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Linearizability conditions of quasi-cubic systems*

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Abstract. In this paper we study the linearizability problem of the two-dimensional complex quasi-cubic system $\dot{z} = z + (zw)^d(a_{30}z^3 + a_{21}z^2w + a_{12}zw^2 + a_{03}w^3)$, $\dot{w} = -w - (zw)^d(b_{30}w^3 + b_{21}w^2z + b_{12}wz^2 + b_{03}z^3)$, where $z, w, a_{ij}, b_{ij} \in \mathbb{C}$ and d is a real number. We find a transformation to change the quasi-cubic system into an equivalent quintic system and then obtain the necessary and sufficient linearizability conditions by the Darboux linearization method or by proving the existence of linearizing transformations.

Key words. Center, period constants, isochronous center, linearizability.

MSC2000: 34C05, 34C07

1 Introduction

Linearizability problem is one of the interesting problems of the investigation of the ordinary differential system

$$\begin{aligned} \frac{dz}{dT} &= z + \sum_{k=2}^{\infty} Z_k(z,w) = P(z,w), \\ \frac{dw}{dT} &= -w - \sum_{k=2}^{\infty} W_k(z,w) = Q(z,w), \end{aligned} \tag{1}$$

where

$$Z_k(z,w) = \sum_{\alpha+\beta=k} a_{\alpha\beta} z^{\alpha} w^{\beta},\ W_k(z,w) = \sum_{\alpha+\beta=k} b_{\alpha\beta} w^{\alpha} z^{\beta},$$

$a_{\alpha\beta}, b_{\alpha\beta} \in \mathbb{C}$, z, w, T are complex variables. As concerned in [1], system (1) is called *linearizable* if there is an analytic transformation

$$\xi = z + o(|(z,w)|), \quad \eta = w + o(|(z,w)|) \tag{2}$$

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such that

$$\frac{d\xi}{dT} = \xi, \quad \frac{d\eta}{dT} = -\eta. \tag{3}$$

By the transformation

$$z = x + yi, w = x - yi, T = it, i = \sqrt{-1} \tag{4}$$

system (1) becomes

$$\begin{aligned} \tfrac{dx}{dt} &= -y + \sum_{k=2}^{\infty} X_k(x,y), \\ \tfrac{dy}{dt} &= x + \sum_{k=2}^{\infty} Y_k(x,y), \end{aligned} \tag{5}$$

where $x, y \in \mathbb{R}$, $X_k(x,y)$ and $Y_k(x,y)$ are homogeneous polynomials of degree k in x and y. We say that system (1) is the associated system of system (5). It is obvious that system (5) is real if and only if t is a real variable and the coefficients of system (1) satisfy conjugate conditions, i.e.,

$$\overline{a_{\alpha\beta}} = b_{\alpha\beta}, \ \alpha \geq 0, \ \beta \geq 0, \ \alpha + \beta \geq 2. \tag{6}$$

When system (5) is real, the critical point at the origin is called a *center* if every solution in a neighborhood of the origin is periodic and, furthermore, an *isochronous center* if these periodic solutions have the same period. It is well known that the origin of system (5) is an isochronous center if and only if system (1)or (5) can be linearized by an analytic substitution (see, e.g., [2, 3, 4, 5]). Thus, in such sense, linearizability problem is an extended problem of the isochronous center problem. For polynomial systems of form (5), a lot of works have been done in the research of centers and isochronous centers (see, e. g., survey publications [1, 3, 6, 7]).

In recent years, some mathematicians consider the following system

$$\begin{aligned} \tfrac{dx}{dt} &= -y + (x^2+y^2)^d \sum_{i=0}^{m} A_{m-i,i} x^{m-i} y^i, \\ \tfrac{dy}{dt} &= x + (x^2+y^2)^d \sum_{i=0}^{m} B_{m-i,i} x^{m-i} y^i, \end{aligned} \tag{7}$$

where $A_{i,j}, B_{i,j}, d \in \mathbb{R}$, and obtain some results about center problem and bifurcation of limit cycles ([8, 9, 10, 11]). The center problem and bifurcation of limit cycles are studied in [8] for system $(7)|_{m=2}$ and d is a real number. The linearizability problem (or equivalently, isochronous center problem) is investigated in [9] for system $(7)|_{m=2}$ and d is a non-negative integer, in [10] for some special form $(7)|_{m=4}$. For the case that $m = 3$, center problem of system (7) is solved in [9] and [11] independently. However, there is no results about the linearizability problem of the associated system of $(7)|_{m=3}$.

Consider the following system

$$\begin{aligned} \tfrac{dz}{dT} &= z + (zw)^d(a_{30}z^3 + a_{21}z^2w + a_{12}zw^2 + a_{03}w^3), \\ \tfrac{dw}{dT} &= -w - (zw)^d(b_{30}w^3 + b_{21}w^2z + b_{12}wz^2 + b_{03}z^3), \end{aligned} \tag{8}$$

where $a_{\alpha\beta}, b_{\alpha\beta}, T \in \mathbb{C}$ and d is a positive real number, (z, w) lies in $\Xi := \{(z, w) \in \mathbb{C}^2 : zw \neq 0\} \cup \{(0, 0)\}$. As indicated in [12] the linearizability problem of system (8) is: Can (8) be linearized to linear system (3) by an analytic near-identity transformation (2) near the origin in Ξ? In this paper, we study the linearizability problem of (8). We find a substitution to transform the quasi-cubic system (8) into an equivalent quintic system and then obtain the necessary and sufficient linearizability conditions by the Darboux linearization method or by proving the existences of linearizing transformations.

2 Preliminaries

In this section we introduce some methods about linearization, which will be used in the next section.

Lemma 2.1 (see [13]) *For system (1) one can derive uniquely the following formal series:*

$$f(z,w) = z + \sum_{k+j=2}^{\infty} c'_{k,j} z^k w^j, \quad g(z,w) = w + \sum_{k+j=2}^{\infty} d'_{k,j} w^k z^j, \tag{9}$$

where $c'_{1,0} = d'_{1,0} = 1$, $c'_{0,1} = d'_{0,1} = c'_{k+1,k} = d'_{k+1,k} = 0, k = 1, 2, \ldots,$ *and*

$$\begin{aligned} c'_{k,j} &= \tfrac{1}{j+1-k} \sum_{\alpha+\beta=3}^{k+j+1} [(k-\alpha+1)a_{\alpha,\beta-1} - (j-\beta+1)b_{\beta,\alpha-1}]c'_{k-\alpha+1,j-\beta+1}, \\ d'_{k,j} &= \tfrac{1}{j+1-k} \sum_{\alpha+\beta=3}^{k+j+1} [(k-\alpha+1)b_{\alpha,\beta-1} - (j-\beta+1)a_{\beta,\alpha-1}]d'_{k-\alpha+1,j-\beta+1}, \end{aligned} \tag{10}$$

such that

$$\frac{df}{dT} = f(z,w) + \sum_{j=1}^{\infty} p'_j z^{j+1} w^j, \quad \frac{dg}{dT} = -g(z,w) - \sum_{j=1}^{\infty} q'_j w^{j+1} z^j \tag{11}$$

and p'_j *and* q'_j *are determined by following recursive formulas:*

$$\begin{aligned} p'_j &= \sum_{\alpha+\beta=3}^{2j+2} [(j-\alpha+2)a_{\alpha,\beta-1} - (j-\beta+1)b_{\beta,\alpha-1}]c'_{j-\alpha+2,j-\beta+1}, \\ q'_j &= \sum_{\alpha+\beta=3}^{2j+2} [(j-\alpha+2)b_{\alpha,\beta-1} - (j-\beta+1)a_{\beta,\alpha-1}]d'_{j-\alpha+2,j-\beta+1}. \end{aligned} \tag{12}$$

Evidently, system (8) is linearizable if and only if all p'_k's and q'_k's given by Lemma 2.1 are zeroes. Therefore, in order to find the linearizability conditions of system (1), we use formula (12) to compute p'_k's and q'_k's and then decompose the variety of the first several quantities.

For system (1), one of efficient methods to investigate the linearizability problem is the so-called Darboux linearization (see [3]). An analytic function $f(z, w)$ is called a *Darboux factor* if there exists $K(z, w) \in \mathbb{C}[z, w]$, called the *cofactor* of $f(z, w)$, such that

$$\frac{\partial f}{\partial z} P(z,w) + \frac{\partial f}{\partial w} Q(z,w) = K(z,w) f(z,w). \tag{13}$$

If $f(z,w)$ is a polynomial, the curve $f(z,w)=0$ is called an *invariant algebraic curve*. Straight computation shows that if there are Darboux factors $f_1, f_2, ..., f_k$ with the cofactors $K_1, K_2, ..., K_k$ satisfying

$$\sum_{i=1}^{k} \alpha_i K_i = 0, \tag{14}$$

then $H = f_1^{\alpha_1} ... f_k^{\alpha_k}$ is a first integral of system (1), and if

$$\sum_{i=1}^{k} \alpha_i K_i + P'_z + Q'_w = 0, \tag{15}$$

then system (1) has an integrating factor $\mu = f_1^{\alpha_1} ... f_k^{\alpha_k}$.

Lemma 2.2 (see[3, 14]) *Assume that system (1) has a Lyapunov first integral $\Psi(z,w)$, that is,*

$$\Psi(z,w) = zw + o(|\,(z,w)\,|^2), \tag{16}$$

and Darboux factor $f_i(z,w)$ satisfying $f_i(0,0)=1$ with the cofactor $K_i(z,w), i=1,...,k$. If $(1-c)\frac{P(z,w)}{z} - c\frac{Q(z,w)}{w} + \sum_{i=1}^{k} \alpha_i K_i = 1$ for some $c, \alpha_1, ..., \alpha_k \in \mathbb{C}$, then the first equation of (1) can be linearized by the substitution $Z = z^{1-c} w^{-c} \psi^c f_1^{\alpha_1} ... f_k^{\alpha_k}$. If $(-c)\frac{P(z,w)}{z} + (1-c)\frac{Q(z,w)}{w} + \sum_{i=1}^{k} \beta_i K_i = -1$ for some $c, \beta_1, ..., \beta_k \in \mathbb{C}$, then the second equation of (1) can be linearized by the substitution $W = z^{-c} w^{1-c} \psi^c f_1^{\beta_1} ... f_k^{\beta_k}$.

Another way to prove the linearizability of (1) is given in [15] if only one transformation is found for one equation of system (1).

Lemma 2.3 (see[15]) *Assume that system (1) has a Lyapunov first integral $\Psi(z,w)$ of the form (16), If the first equation (second equation, respectively) of (1) is linearizable by the change $Z = Z(z,w)(W = W(z,w), respectively)$, them the second equation (first equation,respectively) of (1) can be linearized by the substitution $W = \frac{\psi(z,w)}{Z}$ $(Z = \frac{\psi(z,w)}{W}$, respectively).*

3 The linearizability conditions

By substitution

$$(\xi,\eta) = \begin{cases} \left(z^{\frac{d+3}{4}} w^{\frac{d-1}{4}}, w^{\frac{d+3}{4}} z^{\frac{d-1}{4}}\right), & \text{if } zw \neq 0, \\ (0,0), & \text{if } (z,w) = (0,0), \end{cases} \tag{17}$$

system (8) can be transformed into

$$\begin{aligned} \frac{d\xi}{dT} &= \xi + \tfrac{1}{4}(1-d)b_{03}\xi^5 + \tfrac{1}{4}((1-d)b_{12} + (3+d)a_{30})\xi^4\eta + \tfrac{1}{4}((1-d)b_{21} + (3+d)a_{21})\xi^3\eta^2 \\ &\quad + \tfrac{1}{4}((1-d)b_{30} + (3+d)a_{12})\xi^2\eta^3 + \tfrac{1}{4}(d+3)a_{03}\xi\eta^4, \\ \frac{d\eta}{dT} &= -\eta - \tfrac{1}{4}(1-d)a_{03}\eta^5 - \tfrac{1}{4}((1-d)a_{12} + (3+d)b_{30})\xi\eta^4 - \tfrac{1}{4}((1-d)a_{21} + (3+d)b_{21})\xi^2\eta^3 \\ &\quad - \tfrac{1}{4}((1-d)a_{30} + (3+d)b_{12})\xi^3\eta^2 - \tfrac{1}{4}(d+3)b_{03}\xi^4\eta. \end{aligned} \tag{18}$$

Similarly to [12, Theorem 2.1], it is easy to check that if system (8) is linearizable, i. e., there exists a substitution $Z = z + o(|(z, w)|)$, $W = w + o(|(z, w)|)$ such that $\dot{Z} = Z$, $\dot{W} = -W$, then

$$X = Z^{\frac{d+3}{4}} W^{\frac{d-1}{4}} = \xi + o(|(\xi, \eta)|), \quad Y = Z^{\frac{d-1}{4}} W^{\frac{d+3}{4}} = \eta + o(|(\xi, \eta)|)$$

is a linearizing substitution of system (18). By the same method, one can prove that (8) is linearizable when (18) is linearizable. Thus, in order to obtain linearizability conditions of system (8), we need only to find the linearizability conditions of the quintic system (18).

Theorem 3.1 *If system (8) is linearizable, then one of the following six conditions holds:*
(I) $b_{03} = b_{12} = a_{21} = b_{21} = a_{30} = 0$,
(II) $a_{03} = a_{12} = b_{21} = a_{21} = b_{30} = 0$,
(III) $a_{30} = a_{21} = b_{21} = a_{03} = b_{12} = (b_{30} - a_{12})d - 3a_{12} - b_{30} = 0$,
(IV) $b_{03} = a_{12} = a_{21} = b_{21} = b_{30} = (a_{30} - b_{12})d - a_{30} - 3b_{12} = 0$,
(V) $a_{03} = b_{03} = a_{21} = b_{21} = b_{12}b_{30} - a_{12}a_{30} = (a_{30} - b_{12})d - 2b_{12} = 0$,
(VI) $a_{03} = b_{03} = a_{21} = b_{21} = a_{12} + b_{30} = a_{30} + b_{12} = 0$.

Proof. As mentioned in last section, system (18) is linearizable if and only if all p'_k's and q'_k's given by Lemma 2.1 are zeroes. However, it is difficult to find the common zeroes of infinite polynomials. The usual way is to compute the common zeroes of the first several quantities to obtain the necessary conditions for the linearizability of system (18) and then prove the sufficiency of these necessary conditions by some special methods such as Darboux linearization.

Using the formulas given in Lemma 2.1, for system (18) we compute the first 10 pairs of p'_k and q'_k with computer algebra system *Mathematica* and obtain that $p'_{2i-1} = q'_{2i-1} \equiv 0$ for $i = 1, ..., 5$ and

$$\begin{aligned}
p'_2 &= \frac{1}{4}(2a_{21} + 2b_{21} + a_{21}d - b_{21}d), \\
q'_2 &= \frac{1}{4}(2a_{21} + 2b_{21} - a_{21}d + b_{21}d), \\
p'_4 &= \frac{1}{4}(2a_{30}b_{30}d - 2a_{12}a_{30} - 2a_{03}b_{03} - 2a_{12}b_{12} - 2a_{30}b_{30} - 2b_{12}b_{30} - a_{12}a_{30}d - a_{03}b_{03}d - 2a_{12}b_{12}d + b_{12}b_{30}d), \\
q'_4 &= \frac{1}{4}(2a_{30}b_{30}d - 2a_{12}a_{30} - 2a_{03}b_{03} - 2a_{12}b_{12} - 2a_{30}b_{30} - 2b_{12}b_{30} + a_{12}a_{30}d - a_{03}b_{03}d - 2a_{12}b_{12}d - b_{12}b_{30}d)
\end{aligned}$$

and $p'_6, q'_6, ..., p'_{10}, q'_{10}$ have 146, 146, 312, 312, 674, 674 terms, respectively. We do not present them here, but the reader can easily calculate them using formula (12) with any computer algebra system.

Using *minAssChar* of *Singular* ([16]), we find the decomposition

$$V(p'_2, q'_2, ..., p'_{10}, q'_{10}) = \cup_{i=1}^{6} \Lambda_i,$$

where $\mathrm{V}(p'_2, q'_2, ..., p'_{10}, q'_{10})$ is the variety of the ideal generated by $p'_2, q'_2, ..., p'_{10}, q'_{10}$ and Λ_i means the set determined by condition (i) given in the theorem. Thus, one of the six conditions must hold if system (8) is linearizable. □

In Theorem 3.1, the necessity of the six conditions for system (8) to be linearizable is proved. In the following theorem, we prove their sufficiency one by one. Therefore, actually we obtain the necessary and sufficient linearizability conditions of system (8).

Theorem 3.2 *System (8) is linearizable if and only if one of the six conditions given in Theorem 3.1 holds.*

Proof. By Theorem 3.1, we need only to prove that system (18) is linearizable if one of the six conditions holds.

The system satisfying condition (I) takes form

$$\begin{aligned}\dot{\xi} &= \xi + \tfrac{1}{4}(3+d)a_{03}\xi\eta^4 + \tfrac{1}{4}((3+d)a_{12} + (1-d)b_{30})\xi^2\eta^3,\\ \dot{\eta} &= -\eta - \tfrac{1}{4}(1-d)a_{03}\eta^5 - \tfrac{1}{4}((1-d)a_{12} + (3+d)b_{30})\xi\eta^4.\end{aligned} \tag{19}$$

Though we are unable to find an explicit linearizing transformation for (19), we can prove its existence. we look for a linearizing substitution for the second equation of (19) in the form

$$z_2 = \sum_{k=1}^{\infty} f_k(\xi)\eta^k, \tag{20}$$

where $f_k(\xi)$ $(k = 2, 3, ...)$ are some polynomials of degree $k-1$ and $f_1(\eta) \equiv 1$. (20) provides a linearization of the second equation of (19) if and only if there exist $f_k(\xi)$'s satisfying the differential equation

$$\begin{aligned}&4\xi f_k' - 4(k-1) f_k - (k-3)(-(d-1)a_{12} + (d+3)b_{30})\xi f_{k-3} - (-(d-1)a_{12} + (d+3)b_{30})\xi^2 f_{k-3}'\\ &+(k-4)(d-1)a_{03} f_{k-4} + (d+3)a_{03}\xi f_{k-4}' = 0,\end{aligned} \tag{21}$$

where $f_n(\xi) \equiv 0$ for all $n \leq 0$. $f_2, f_3, ...f_5$ can be obtained from (21) directly. Assume that for $k = 6, ..., m$ there are polynomials f_k of degree $k-1$ satisfying (20) yielding a linearization. Solving the linear differential equation (21), we obtain

$$f_(m+1)(\xi) = \xi^m(C + \int \xi^{-m-1} h_{m-2} d\xi) = C\xi^m + \overline{h}_{m-2}(\xi)$$

because h_{m-2} is a polynomials of degree $m-2$.

In order to prove the linearizability of the first equation of (19), we show that a Lyapunov first integral of the system can be found in the form $\psi(\xi, \eta) = \sum_{k=1}^{\infty} g_k(\xi)\eta^k$, where $g_1(\xi) = \xi$, $g_2(\xi) = \xi^2$ and $g_k(\xi)$ are polynomial of degree k satisfying the linear differential equation

$$\begin{aligned}&4\xi g_k' - 4k g_k - (k-3)(-(d-1)a_{12} + (d+3)b_{30})\xi g_{k-3} - (-(d-1)a_{12} + (d+3)b_{30})\xi^2 g_{k-3}'\\ &+(k-4)(d-1)a_{03} g_{k-4} + (d+3)a_{03}\xi g_{k-4}' = 0.\end{aligned} \tag{22}$$

Similarly, polynomials g_k's can be determined recursively by (22) and, therefore, by Theorem 2.3 the first equation of (19) can be linearized by the change $z_1 = \Psi(\xi, \eta)/z_2$.

For the system satisfying condition (III), we firstly consider $a_{12} = b_{30}$, i. e.,

$$\begin{aligned} \dot{\xi} &= \xi + \tfrac{1}{4}(1-d)b_{03}\xi^5, \\ \dot{\eta} &= -\eta - \tfrac{1}{4}(3+d)b_{03}\xi^4\eta. \end{aligned} \tag{23}$$

Moreover, we assume that $b_{03} \neq 0$. Otherwise, (23) is a linear system. When $d = -1$, (23) has three Darboux factors

$$f_1 = \xi, \quad f_2 = \eta, \quad f_3 = 1 + \frac{1}{2}b_{03}\xi^4$$

with cofactors

$$K_1 = 1 + \frac{1}{2}b_{03}\xi^4, \;\; K_2 = -1 - \frac{1}{2}b_{03}\xi^4, \;\; K_3 = 2b_{03}\xi^4,$$

respectively. By Lemma 2.2, system (23) can be linearized by the substitution

$$z_1 = \xi f_1^{-\frac{3}{2}} f_2^{\frac{1}{2}} f_3^{\frac{2+b_{03}}{4b_{03}}}, \quad z_2 = \eta f_1^{\frac{5}{6}} f_2^{\frac{5}{6}} f_3^{\frac{2+b_{03}}{4b_{03}}}.$$

When $d \neq -1$, system (23) has three Darboux factors

$$f_1 = \xi, \quad f_2 = \eta, \quad f_3 = 1 + \frac{1}{4}(1-d)b_{03}\xi^4$$

with cofactors

$$K_1 = 1 + \frac{1}{4}(1-d)b_{03}\xi^4, \;\; K_2 = -1 - \frac{1}{4}(3+d)b_{03}\xi^4, \;\; K_3 = -b_{03}(-1+d)\xi^4,$$

respectively. By Lemma 2.2, system (23) can be linearized by the substitution

$$z_1 = \xi f_1^{\alpha_1} f_2^{\alpha_2} f_3^{\alpha_3}, \quad z_2 = \eta f_1^{\beta_1} f_2^{\beta_2} f_3^{\beta_3},$$

where

$$\alpha_1 = -\frac{-1+3d}{1+d}, \quad \alpha_2 = -\frac{2(-1+d)}{1+d}, \quad \alpha_3 = 1, \quad \beta_1 = -\frac{2(-1+d)}{1+d}, \quad \beta_2 = -\frac{-1+3d}{1+d}, \quad \beta_3 = 1.$$

The system satisfying condition (III) and $a_{12} \neq b_{30}$ is of the form

$$\dot{\xi} = \frac{(a_{12} - b_{30} + a_{12}b_{03}\xi^4)\xi}{a_{12} - b_{30}}, \;\; \dot{\eta} = \frac{(-a_{12} - a_{12}^2\xi\eta^3 + b_{30}(1 + b_{03}\xi^4 + b_{30}\xi\eta^3))\eta}{a_{12} - b_{30}}, \tag{24}$$

which has three Darboux factors

$$f_1 = \xi, \quad f_2 = \eta, \quad f_3 = 1 + \frac{a_{12}b_{03}}{a_{12} - b_{30}}\xi^4$$

with cofactors

$$K_1 = \frac{a_{12} - b_{30} + a_{12}b_{03}\xi^4}{a_{12} - b_{30}}, \;\; K_2 = \frac{-a_{12} - a_{12}^2\xi\eta^3 + b_{30}(1 + b_{03}\xi^4 + b_{30}\xi\eta^3)}{a_{12} - b_{30}}, \;\; K_3 = \frac{4a_{12}b_{03}\xi^4}{a_{12} - b_{30}},$$

respectively. We find a linearizing substitution $z_1 = \xi f_3^{-\frac{1}{4}}$ for the first equation of (24). On the other hand, it is easy to check that (15) holds with $\alpha_1 = \alpha_2 = -4$ and $\alpha_3 = -\frac{1}{4} + \frac{3b_{30}}{4a_{12}}$, which implies that (24) has an integrating factor $\mu = \xi^{-4}\eta^{-4} f_3^{-\frac{1}{4}+\frac{3b_{30}}{4a_{12}}}$. Furthermore, we find a first integral

$$H(\xi,\eta) = -\eta^3 f_3^{\frac{7a_{12}-3b_{30}}{4a_{12}}} - \frac{(b_{30}+a_{12})\xi}{2} F_1\left(\frac{1}{2}, \frac{a_{12}-3b_{30}}{4a_{12}}, \frac{3}{2}, \frac{a_{12}b_{03}\xi^4}{b_{30}-a_{12}}\right),$$

where F_1 is the Gauss hypergeometric function. Then $\Psi(\xi,\eta) = (H(\xi,\eta))^{\frac{1}{3}}$ is a Lyapunov first integral of (24) yielding the linearization of the second equation of (24) of the form $z_2 = \xi^{-1} f_3^{\frac{1}{4}} \Psi(\xi,\eta)$ by Lemma2.3.

By the substitution $(\xi,\eta,T) \to (\eta,\xi,-T)$, the system satisfying condition (II) (resp. condition (IV)) can be transformed into the system satisfying condition (I) (resp. system satisfying condition (III)), which implies that it is linearizable.

For system satisfying condition (V), we only consider $b_{30} \neq 0$, i. e.,

$$\begin{aligned} \dot{\xi} &= \xi + \tfrac{1}{4}(a_{12}+b_{30})\xi^2\eta^3 + \tfrac{3a_{30}(a_{12}+b_{30})}{4b_{30}}\xi^4\eta, \\ \dot{\eta} &= -\eta - \tfrac{a_{30}}{4b_{30}}(a_{12}+b_{30})\xi^3\eta^2 - \tfrac{3}{4}(a_{12}+b_{30})\xi\eta^4, \end{aligned} \tag{25}$$

because (18) is linear if condition (V) holds and $b_{30} = 0$. System (25) has three Darboux factors

$$f_1 = 1 + (1+\frac{a_{12}}{b_{30}})a_{30}\xi^3\eta, \;\; f_2 = 1 + (a_{12}+b_{30})\xi\eta^3, \;\; f_3 = 1 + (a_{12}+b_{30})\xi\eta^3 + (1+\frac{a_{12}}{b_{30}})a_{30}\xi^3\eta,$$

which yield a linearizing substitution

$$z_1 = \xi f_1^{-\frac{7}{16}} f_2^{\frac{1}{16}} f_3^{\frac{1}{16}}, \quad z_2 = \eta f_1^{-\frac{7}{8}} f_2^{-\frac{11}{8}} f_3$$

of (25) by Lemma 2.2.

System satisfying condition (VI) is of the form

$$\begin{aligned} \dot{\xi} &= \xi - \tfrac{1}{2}(b_{12}(1+d)\xi^4\eta - a_{12}(1+d)\xi^2\eta^3), \\ \dot{\eta} &= -\eta - \tfrac{1}{2}(b_{12}(1+d)\xi^3\eta^2 - a_{12}(1+d)\xi\eta^4), \end{aligned} \tag{26}$$

which has a Darboux factor $f_1(\xi,\eta) = 1 - b_{12}(1+d)\xi^3\eta - a_{12}(1+d)\xi\eta^3$. Let

$$g(\xi,\eta) = f_1 - 1, \quad X = \xi f_1^{-\frac{1}{6}}, \quad Y = \eta f_1^{-\frac{1}{6}}.$$

Then, $g(\xi,\eta) = g(X,Y)\sqrt{f_1(\xi,\eta)}$, that is, $g(X,Y)^2 = g(\xi,\eta)^2/(1+g(\xi,\eta))$. From (26), we obtain

$$\dot{X} = X(1+g(\xi,\eta)/2), \dot{Y} = -Y(1+g(\xi,\eta)/2). \tag{27}$$

We clam that there exists a function $m(X,Y)$ such that $\dot{m} = g(\xi,\eta)$. In fact,we need only to solve the equation

$$X\tfrac{\partial m}{\partial X} - Y\tfrac{\partial m}{\partial Y} = \tfrac{g(\xi,\eta)}{1+g(\xi,\eta)/2} = \tfrac{g(X,Y)}{\sqrt{1+g(X,Y)^2/4}}. \tag{28}$$

Since the right-hand side of (28) can be expanded in odd powers of $g(X, Y)$, there is no term $X^k Y^k$ in the expansion. Thus, we can solve (28) for $m(X, Y)$ satisfying $m(0, 0) = 0$. Therefore, substitution $x_1 = Xe^{-m(X,Y)/2}, y_1 = Ye^{m(X,Y)/2}$ linearizes system (26). $\square$

In [17] all linearizability conditions of quintic systems with homogeneous nonlinearities are given by computing the linearizability quantities ([5]). From Theorem 2 of [17] we also obtain the same linearizability conditions of system (18) as given in Theorem 3.1. But in this paper we find these conditions by calculating the first ten pairs of singular point values and period constants, which is a different method from that used in [17]. On the other hand, we also use different methods to prove the sufficiency of these conditions. For instance, for the system satisfying condition (I) we prove the existence of linearizing transformations directly and in [17] it is to find a transversal commuting system, which usually is more difficult because there is no general methods to do this. For the system satisfying condition (III) the existence of a linearizing transformation is proved in [17], but in this paper we find the explicit expression of the linearizing transformation.

If $a_{\alpha\beta}$ and $b_{\alpha\beta}$ satisfy conjugate conditions (6), then by substitution (4) system (8) can be transformed into system $(7)|_{m=3}$. The linearizability condition is actually the isochronous center condition and Our results are consistent with that of [9] and [11].

References

[1] J. Chavarriga, M. Sabatini, A survey of isochronous centers, *Qual. Theory Dyn. Syst.* **1**(1999) 1-70.

[2] J. Chavarriga, I. A. García, J. Giné, Isochronous centers of a linear center perturbed by fifth degree homogeneous polynomials, *J. Comput. Appl. Math.* **126**(2000) 351-368.

[3] P. Mardešić, C. Rousseau, B. Toni, Linearization of isochronous centers, *J.Diff. Equa.* **121**(1995) 67-108.

[4] V. G. Romanovski, The Linearizable Centers of Time-Reversible Polynomial Systems, *Prog. Theor. Phys. Suppl.* **150**(2003) 243-254.

[5] V. G. Romanovski, D. S. Shafer, *The Center and Cyclicity Problems: A Computational Algebra Approach*, Birkhüser, Boston, 2009.

[6] V. V. Amelkin, N. A. Lukashevich, A. P. Sadovskii, *Nonlinear Oscillations in Second Order Systems*, BSU, Minsk, 1982.

[7] I. Pleshkan, A new method of investigating the isochronicity of a system of two differential equations, *Diff. Equa.* **5**(1969) 796-802.

[8] Y. Liu, The generalized focal values and bifurcation of limit cycles for quasi quadratic systems, in Chinese, *Acta Math. Sin.* **45**(2002) 671-682.

[9] J. Llibre, C. Valls, Classification of the centers, their cyclicity and isochronicity for a class of polynomial differential systems generalizing the linear systems with cubic homogeneous nonlinearities, *J. Diff. Equa.* **246**(2009) 2192-2204.

[10] J. Llibre, C. Valls, Classification of the centres and isochronous centers for a class of quartic-like systems, *Nonlinear Anal. TMA* **71**(2009) 3119-3128.

[11] P. Xiao, *Critical Point Quantities and Integrability Conditions for Complex Planar Resonant Polynomial Systems*, in Chinese, PHD thesis of Central South University, 2005.

[12] X. Chen, W. Huang, V. G. Romanovski, W. Zhang, Linearizability conditions of a time-reversible quartic-like system, *J. Math. Anal. Appl.* **383**(2011) 179-189.

[13] Y. Liu, W. Huang, A new method to determine isochronous center conditions for polynomial differential systems, *Bull. Sci. math.* **127**(2003) 133-148.

[14] D. Dolićanin, G. Milovanović, V. G. Romanovski, Linearizability conditions for a cubic system, *Appl. Math. Comput.* **190**(2007) 937-945.

[15] C. Christopher, C. Rousseau, Nondegenerate linearizable centres of complex planar quadratic and symmetric cubic systems in $\mathbb{C}^2$, *Publications Matematiques* **45**(2001) 95-123.

[16] G. M. Greuel, G. Pfister, H. Schönemann, Singular version 1.2 User Manual, in: Reports On Computer Algebra, number 21, Centre for Computer Algebra, University of Kaiserslautern, June 1998. http:www.mathematik.uni-kl.de/ zca/Singular.

[17] V. G. Romanovski, X. Chen, Z. Hu, Linearizability of linear systems perturbed by 5th degree homogeneous polynomials, *J. Phys. A: Math. Theor.* **40**(2007) 5905-5919.

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