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Periodic solutions for a neutral delay predator-prey model with nonmonotonic functional response*

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Abstract: By using a continuation theorem based on coincidence degree theory, some new sufficient conditions are obtained for the existence of positive periodic solutions of the following neutral delay predator-prey model with nonmonotonic functional response:

$$\begin{cases} x'(t) = x(t)[r(t) - a(t)x(t-\sigma(t)) - b(t)x'(t-\sigma(t))] - g(x(t))y(t), \\ y'(t) = y(t)[-d(t) + \mu(t)g(x(t-\tau(t))]. \end{cases}$$

Moreover, an example is employed to illustrate the main results.

Keywords: Predator-prey model; neutral delay; nonmonotonic functional response; positive periodic solution; coincidence degree.

AMS(2000) Subject Classification: 34C25; 34K13

1 Introduction

In a classic study of population dynamics, the predator-prey models have been studied extensively. We refer the reader to [1−5] and the references cited therein. Up to the present, most authors just studied systems with monotonic functional response, such as [6,7]. However, the actual living environments of species are not always like this due to the ecological effects of

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human activities and industry, e.g., the location of manufacturing industries and pollution of the atmosphere, rivers, and soil etc. In view of such kinds of situations, Fan and Quan [8] investigated the existence and uniqueness of limit cycle of such a type of predator-prey system, in which the predator would decrease its grasping ability while the prey has group defence ability, namely,

$$\begin{cases} \dot{x} = \Phi(x) - y\Psi(x), \\ \dot{y} = y[\mu\Psi(x) - D]. \end{cases}$$

where

$$\Phi(0) = 0, \quad \lim_{x\to\infty} \Phi(x) < 0, \quad \Psi(x), \Phi(x) \in C^1[0, +\infty), \quad \Psi(0) = 0,$$

and

$$\exists k > 0, \text{such that } (x - k)\Psi'(x) < 0 \text{ and } \lim_{x\to\infty} \Psi(x) = 0,$$

μ, D are positive constants. For a special case of this system, in view of time delay effect, Ruan [9] and Xiao [10] considered the bifurcation and stability of the following predator-prey model with nonmonotonic functional response

$$\begin{cases} x'(t) = x(t)[a - bx(t)] - \frac{cx(t)y(t)}{m^2+x^2(t)}, \\ y'(t) = y(t)[-d + \frac{\mu x(t-\tau)}{m^2+x^2(t-\tau)}]. \end{cases} \tag{1.1}$$

where $x(t)$ and $y(t)$ represent predator and prey densities respectively, a, b, m, μ and d are all positive constants, and τ is a nonnegative constant. Furthermore, Fan and Wang [11] established verifiable criteria for the global existence of positive periodic solutions of a more general delayed predator-prey model with nonmonotonic functional response with periodic coefficients of the form

$$\begin{cases} x'(t) = x(t)[a(t) - b(t)x(t)] - g(x(t))y(t), \\ y'(t) = y(t)[-d(t) + \mu(t)g(x(t - \tau))]. \end{cases} \tag{1.2}$$

In particular, Kuang [12] studied the local stability and oscillation of the following neutral delay Gause-type predator-prey system:

$$\begin{cases} x'(t) = rx(t)[1 - \frac{x(t-\tau)+\rho x'(t-\tau)}{K}] - y(t)p(x(t)), \\ y'(t) = y(t)[-\alpha + \beta p(x(t - \sigma))]. \end{cases} \tag{1.3}$$

Since the coefficients and delays in differential equations of population and ecology problems are usually time-varying in the real world, the model (1.3) can be naturally extended to the following neutral delay predator-prey model with nonmonotonic functional response:

$$\begin{cases} x'(t) = x(t)[r(t) - a(t)x(t-\sigma(t)) - b(t)x'(t-\sigma(t))] - g(x(t))y(t), \\ y'(t) = y(t)[-d(t) + \mu(t)g(x(t-\tau(t))]. \end{cases} \tag{1.4}$$

where $x(t)$ and $y(t)$ represent predator and prey densities respectively, $r(t), a(t), b(t), d(t)$, and $\mu(t)$ are all positive periodic continuous functions with period $\omega > 0$, $\sigma(t), \tau(t)$ are ω-periodic continuous functions, the function g satisfying the following conditions:

(i) $g \in C^1[0, +\infty)$, $g(0) = 0$;

(ii) There exists a constant $k > 0$ such that $(x - k)g'(x) < 0$ for $x \neq k$;

(iii) $\lim_{x \to +\infty} g(x) = 0$,

where C^n is the nth order continuous function space, $n = 1, 2$.

As pointed out by Kuang [13], it would be of interest to study the existence of periodic solutions for periodic systems with time delay. The periodic solutions play the same role as is played by the equilibria in autonomous systems. In addition, in view of the fact that many predator-prey systems display sustained fluctuations, it is thus desirable to construct predator-prey models capable of producing periodic solutions. To our knowledge, no such work has been done on the global existence of positive periodic solutions of (1.4). Motivated by this, our aim in this paper is, using the coincidence degree theory developed by Gaines and Mawhin [14], to derive a set of easily verifiable sufficient conditions for the existence of positive periodic solutions of system (1.4). For convenience, we will use the following notations

$$|f|_0 = \max_{t \in [0,\omega]} \{|f(t)|\}, \quad f^+ = \max_{t \in [0,\omega]} \{f(t)\}, \quad f^- = \min_{t \in [0,\omega]} \{f(t)\}, \quad \overline{f} = \frac{1}{\omega} \int_0^\omega f(t)dt.$$

In this paper, we always make the following assumptions for system (1.4).

(H_1) $b \in C^1(R, (0, +\infty)), \sigma \in C^2(R, R), 1 - \sigma'(t) > 0$ and $c(t) > 0$, where

$$c(t) = a(t) - B'(t), \quad B(t) = \frac{b(t)}{1 - \sigma'(t)}, \quad t \in R.$$

(H_2) $1 - \tau'(t) > 0$, $\overline{r} L \Lambda^- > C^+ \overline{d}$, $\max_{t \in [0,\omega]} \{b^+, B^+\} e^{\beta_1} < 1$, where

$$C(t) = \frac{c(\varphi(t))}{1 - \sigma'(\varphi(t))}, \quad \Lambda(t) = \frac{\mu(\psi(t))}{1 - \tau'(\psi(t))}, \quad L = \min_{x \in [\beta_2, \beta_1]} h(e^x), \quad h(x(t)) = \frac{g(x(t))}{x(t)},$$

$t = \varphi(p)$ is the inverse function of $p = t - \sigma(t)$, $t = \psi(q)$ is the inverse function of $q = t - \tau(t)$, and

$$\beta_1 = \ln \frac{2\overline{r}}{C^-} + B^+ \frac{2\overline{r}}{c^-} + 2\overline{r}\omega, \quad \beta_2 = \ln(\frac{\overline{d}}{\Lambda^+ M}) - \frac{2\overline{r}\omega + |B'|_0 e^{\beta_1}\omega}{1 - B^+ e^{\beta_1}}.$$

(H_3) $g(k)\overline{\mu} > \overline{d}$.

(H_4) For $g(u_1) = g(u_2) = \frac{\overline{d}}{\overline{\mu}}$, we have

$$0 < u_1 < \frac{\overline{r}}{\overline{a}} < u_2.$$

2 The existence of a positive periodic solution

In this section, we shall study the existence of at least one positive periodic solution of system (1.4). The method to be used in this paper involves the applications of the continuation theorem of the coincidence degree. For the readers' convenience, we introduce some concepts and results concerning the coincidence degree as follows.

Let X, Z be real Banach spaces, $L : DomL \subset X \to Z$ be a linear mapping, and $N : X \to Z$ be a continuous mapping. The mapping L is called a Fredholm mapping of index zero if $dimKerL = CodimImL < +\infty$ and ImL is closed in Z.

If L is a Fredholm mapping of index zero and there exist continuous projectors $P : X \to X$, and $Q : Z \to Z$ such that $ImP = KerL, KerQ = ImL = Im(I - Q)$. It follows that $L \mid DomL \cap KerP : (I - P)X \to ImL$ is invertible. We denote the inverse of that map by K_P.

If Ω be an open bounded subset of X, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \to X$ is compact.

Since ImQ is isomorphic to $KerL$, there exists an isomorphism $J : ImQ \to KerL$.

Lemma 2.1 (Mawhin's continuous theorem [14])**.** *Let $\Omega \subset X$ be an open bounded set. Let L be a Fredholm mapping of index zero and N be L-compact on $\bar{\Omega}$. Suppose further*

(i) for each $\lambda \in (0, 1), x \in \partial\Omega \cap DomL, Lx \neq \lambda Nx$;

(ii) for each $x \in \partial\Omega \cap KerL, QNx \neq 0$;

(iii) $deg\{JQN, \Omega \cap KerL, 0\} \neq 0$.

Then the operator equation $Lx = Nx$ has at least one solution in $\bar{\Omega} \cap DomL$.

Lemma 2.2 (See [11]). *Suppose (H_3) holds, the algebraic equations*

$$\begin{cases} \overline{r}u - \overline{a}u^2 - h(u)v = 0, \\ -\overline{d} + \overline{\mu}g(u) = 0 \end{cases}$$

has a unique positive solution if and only if, there exist two positive constants u_1 and u_2 such that

$$u_1 < \frac{\overline{r}}{\overline{a}} < u_2,$$

and

$$0 < u_1 < u_2, \quad \text{and} \quad g(u_1) = g(u_2) = \frac{\overline{d}}{\overline{\mu}}.$$

Theorem 2.1. *Assume that $(H_1) - (H_4)$ hold. Then system (1.4) has at least one ω-periodic solution with strictly positive components.*

Proof. Consider the following system:

$$\begin{cases} u_1'(t) = r(t) - a(t)e^{u_1(t-\sigma(t))} - b(t)e^{u_1(t-\sigma(t))}u_1'(t-\sigma(t)) - h(e^{u_1(t)})e^{u_2(t)}, \\ u_2'(t) = -d(t) + \mu(t)g(e^{u_1(t-\tau(t))}). \end{cases} \tag{2.1}$$

where all functions are defined as ones in system (1.4). It is easy to see that if system (2.1) has one ω-periodic solution $(u_1^*(t), u_2^*(t))^T$, then $(x^*(t), y^*(t))^T = (e^{u_1^*(t)}, e^{u_2^*(t)})^T$ is a positive ω-periodic solution of system (1.4). Therefore, to complete the proof it suffices to show that system (2.1) has one ω-periodic solution.

Take

$$X = Z = \{u = (u_1(t), u_2(t))^T \in C^1(R, R^2) : u_i(t+\omega) = u_i(t), t \in R, i = 1, 2\},$$

and define

$$|u|_\infty = \max_{t\in[0,\omega]}\{|u_1(t)| + |u_2(t)|\}, \qquad \|u\| = |u|_\infty + |u'|_\infty.$$

Then X and Z are Banach spaces when they are endowed with the norms $\|\cdot\|$ and $|\cdot|_\infty$, respectively. Let $L : X \to Z$ and $N : X \to Z$ be

$$L(u_1(t), u_2(t))^T = (u_1'(t), u_2'(t))^T$$

and

$$N\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} r(t) - a(t)e^{u_1(t-\sigma(t))} - b(t)e^{u_1(t-\sigma(t))}u_1'(t-\sigma(t)) - h(e^{u_1(t)})e^{u_2(t)} \\ -d(t) + \mu(t)g(e^{u_1(t-\tau(t))}) \end{bmatrix}$$

With these notations system (2.1) can be written in the form

$$Lu = Nu, \quad u \in X.$$

Obviously, $KerL = R^2, ImL = \{(u_1,u_2)^T \in Z : \int_0^\omega u_i(t)dt = 0, i = 1,2\}$ is closed in Z, and $dimKerL = codimImL = 2$. Therefore L is a Fredholm mapping of index zero. Now define two projectors $P : X \to X$ and $Q : Z \to Z$ as

$$P\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} \overline{u}_1 \\ \overline{u}_2 \end{bmatrix}, \quad \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \in X$$

and

$$Q\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} \overline{u}_1 \\ \overline{u}_2 \end{bmatrix}, \quad \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \in Z.$$

Then P and Q are continuous projectors such that

$$ImP = KerL, KerQ = ImL = Im(I-Q).$$

Furthermore, the generalized inverse (to L) $K_P : ImL \to DomL \cap KerP$ exists and has the form

$$K_p(u) = \int_0^t u(s)ds - \frac{1}{\omega}\int_0^\omega \int_0^t u(s)dsdt.$$

Then $QN : X \to Z$ and $K_P(I-Q)N : X \to X$ can be read as

$$(QN)u = \begin{bmatrix} \frac{1}{\omega}\int_0^\omega \Big[r(t) - (a(t) - B'(t))e^{u_1(t-\sigma(t))}) - h(e^{u_1(t)})e^{u_2(t)}\Big]dt \\ \frac{1}{\omega}\int_0^\omega \Big[-d(t) + \mu(t)g(e^{u_1(t-\tau(t))})\Big]dt \end{bmatrix}$$

and

$$K_P(I-Q)Nu = \begin{bmatrix} \int_0^t \Big[r(s) - c(s)e^{u_1(s-\sigma(s))} - h(e^{u_1(s)})e^{u_2(s)}\Big]ds - b(t)e^{u_1(t-\sigma(t))} + b(0)e^{u_1(-\sigma(0))} \\ \int_0^t \Big[-d(s) + \mu(s)g(e^{u_1(s-\tau(s))})\Big]ds \end{bmatrix}$$

$$
-\begin{bmatrix}
\frac{1}{\omega}\int_0^\omega\int_0^t\Big[r(s)-c(s)e^{u_1(s-\sigma(s))})-h(e^{u_1(s)})e^{u_2(s)}\Big]dsdt \\
-\frac{1}{\omega}\int_0^\omega\Big[b(t)e^{u_1(t-\sigma(t))}+b(0)e^{u_1(-\sigma(0))}\Big]dt \\
\frac{1}{\omega}\int_0^\omega\int_0^t\Big[-d(s)+\mu(s)g(e^{u_1(s-\tau(s))})\Big]dsdt
\end{bmatrix}
$$

$$
-\begin{bmatrix}
(\frac{t}{\omega}-\frac{1}{2})\int_0^\omega\Big[r(s)-c(s)e^{u_1(s-\sigma(s))}-h(e^{u_1(s)})e^{u_2(s)}\Big]ds \\
(\frac{t}{\omega}-\frac{1}{2})\int_0^\omega\Big[-d(s)+\mu(s)g(e^{u_1(s-\tau(s))})\Big]ds
\end{bmatrix}
$$

Obviously, QN and $K_P(I-Q)N$ are continuous by the Lebesgue theorem, and it is not difficult to show that $\overline{K_P(I-Q)N(\bar{\Omega})}$ is compact for any open bounded $\Omega\subset X$ by using Arzela-Ascoli theorem. Moreover, $QN(\bar{\Omega})$ is clearly bounded. Thus, N is $L-$compact on $\bar{\Omega}$ for any open bounded set $\Omega\subset X$.

Now we reach the position to search for an appropriate open bounded subset Ω for the application of Lemma 2.1. Corresponding to operator equation $Lu=\lambda Nu,\ \ \lambda\in(0,1)$, we have

$$
\begin{cases}
u_1'(t)=\lambda[r(t)-a(t)e^{u_1(t-\sigma(t))}-b(t)e^{u_1(t-\sigma(t))}u_1'(t-\sigma(t))-h(e^{u_1(t)})e^{u_2(t)}], \\
u_2'(t)=\lambda[-d(t)+\mu(t)g(e^{u_1(t-\tau(t))})].
\end{cases}
\tag{2.2}
$$

Suppose that $(u_1(t),u_2(t))^T\in X$ is a solution of (2.2) for a certain $\lambda\in(0,1)$. Integrating (2.2) over the interval $[0,\omega]$ leads to

$$
\int_0^\omega\big[r(t)-a(t)e^{u_1(t-\sigma(t))}-b(t)e^{u_1(t-\sigma(t))}u_1'(t-\sigma(t))-h(e^{u_1(t)})e^{u_2(t)}\big]dt=0 \tag{2.3}
$$

and

$$
\int_0^\omega[-d(t)+\mu(t)g(e^{u_1(t-\tau(t))})]dt=0. \tag{2.4}
$$

Note that

$$
\begin{aligned}
\int_0^\omega b(t)e^{u_1(t-\sigma(t))}u_1'(t-\sigma(t))dt=&\int_0^\omega\frac{b(t)}{1-\sigma'(t)}(e^{u_1(t-\sigma(t))})'dt=\int_0^\omega B(t)(e^{u_1(t-\sigma(t))})'dt \\
=&B(t)e^{u_1(t-\sigma(t))}|_0^\omega-\int_0^\omega B'(t)e^{u_1(t-\sigma(t))}dt=-\int_0^\omega B'(t)e^{u_1(t-\sigma(t))}dt,
\end{aligned}
$$

which, together with (2.3), implies

$$
\int_0^\omega[c(t)e^{u_1(t-\sigma(t))}+h(e^{u_1(t)})e^{u_2(t)}]dt=\overline{r}\omega. \tag{2.5}
$$

From (2.4), we have

$$\int_0^\omega \mu(t)g(e^{u_1(t-\tau(t))})dt = \overline{d}\omega. \tag{2.6}$$

In view of (2.2),(2.5) and (H_1), one can find

$$\begin{aligned}\int_0^\omega \Big|\frac{d}{dt}[u_1(t)+\lambda B(t)e^{u_1(t-\sigma(t))}]\Big|dt =&\lambda\int_0^\omega \Big|r(t)-c(t)e^{u_1(t-\sigma(t))}-h(e^{u_1(t)})e^{u_2(t)}\Big|dt\\ \le&\int_0^\omega r(t)dt+\int_0^\omega [c(t)e^{u_1(t-\sigma(t))}+h(e^{u_1(t)})e^{u_2(t)}]dt \qquad (2.7)\\ =&2\overline{r}\omega.\end{aligned}$$

Let $t=\varphi(p)$ be the inverse function of $p=t-\sigma(t)$. It is easy to see that $c(\varphi(p))$ and $\sigma'(\varphi(p))$ are all ω-periodic functions. Furthermore, it follows from (2.5) and (H_1) that

$$\begin{aligned}\overline{r}\omega \ge \int_0^\omega c(t)e^{u_1(t-\sigma(t))}dt &= \int_{-\sigma(0)}^{\omega-\sigma(\omega)} c(\varphi(p))e^{u_1(p)}\frac{1}{1-\sigma'(\varphi(p))}dp\\ =\int_0^\omega \frac{c(\varphi(p))}{1-\sigma'(\varphi(p))}e^{u_1(p)}dp &= \int_0^\omega \frac{c(\varphi(t))}{1-\sigma'(\varphi(t))}e^{u_1(t)}dt,\end{aligned}$$

which yields

$$\int_0^\omega \Big[\frac{c(\varphi(t))}{1-\sigma'(\varphi(t))}e^{u_1(t)}+c(t)e^{u_1(t-\sigma(t))}\Big]dt \le 2\overline{r}\omega.$$

According to the mean value theorem of differential calculus, we see that there exists $\xi\in[0,\omega]$ such that

$$\frac{c(\varphi(\xi))}{1-\sigma'(\varphi(\xi))}e^{u_1(\xi)}+c(t)e^{u_1(\xi-\sigma(\xi))}\le 2\overline{r}.$$

This, together with (H_1), yields

$$u_1(\xi)\le \ln\frac{2\overline{r}}{C^-}$$

and

$$e^{u_1(\xi-\sigma(\xi))}\le\frac{2\overline{r}}{c^-},$$

which, together with (2.7), imply that, for any $t\in[0,\omega]$,

$$\begin{aligned}u_1(t)+\lambda B(t)e^{u_1(t-\sigma(t))} \le& u_1(\xi)+\lambda B(\xi)e^{u_1(\xi-\sigma(\xi))}+\int_0^\omega \Big|\frac{d}{dt}[u_1(t)+\lambda B(t)e^{u_1(t-\sigma(t))}]\Big|dt\\ \le&\ln\frac{2\overline{r}}{C^-}+B^+\frac{2\overline{r}}{c^-}+2\overline{r}\omega =: \beta_1.\end{aligned}$$

As $\lambda B(t)e^{u_1(t-\sigma(t))}\ge 0$, one can find that

$$u_1(t)\le\beta_1,\quad t\in[0,\omega]. \tag{2.8}$$

Since $(u_1(t), u_2(t))^T \in X$, there exist $\xi_i, \eta_i \in [0, \omega] \quad (i = 1, 2)$ such that

$$u_i(\xi_i) = \min_{t\in[0,\omega]}\{u_i(t)\}, \quad u_i(\eta_i) = \max_{t\in[0,\omega]}\{u_i(t)\}, \quad i = 1, 2. \tag{2.9}$$

According to (2.2), (2.5) and (2.8), for any $t \in [0, \omega]$, we obtain

$$\begin{aligned}\int_0^\omega |u_1'(t)|dt =&\lambda \int_0^\omega |r(t) - a(t)e^{u_1(t-\sigma(t))} - b(t)e^{u_1(t-\sigma(t))}u_1'(t - \sigma(t)) - h(e^{u_1(t)})e^{u_2(t)}|dt \\ \leq& \int_0^\omega r(t)dt + \int_0^\omega [c(t)e^{u_1(t-\sigma(t))} + h(e^{u_1(t)})e^{u_2(t)}]dt + \int_0^\omega |B'(e^{u_1(t)})e^{u_1(t-\sigma(t))}|dt \\ &+ \int_0^\omega |b(t)e^{u_1(t-\sigma(t))}u_1'(t - \sigma(t))|dt \\ \leq& 2\overline{r}\omega + |B'|_0e^{\beta_1}\omega + \int_0^\omega |b(t)e^{u_1(t-\sigma(t))}u_1'(t - \sigma(t))|dt.\end{aligned}$$

In addition,

$$\begin{aligned}\int_0^\omega |b(t)e^{u_1(t-\sigma(t))}u_1'(t - \sigma(t))|dt =& \int_{-\sigma(0)}^{\omega-\sigma(\omega)} |b(\varphi(p))e^{u_1(p)}u_1'(p)|\frac{1}{1 - \sigma'(\varphi(p))}dp \\ =& \int_{-\sigma(0)}^{\omega-\sigma(\omega)} |\frac{b(\varphi(p))}{1 - \sigma'(\varphi(p))}e^{u_1(p)}u_1'(p)|dp \\ =& \int_{-\sigma(0)}^{\omega-\sigma(\omega)} |B(\varphi(p))e^{u_1(p)}u_1'(p)|dp \\ \leq& B^+e^{\beta_1} \int_{-\sigma(0)}^{\omega-\sigma(\omega)} |u_1'(p)|dp \\ =& B^+e^{\beta_1} \int_0^\omega |u_1'(t)|dt\end{aligned}$$

which implies that

$$\int_0^\omega |u_1'(t)|dt \leq 2\overline{r}\omega + |B'|_0e^{\beta_1}\omega + B^+e^{\beta_1} \int_0^\omega |u_1'(t)|dt. \tag{2.10}$$

From (H_2), we obtain

$$\int_0^\omega |u_1'(t)|dt \leq \frac{2\overline{r}\omega + |B'|_0e^{\beta_1}\omega}{1 - B^+e^{\beta_1}}. \tag{2.11}$$

Since

$$\lim_{x\to 0} h(x) = \lim_{x\to 0} \frac{g(x)}{x} = g'(0) \quad \text{and} \quad \lim_{x\to+\infty} h(x) = 0,$$

there exists a constant $M > 0$ such that

$$h(x) \leq M, \quad \text{for } \ x \in [0, +\infty). \tag{2.12}$$

Let $t=\psi(q)$ be the inverse function of $q=t-\tau(t)$. It is easy to see that $\mu(\psi(q))$ and $\tau'(\psi(q))$ are all ω-periodic functions. By virtue of (2.6), (2.9), (2.12) and (H_2), we have

$$\overline{d}\omega=\int_0^\omega \mu(t)g(e^{u_1(t-\tau(t))})dt=\int_0^\omega \frac{\mu(\psi(t))}{1-\tau'(\psi(t))}g(e^{u_1(t)})dt\le \Lambda^+M\omega e^{u_1(\eta_1)},$$

and so

$$u_1(\eta_1)\ge \ln(\frac{\overline{d}}{\Lambda^+M}).$$

Then

$$u_1(t)\ge u_1(\eta_1)-\int_0^\omega |u_1'(t)|dt\ge \ln(\frac{\overline{d}}{\Lambda^+M})-\frac{2\overline{r}\omega+|B'|_0e^{\beta_1}\omega}{1-B^+e^{\beta_1}}=:\beta_2. \tag{2.13}$$

It follows from (2.8) and (2.13) that

$$\max_{t\in[0,\omega]}|u_1(t)|\le \max\{|\beta_1|,|\beta_2|\}=:D_1. \tag{2.14}$$

From (2.5) and (H_2), one can find that

$$u_2(\xi_2)\le \ln(\frac{\overline{r}}{L}). \tag{2.15}$$

In view of (2.6)

$$\overline{d}\omega=\int_0^\omega \mu(t)g(e^{u_1(t-\tau(t))})dt\ge L\int_0^\omega \mu(t)e^{u_1(t-\tau(t))}dt=L\int_0^\omega \frac{\mu(\psi(t))}{1-\tau'(\psi(t))}e^{u_1(t)}dt\ge L\Lambda^-\int_0^\omega e^{u_1(t)}dt,$$

This implies that

$$\int_0^\omega e^{u_1(t)}dt\le \frac{\overline{d}\omega}{L\Lambda^-}.$$

Notice that

$$\int_0^\omega h(e^{u_1(t)})e^{u_2(t)}dt\le M\int_0^\omega e^{u_2(t)}dt,$$

$$\int_0^\omega c(t)e^{u_1(t-\sigma(t))}dt\le \int_0^\omega \frac{c(\varphi(t))}{1-\sigma'(\varphi(t))}e^{u_1(t)}dt\le C^+\int_0^\omega e^{u_1(t)}dt,$$

we can get from (2.5) and (H_2) that

$$e^{u_2(\eta_2)}\omega\ge \int_0^\omega e^{u_2(t)}dt\ge \frac{\overline{r}\omega-C^+\int_0^\omega e^{u_1(t)}dt}{M}\ge \frac{\overline{r}L\Lambda^-\omega-C^+\overline{d}\omega}{LM\Lambda^-} \tag{2.16}$$

i.e.

$$u_2(\eta_2)\ge \ln\big(\frac{\overline{r}L\Lambda^--C^+\overline{d}}{LM\Lambda^-}\big). \tag{2.17}$$

In addition, it follows from (2.2), (2.6) that, for any $t \in [0, \omega]$,

$$\int_0^\omega |u_2'(t)|dt = \lambda \int_0^\omega \Big| - d(t) + \mu(t)g(e^{u_1(t-\tau(t))})\Big|dt \le \int_0^\omega d(t)dt + \int_0^\omega \mu(t)g(e^{u_1(t-\tau(t))})dt = 2\overline{d}\omega,$$

which, together with (2.15) and (2.17), implies that for $t \in [0, \omega]$,

$$u_2(t) \le u_2(\xi_2) + \int_0^\omega |u_2'(t)|dt \le \ln(\frac{\overline{r}}{L}) + 2\overline{d}\omega =: \beta_3$$

and

$$u_2(t) \ge u_2(\eta_2) - \int_0^\omega |u_2'(t)|dt \ge \ln\big(\frac{\overline{r}L\Lambda^- - C^+\overline{d}}{LM\Lambda^-}\big) - 2\overline{d}\omega =: \beta_4.$$

Hence

$$\max_{t\in[0,\omega]} |u_2(t)| \le \max\{|\beta_3|, |\beta_4|\} =: D_2. \tag{2.18}$$

From (2.2), (2.8), (2.12) and (2.18), one can find that for any $t \in [0, \omega]$,

$$\begin{aligned} |u_1'(t)| =& \Big|\lambda\Big[r(t) - a(t)e^{u_1(t-\sigma(t))} - b(t)e^{u_1(t-\sigma(t))}u_1'(t-\sigma(t)) - h(e^{u_1(t)})e^{u_2(t)}\Big]\Big| \\ \le& r^+ + a^+e^{\beta_1} + b^+e^{\beta_1}|u_1'|_0 + Me^{D_2} \end{aligned}$$

and

$$|u_2'(t)| = \Big|\lambda\Big[- d(t) + \mu(t)g(e^{u_1(t-\tau(t))})\Big]\Big| \le d^+ + \mu^+Me^{\beta_1}.$$

These, together with (H_2), yield

$$|u_1'|_0 \le \frac{r^+ + a^+e^{\beta_1} + Me^{D_2}}{1 - b^+e^{\beta_1}} =: D_3, \tag{2.19}$$

and

$$|u_2'|_0 \le d^+ + \mu^+Me^{\beta_1} =: D_4. \tag{2.20}$$

From (2.14), (2.18)-(2.20), we have

$$\|u\| = |u|_\infty + |u'|_\infty \le D_1 + D_2 + D_3 + D_4.$$

Furthermore, it follows from (H_4) and Lemma 2.2 that the algebraic equations

$$\begin{cases} \overline{r} - \overline{a}u - h(u)v = 0, \\ -\overline{d} + \overline{\mu}g(u) = 0. \end{cases}$$

has a unique solution $(u^*, v^*)^T \in R^2_+$ with $u^*, v^* > 0$. Denote $D = D_1 + D_2 + D_3 + D_4 + D_0$, where $D > 0$ is taken sufficiently large such that

$$\|(\ln\{u^*\}, \ln\{v^*)\| = \max\{|\ln\{u^*\}|, |\ln\{v^*)|\} < D_0.$$

We now take

$$\Omega = \{x(t) \in X : \|x\| < D\}.$$

This satisfies condition(i) in Lemma 2.1. When $(u_1(t), u_2(t))^T \in \partial\Omega \cap KerL = \partial\Omega \cap R^2, (u_1(t), u_2(t))^T$ is a constant vector in R^2 with $|u_1| + |u_2| = D$. Thus, we have

$$QN \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \overline{r} - \overline{a}e^{u_1} - h(e^{u_1})e^{u_2} \\ -\overline{d} + \overline{\mu}g(e^{u_1}) \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This proves that condition (ii) in Lemma 2.1 is satisfied.

Taking $J = I : ImQ \to KerL, (u_1, u_2)^T \to (u_1, u_2)^T$, in view of the assumptions in Theorem 2.1, a direct computation gives

$$deg\{JQN, \Omega \cap KerL, 0\} \neq 0.$$

By now we have proved that Ω satisfies all the requirements in Lemma 2.1. Hence, (2.1) has at least one ω-periodic solution. Accordingly, system (1.4) has at least one ω-periodic solution with strictly positive components. The proof of Theorem 2.1 is complete. □

Remark 2.1. *It is easy to see that (H_3) is also the necessary condition for the existence of positive ω-periodic solutions of system (1.4).*

Remark 2.2. *The time delays $\sigma(t)$ and $\tau(t)$ have influence on the existence of positive periodic solutions to system (1.4).*

Remark 2.3. *If $\sigma(t) \equiv \sigma, \tau(t) \equiv \tau$ are positive constant, the result is still holds. But the priori bounds of all positive periodic solutions are different, The $C(t) = \frac{c(\varphi(t))}{1-\sigma'(\varphi(t))}, \Lambda(t) = \frac{\mu(\psi(t))}{1-\psi'(\varphi(t))}$ should be replaced by $B(t) = b(t+\sigma), C(t) = c(t+\sigma), \Lambda(t) = \mu(t+\tau)$.*

3 An Example

In this section, we give an example to illustrate the results obtained in previous sections.

Example 3.1. Consider the following system:

$$\begin{cases} x'(t) = x(t)\big[(3+2\sin(20\pi t)) - (\frac{1}{2} - \frac{1}{4}\cos(20\pi t))x(t - \frac{1}{20\pi}\sin(20\pi t)) \\ \qquad - \frac{1}{100}x'(t - \frac{1}{20\pi}\sin(20\pi t))\big] - \frac{x(t)y(t)}{9+x^2(t)}, \\ y'(t) = y(t)\big[-\frac{1}{200}(1 - \frac{1}{3}\cos(20\pi t) + \frac{x(t-\frac{1}{60\pi}\sin(20\pi t))}{9+x^2(t)}\big]. \end{cases} \tag{3.1}$$

A straightforward calculation shows that

$$\overline{r} = 3, \quad \overline{a} = \frac{1}{2}, \quad \overline{d} = \frac{1}{200}, \quad \overline{\mu} = 1, \quad a^- = \frac{1}{4}, \quad b^+ = \frac{3}{200}, \quad k = 3, \quad \omega = \frac{1}{10}$$

and

$$B(t) = \frac{1}{100}, \quad C(t) = \frac{1}{2}, \quad \Lambda(t) = 1, \quad c(t) = a(t) = \frac{1}{2} - \frac{1}{4}\cos(20\pi t),$$

Further,

$$\beta_1 = \ln 12 + 0.84, \qquad \beta_2 = -2.1276, \qquad L = \min_{t\in[\beta_2,\beta_1]} h(e^x) = 0.0013,$$

Hence,

$$g(k)\overline{\mu} = \frac{1}{6} > \frac{1}{200}.$$

In addition,

$$\max_{t\in[0,\omega]}\{b^+, B^+\}e^{\beta_1} = \frac{3}{200} \times 12 \times e^{0.84} = 0.4170 < 1$$

and

$$\overline{r}L\Lambda^- = 3 \times 0.0013 \times 1 = 0.0039 > C^+\overline{d} = \frac{1}{2} \times \frac{1}{200} = 0.0025.$$

Consequently, all the conditions in Theorem 2.1 hold. Therefore, system 3.1 has at least one $\frac{1}{10}$-periodic solution with strictly positive components.

Remark 3.1. To the best of our knowledge, few authors have considered the problems of periodic solutions of neutral delay predator-prey model with nonmonotonic functional response. One can easily see that all the results in [15-17] and the references therein cannot be applicable to Eq. (3.1) to obtain the existence of $\frac{1}{10}$-periodic solutions. This implies that the results of this paper are new.

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