z)}$, which is clearly dissipative. Moreover, since $S(t)$ is of contractions, then for all $z \in \mathcal{B}_\rho$, we have $\|S(t)z\| \leq \|z\| \leq \rho$ and so

$$\langle NS(t)z, S(t)z\rangle = c\frac{\|B^*S(t)z\|_U^2}{r(S(t)z)} \geq c\frac{\|B^*S(t)z\|_U^2}{M(\rho)}.$$

Then for all $z \in \mathcal{B}_\rho$, we have $\int_0^T \langle NS(t)z, S(t)z\rangle \, dt \geq \delta_\rho \|z\|^2$ with $\delta_\rho = \frac{c\alpha(B)}{M(\rho)}$. In other words, N verifies (4) with $\delta_\rho(N) \geq \frac{c\alpha(B)}{M(\rho)}$. Furthermore, the operator N is locally Lipschitz. Indeed, let $x \in H$ and let $R, L_{R,x}(r) > 0$ such that for all $z, y \in H$; $\|x - y\|$, $\|x - z\| \leq R$, we have $\|r(y) - r(z)\| \leq L_{R,x}(r)\|y - z\|$. Then, letting $R_x = R + \|x\|$, we obtain

$$\begin{aligned}
\|Nz - Ny\| &= \frac{\|cr(y)BB^*z - cr(z)BB^*y\|}{r(z)r(y)} \\
&\leq \frac{c}{m^2(R_x)}\|r(y)BB^*z - r(z)BB^*y\| \\
&\leq \frac{c}{m^2(R_x)}\|r(y)BB^*(z-y) + (r(y) - r(z))BB^*y\| \\
&\leq \frac{c\|BB^*\|(M(R_x) + R_x L_{R,x}(r))}{m^2(R_x)}\|z - y\|.
\end{aligned}$$

This shows that N is locally Lipschitz.
Now, taking $x = 0$, $R = \rho$, and letting $L_{\rho,0}(N) = L_\rho(N)$ in the last inequality, we get

$$L_\rho(N) \leq \frac{c\|BB^*\|(M(\rho) + \rho L_\rho(r))}{m^2(\rho)}. \tag{20}$$

We have $\delta_\rho(N) \geq \delta_\rho = \frac{c\alpha(B)}{M(\rho)}$. Then

$$\begin{aligned}
(19) \Rightarrow c^2 &< \frac{m^4(\rho)\delta_\rho(N)}{2T^2\|BB^*\|^2(M(\rho) + \rho L_\rho(r))^2} \\
\Rightarrow 2T^2 &\frac{c^2\|BB^*\|^2(M(\rho) + \rho L_\rho(r))^2}{m^4(\rho)} < \delta_\rho(N)
\end{aligned}$$

This, together with (20), implies that

$$(19) \Rightarrow L_\rho(N) < \frac{1}{T}\left(\frac{\delta_\rho(N)}{2}\right)^{\frac{1}{2}}.$$

The result of Theorem 1 implies the uniform exponential stabilizability of the system (1) on B_ρ with the control (18).
2) Let $\rho > \|z_0\|$ and let c be such that :

$$0 < c < \frac{\alpha(B)m^4}{2T^2M\left(M + \rho L(r)\right)^2\|BB^*\|^2}. \tag{21}$$

It follows from the first point that the control (18) exponentially stabilizes the system (1) on B_ρ. The choice of ρ implies that $z_0 \in B_\rho$, and hence the solution of system (1) with z_0 as initial state exponentially converges to 0, as $t \to +\infty$. This achieves the proof. ■

3.2 Constrained controls

Let us consider the two bounded controls

$$u_1(t) = \frac{-c}{1 + \|B^*z(t)\|_U} B^*z(t), \tag{22}$$

and

$$u_2(t) = \frac{-c}{\sup\left(1, \|B^*z(t)\|_U\right)} B^*z(t), \tag{23}$$

where $c > 0$ is the gain control.

As applications to constrained stabilization of the system (1), we have the following result

Theorem 4 *Let A generate a semigroup $S(t)$ of contractions on H and let $B \in \mathcal{L}(U, H)$ such that (17) holds. Then*

1) for all $\rho > 0$, there exists $c >$ such that both the controls (22) and (23) uniformly exponentially stabilizes (1) on $\mathcal{B}_\rho$.

2) both the controls (22) and (23) globally exponentially stabilizes (1) for some $c > 0$.

Proof. 1) First, let us note that the controls (22) and (23) have respectively the form of (18) with

$$r(z) = 1 + \|B^*z\|_U \text{ and } r(z) = \sup(1, \|B^*z\|_U).$$

Here, the map r is Lipschitz with $L_\rho(r) = \|B^*\| = \|B\|$. (This is clear for (22) and for (23), one can remark that : $2\sup\left(1, \|B^*z\|_U\right) = |1 - \|B^*z\|_U| + 1 + \|B^*z\|_U$).

Also we have $1 \leq r(z) \leq 1 + \rho\|B\|$, $\forall z \in \mathcal{B}_\rho$. Hence we can take $M(\rho) = 1 + \rho\|B\|$ and $m(\rho) = 1\cdot$

Now remarking that the inequality (19) is equivalent to the following one

$$0 < c < \frac{\alpha(B)}{2T^2(1 + \rho\|B^*\|)(1 + 2\rho\|B^*\|)^2\|BB^*\|^2}, \tag{24}$$

we deduce from Theorem 3 that (22) and (23) uniformly exponentially stabilize (1) on $\mathcal{B}_\rho$ for all c satisfying (24).

2) It follows from the same techniques as in 1) by taking $\rho > \|z_0\|$.■

Remark 3 *Taking* $0 < c < \dfrac{\epsilon}{(1 + \rho\|B^*\|)(1 + 2\rho\|B^*\|)^2}$ *with* $0 < \epsilon < \dfrac{\alpha(B)}{2T\|BB^*\|}$ *we have* $|u_i(t)| \leq \epsilon \leq \dfrac{\alpha(B)}{2T\|BB^*\|}$. *In other words, the controls (22) and (23) are uniformly bounded with respect to the initial states.*

3.3 Necessary conditions for exponential stabilization

In the case of the system (1), the results of Theorem 2 can be reformulated as follows :

Theorem 5 *1) The condition*

$$\forall y \in H,\ S(t-s)BB^*S(s)y = 0,\ \forall t \geq 0,\ \forall s \in [0,t] \text{ implies } y \in \Lambda,$$

is necessary for the exponential stability of (1) with the control (18).

2) If A generates a semigroup $S(t)$ of isometries on H, then the condition

$$\forall y \in H,\ S(t-s)BB^*S(s)y = 0,\ \forall t \geq 0,\ s \in [0,t] \text{ implies } y = 0,$$

is necessary for the exponential stability of (1) with the control (18).

Proof. It follows from Theorem 2 by taking $N = \dfrac{-cBB^*}{r}$.■

Remark 4 *1. The results of Theorem 5 can be applied to avoid the "bad" actuators, i.e, the ones that do not guarantee the exponential stability.*

We recall that an actuator can be defined as a couple $(\omega, a(\cdot))$ of a function f, which indicates the spatial distribution of the action on the support ω which is a part of the closure $\overline{\Omega}$ of the domain Ω (see [5, 6, 7, 10]).

2. As a consequence of the above theorem, a necessary condition for exponential stabilization of the system (1) with the control (18) is that all the modes of A corresponding to eigenvalues λ such that $\mathcal{R}e(\lambda) \geq 0$ are actives. In other words, for all $\lambda \in Sp(A)$; $\mathcal{R}e(\lambda) \geq 0$ and for all corresponding eigenfunction $\varphi \in \ker(A - \lambda I) - (0)$, we have $BB^*\varphi \neq 0$. As an example; for $H = L^2(0,1)$ and $A = \dfrac{\partial^2}{\partial x^2}$, $\forall z \in \mathcal{D}(A) = \{z \in H^2(0,1)/\ z'(0) = z'(1) = 0\}$, a necessary condition for exponential stability is $BB^*(\mathbf{1}) \neq 0$. In term of actuators, if we take $B : u \in U = I\!R \mapsto (a(\cdot)\chi_\omega)u \in L^2(0,1)$, i.e, the action applies in the subregion ω of Ω with the spatial repartition $a(x)$, then we have $B^*y = \int_\omega a(x)y(x)dx$. Thus, an actuator $(\omega, a(\cdot))$ such that $\int_\omega a(x)dx = 0$ is a "bad" one.

4 Robustness of constrained controls

Let us now proceed to robustness question of the controls (22) and (23) to small perturbations of the parameters system. Consider the following perturbed system :

$$\frac{dz(t)}{dt} = Az(t) + az + Bu(t),\quad z(0) = z_0, \tag{25}$$

where A and B are as in (1) and the perturbation a is a nonlinear operator from H to itself.

Consider the nominal system :

$$\frac{dz(t)}{dt} = Az(t) + N_i z(t), \;\; z(0) = z_0, \; (i = 1, 2), \tag{26}$$

where $N_i z = \dfrac{-BB^* z}{r_i(z)}$, $r_1(z) = 1 + \|B^* z\|_U$ and $r_2(z) = \sup(1, \|B^* z\|_U)$, for all $z \in H$.

Let us define the set of admissible perturbations : $\Omega_A = \{a : H \to H /\ a$ is dissipative, locally Lipschitz such that $a(0) = 0$ and $L_\rho(a) < \dfrac{\sqrt{\delta_\rho(N_i)}}{T\sqrt{2}} - L_\rho(N_i), \; i = 1, 2\}$. Note that the assumption $a(0) = 0$ implies that 0 remains an equilibrium for (25).

We have the following result

Theorem 6 *Let assumptions of Theorem 4 hold. Then for any perturbation $a \in \Omega_A$, the controls (22) and (23) uniformly exponentially stabilize the system (25) on $\mathcal{B}_\rho$.*

If a is Lipschitz, then the controls (22) and (23) globally exponentially stabilize (25).

Proof. First let us note that from Theorem 4, one deduce that $\Omega_A \neq \emptyset$. Let $a \in \Omega_A$ and let $\tilde{N}_i = a - N_i$. We have

$$| < \tilde{N}_i S(t)y, S(t)y > | \geq | < N_i S(t)y, S(t)y > | \cdot$$

It follows that

$$\int_0^T | < \tilde{N}_i S(t)y, S(t)y > | dt \geq \delta_\rho(N_i) \|y\|^2,$$

so that $\tilde{N}_i$ satisfies (4) with $\delta_\rho(\tilde{N}_i) \geq \delta_\rho(N_i) = \dfrac{c\alpha(B)}{1 + \rho\|B\|}, \; i = 1, 2.$

Clearly the operator $\tilde{N}_i$ is dissipative, locally Lipschitz and verifies :

$$L_\rho(\tilde{N}_i) \leq L_\rho(N_i) + L_\rho(a) < \frac{\sqrt{\delta_\rho(\tilde{N}_i)}}{T\sqrt{2}}, \text{ for all } a \in \Omega_A.$$

Then from Theorem 1, there exists $c > 0$ for which the controls (22) and (23) uniformly exponentially stabilize (25) on $\mathcal{B}_\rho$.

Now if a is Lipschitz, then $L_{\|z_0\|}(\tilde{N}_i) < \dfrac{\sqrt{\delta_{\|z_0\|}(\tilde{N}_i)}}{T\sqrt{2}}$ provided that

$$2T^2 c(1 + \|z_0\|^2 \|B^*\|)^2 \|BB^*\|^2 + L(a) < \frac{\alpha(B)}{1 + \|z_0\|\|B\|},$$

which holds for c small enough. The global stability follows then from Theorem 1.■

The system (25) may be seen as a perturbation of (1) in its dynamic A. Next, we consider the problem of robustness of controls (22) and (23) with respect to perturbations of B. Let us consider the linear system

$$\frac{dz(t)}{dt} = Az(t) + (B + b)u(t), \;\; z(0) = z_0, \tag{27}$$

where $b \in \mathcal{L}(U, H)$. We have the following result.

Theorem 7 *Let A generate a semigroup $S(t)$ of contractions on H and let $B \in \mathcal{L}(U, H)$ such that (17) holds. Then the controls (22) and (23) are globally exponentially robust to any perturbation $b \in \mathcal{L}(U, H)$ of B such that $\|b^*\| < \dfrac{\alpha(B)}{2T\|B^*\|}$. Furthermore, the robustness is uniform on $\mathcal{B}_\rho$.*

Proof.
We have

$$\|(B^* + b^*)S(t)y\|_U \geq |\|B^*S(t)y\|_U - \|b^*S(t)y\||\cdot$$

Then

$$\begin{aligned}\|(B^* + b^*)S(t)y\|_U^2 &\geq \|B^*S(t)y\|_U^2 - 2\|B^*S(t)y\|\|b^*S(t)y\| + \|b^*S(t)y\|^2 \\ &\geq \|B^*S(t)y\|_U^2 - 2\|B^*\|\|b^*\|\|y\|^2\end{aligned}$$

Integrating this inequality and using (17), we get

$$\int_0^T \|(B^* + b^*)S(t)y\|_U^2 dt \geq (\alpha(B) - 2T\|B^*\|\|b^*\|)\|y\|^2, \ \forall y \in H,$$

which implies that $B + b$ verifies (17) with $\alpha = \alpha(B) - 2T\|B^*\|\|b^*\|\cdot$
From Theorem 4, we deduce that the controls :

$$u_{b1}(t) = \frac{-c}{1 + \|(B^* + b^*)z(t)\|_U}(B^* + b^*)z(t)$$

and

$$u_{b2}(t) = \frac{-c}{\sup(1, \|(B^* + b^*)z(t)\|_U)}(B^* + b^*)z(t)$$

globally exponentially stabilize the perturbed system (27) for some $c > 0$; uniformly on $\mathcal{B}_\rho$.■

Now let us see the problem of robustness associated to linear perturbations acting, jointly, on the dynamic and the operator of control.
Consider the perturbed system :

$$\frac{dz(t)}{dt} = (A + a)z(t) + (B + b)u(t), \quad z(0) = z_0, \tag{28}$$

where $a \in \mathcal{L}(H)$ and $b \in \mathcal{L}(U, H)$. We have the following result.

Theorem 8 *Let A generate a semigroup $S(t)$ of contractions on H and let $B \in \mathcal{L}(U, H)$ such that (17) holds. Then the controls (22) and (23) are globally exponentially robust to any perturbation a and b such that a is dissipative, $\|a\| < \dfrac{-1 + \sqrt{1 + \frac{\alpha(B)}{T\|B\|^2}}}{T}$ and $\|b\| < \dfrac{\alpha_a(B)}{2T\|B^*\|}$, where $\alpha_a(B) = \alpha(B) - T^2\|B\|^2(T\|a\|^2 + 2\|a\|)$.*

Proof.

Under the assumptions on a, the operator $A + a$ is the infinitesimal generator of a semi-group of contractions $S_a(t)$ (see [11]), and for all $t \geq 0$ and $y \in H$, we have

$$S_a(t)y = S(t)y + \int_0^t S(t-s)aS_a(s)yds. \tag{29}$$

The system (28) may be seen as a perturbation of the system (25) in its control operator B by b. Then from Theorem 7, it is sufficient to show that

$$\int_0^T \|B^* S_a(t)y\|_U^2 dt \geq \alpha_a \|y\|^2, \ \forall y \in H, \tag{30}$$

for some $\alpha_a > 0$.

Based on (29), we obtain the following relation

$$\langle BB^* S_a(t)y, S_a(t)y\rangle = \langle BB^* S(t)y, S(t)y\rangle + \phi(t), \tag{31}$$

where $\phi(t)$ is a scalar function such that

$$|\phi(t)| \leq K_a \|y\|^2, \ \forall t \in [0, T],$$

where $K_a = T\|B\|^2 \left(T\|a\|^2 + 2\|a\|\right)$.

Then we have

$$\|B^* S_a(t)y\|^2 \geq \|B^* S(t)y\|^2 - |\phi(t)|$$

Integrating this last inequality and using (17), we deduce

$$\int_0^T \|B^* S_a(t)y\|_U^2 dt \geq (\alpha(B) - TK_a)\,\|y\|^2.$$

Then we obtain (30) provided that $\alpha(B) - TK_a > 0$ i.e.

$$T^2\|B\|^2 \left(T\|a\|^2 + 2\|a\|\right) - \alpha(B) < 0,$$

which is equivalent to $\|a\| < \dfrac{-1 + \sqrt{1 + \frac{\alpha(B)}{T\|B\|^2}}}{T}$. Then we conclude by Theorem 7.■

5 An example

Let $\Omega = (0, 1)$ and let $Q = \Omega\times]0, +\infty[$. Consider the following wave equation

$$\begin{cases} \dfrac{\partial^2 z(x,t)}{\partial t^2} = \dfrac{\partial^2 z(x,t)}{\partial x^2} + u(t), & on\ Q \\ z = 0, & on\ \partial\Omega\times]0, +\infty[\end{cases} \tag{32}$$

and let $H = H_0^1(\Omega) \times L^2(\Omega)$ with $\langle (y_1, z_1), (y_2, z_2) \rangle = \langle y_1, y_2 \rangle_{H_0^1(\Omega)} + \langle z_1, z_2 \rangle_{L^2(\Omega)}$. The operator $A = \begin{pmatrix} 0 & I \\ \dfrac{\partial^2}{\partial x^2} & 0 \end{pmatrix}$ with domain $D(A) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ is skew-adjoint. The spectrum of the operator $\dfrac{\partial^2}{\partial x^2}$ with Dirichlet boundary conditions is given by the simple eigenvalues $\lambda_j = (j\pi)^2$, corresponding to eigenfunctions $\varphi_j(x) = \sqrt{2}\sin(j\pi x)$, $\forall j \in \mathbb{N}^*$.

Here, we have $B : L^2(0,1) \to H$, $Bz = (0, z)$ and $B^* : H \to L^2(0,1)$, $B^*(y, z) = z$.

Let $y = (y_1, y_2) \in H$ with $y_1 = \sum_{j=1}^{\infty} \alpha_j \varphi_j$ and $y_2 = \sum_{j=1}^{\infty} \lambda_j^{\frac{1}{2}} \beta_j \varphi_j$, where $(\alpha_j, \beta_j) \in \mathbb{R}^2$, $j \geq 1$. We have $\|y\|^2 = \sum_{j=1}^{\infty} \lambda_j (\alpha_j^2 + \beta_j^2)$. Separation of variables yields

$$S(s)y = \sum_{j=1}^{\infty} \begin{pmatrix} \alpha_j \cos(\lambda_j^{\frac{1}{2}} s) + \beta_j \sin(\lambda_j^{\frac{1}{2}} s) \\ -\alpha_j \lambda_j^{\frac{1}{2}} \sin(\lambda_j^{\frac{1}{2}} s) + \beta_j \lambda_j^{\frac{1}{2}} \cos(\lambda_j^{\frac{1}{2}} s) \end{pmatrix} \varphi_j, \quad \forall s \geq 0.$$

Then we have

$$\|B^* S(s) y\|^2 = \sum_{j=1}^{\infty} \lambda_j \left\{ \alpha_j^2 \sin^2(j\pi s) + \beta_j^2 \cos^2(j\pi s) - \sin(2j\pi s)\alpha_j \beta_j \right\}.$$

It follows that

$$\int_0^2 \|B^* S(s) y\|^2 ds = \sum_{j=1}^{\infty} \lambda_j (\alpha_j^2 + \beta_j^2),$$

so the assumption (17) holds with $T = 2$ and we have $\alpha(B) \geq 1$.

We conclude that the feedback controls

$$u_i(t) = \frac{-\dfrac{\partial z(x,t)}{\partial t}}{r_i(\|\dfrac{\partial z(x,t)}{\partial t}\|_{L^2(\Omega)})}, \; i = 1, 2,$$

exponentially stabilize (32), where $r_1(x) = 1 + x$ and $r_2(x) = \sup(1, x)$, $\forall x \in \mathbb{R}$.

Let us now consider the perturbed system :

$$\begin{cases} \dfrac{\partial^2 z(x,t)}{\partial t^2} = \Delta z(x,t) + \lambda \dfrac{\partial z(x,t)}{\partial t} + (1+\mu) u(t), & on\ Q \\ z = 0, & on\ \partial\Omega \times]0, +\infty[\end{cases} \tag{33}$$

The system (32) may be seen as the system (1), perturbed in its dynamic by $a = \begin{pmatrix} 0 & 0 \\ 0 & \lambda I \end{pmatrix}$ and in its operator of control by $b = \begin{pmatrix} 0 \\ \mu \end{pmatrix}$. Applying results of Theorem 8, we deduce that (32) is exponentially stabilizable with the feedback law :

$$u_i(t) = \frac{-(1+\mu)\dfrac{\partial z(x,t)}{\partial t}}{r_i(|1+\mu|\|\dfrac{\partial z(x,t)}{\partial t}\|_{L^2(\Omega)})}, \ i = 1, 2,$$

under the perturbations a, b, provided that $0 < -\lambda < \dfrac{\sqrt{6}-2}{4}$ and $|\mu| < 1 - 8(\lambda^2 + \lambda)$.

6 Conclusion

In this work, sets of necessary and sufficient conditions for exponential stability of nonlinear systems are obtained. Then we have studied the exponential stabilization of distributed linear systems using bounded feedbacks. The established results can be applied to systems which are subject to constraint on the control input. Also sets of allowed perturbations of the parameters system that maintain the exponential stabilization of the considered systems are given.

References

[1] Ball, J. (1978). On the asymptotic behaviour of generalized processes, with applications to nonlinear evolution equations, J. Differential Equations, 27, 224-265.

[2] Ball, J., & Slemrod, M. (1979). Feedback stabilization of distributed semilinear control systems. Appl. Math. Opt., 5, 169-179.

[3] Berrahmoune, L. (2009). Stabilization and decay estimate of linear control systems in Hilbert space with non-linear feedback. IMA Journal of Mathematical Control and Information, 26, 495-507.

[4] Bounit, H. & Hammouri, H. (1999). Feedback stabilization for a class of distributed semilinear control systems. Nonlinear Anal., 37, 953-969.

[5] El Jai, A. & El Yacoubi, S. (1993). On the number of actuators in parabolic systems, Appl. Math. Comp. Sci., 34, 673-686.

[6] El Jai, A. & Pritchard, AJ. Sensors and actuators in distributed systems analysis(Ellis Horwood Series in Applied Mathematics, Wiley, New York, 1988).

[7] Lou, Y. & Christofides, P.D. (2003). Optimal actuator/sensor placement for nonlinear control of the Kuramoto-Sivashinsky equation, IEEE Transactions on Control Systems Technology., 11, 737-745.

[8] Ouzahra, M. (2010). Exponential and weak stabilization of constrained bilinear Systems, SIAM J. Control Optim., 48, 3962-3974.

[9] Ouzahra M. (2012), Global stabilization of semilinear systems using switching controls. Automatica, Automatica, 48, pp. 837-843.

[10] Ouzahra M., & El Ayadi, R. (2008), Strategic functions for feedback stabilization of bilinear systems, Journal International Journal of Robotics and Automation, Volume 23 Issue 1, Pages 40-48.

[11] Pazy, A. (1983). Semi-groups of linear operators and applications to partial differential equations, Springer Verlag, New York.

[12] Prichard, A. J., & Zabczyk, J. (1981). Stability and stabilizability of infinite dimensional systems. Siam Review, 23, 25-51.

[13] Seidman, T. I. & Li, H. (2001). A note on stabilization with saturating feedback. Discrete Continuous Dyn. Syst., 7, 319-328.

[14] Slemrod, M. (1989). Feedback stabilization of a linear control system in Hilbert space with an a priori bounded control, Math. Control Signals Systems, 2, 265-285.

(Received July 25, 2012)