

# Overdetermined boundary value problems with strongly nonlinear elliptic PDE\*

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**Abstract:** We consider the strongly nonlinear elliptic Dirichlet problem in a connected bounded domain, overdetermined with the constant Neumann condition  $F(\nabla u) = c$  on the boundary. Here  $F$  is convex and positively homogeneous of degree 1, and its polar  $F^*$  represents the anisotropic norm on  $\mathbb{R}^n$ . We prove that, if this overdetermined boundary value problem admits a solution in a suitable weak sense, then  $\Omega$  must be of Wulff shape.

**keywords:** Overdetermined boundary value problems; Strongly nonlinear elliptic PDE; Wulff shape; F-mean curvature; P-function; Pohozaev identity.

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## 1 Introduction and statement of the main result

Throughout this paper let  $F : \mathbb{R}^n \rightarrow [0, +\infty)$  be a convex function of class  $C^4(\mathbb{R}^n \setminus \{0\})$  which is even and positively homogeneous of degree 1, i.e.,

$$F(t\xi) = |t|F(\xi), \quad \forall t \in \mathbb{R}, \xi \in \mathbb{R}^n,$$

and  $F_{\xi_i} = \frac{\partial F}{\partial \xi_i}$ . A typical example is  $F(\xi) = (\sum_{i=1}^n |\xi_i|^q)^{\frac{1}{q}}$  for  $q \in (1, +\infty)$ .

Set  $W_F(r) := \{x \in \mathbb{R}^n \mid F^*(x) = r\}$ , where  $r \in \mathbb{R}^+$  and

$$F^*(x) = \sup_{\xi \neq 0} \frac{\langle x, \xi \rangle}{F(\xi)}, \quad \forall x \in \mathbb{R}^n.$$

The set  $W_F(r)$  is usually said to be the Wulff shape of  $F$ . One can easily see that if  $F(\xi) = |\xi|$ , then the corresponding Wulff shape  $W_F(r)$  is the standard sphere in  $\mathbb{R}^n$ . In many problems, the Wulff shape plays a role similarly to that of standard sphere.

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Roughly speaking, the Wulff shape of  $F$  is the "sphere" associated with the norm of  $F^*$  in  $\mathbb{R}^n$ , and  $r \in \mathbb{R}^+$  is the "radius" of the Wulff shape. Further details about Wulff shape can be found in [15, 19, 24, 33] and the references therein.

In this paper, we are interested in the strongly nonlinear elliptic operators

$$Qu := \sum_{i=1}^n \frac{\partial}{\partial x_i} (F^{p-1}(\nabla u) F_{\xi_i}(\nabla u)), \quad (1.1)$$

where  $1 < p < +\infty$ . It is easy to see that some special cases of (1.1) have been extensively discussed and well known to us. For  $F(\xi) = |\xi|$  and  $q = 2$ ,  $Q$  is the p-Laplace operator. For  $F(\xi) = (\sum_{i=1}^n |\xi_i|^p)^{\frac{1}{p}}$ ,  $Q$  is the pseudo-p-Laplace operator that was studied in [5]. The anisotropic elliptic operator, which was studied in [3, 11, 18, 36], is just the operator  $Q$  when  $p = 2$ .

We assume further that  $F(\xi) > 0$  for any  $\xi \neq 0$ , and  $\text{Hess}(F^p)$  is uniformly positive definite in  $\mathbb{R}^n \setminus \{0\}$  for  $1 < p < +\infty$ . For a connected bounded domain  $\Omega$  of  $\mathbb{R}^n$ , we consider the following overdetermined boundary value problem

$$(P) \begin{cases} Qu = -1, & \text{in } \Omega, & (P.1) \\ u = 0, & \text{on } \partial\Omega, & (P.2) \\ F(\nabla u) = c, & \text{on } \partial\Omega, & (P.3) \end{cases}$$

where  $c$  is a positive constant.

In [4], M. Belloni, V. Ferone and B. Kawohl obtained the symmetry of positive solutions to the problem (P.1)-(P.2) as follows.

**Theorem [4]** If  $\Omega \subset \mathbb{R}^n$  has the Wulff shape of  $F$  and  $p = n$ , the level sets of any solution to (P.1)-(P.2) have the Wulff shape of  $F$ .

In general, we know that the overdetermined problem (P) has no solution. From the Theorem [4], for  $p = n$ , it is not difficult to verify that problem (P) admits a solution if  $\Omega$  has the Wulff shape of  $F$ .

A natural question is that, for general  $1 < p < +\infty$ , whether the following statement holds true:

$$\text{if } (P) \text{ admits a solution, then } \Omega \text{ has the Wulff shape.} \quad (1.2)$$

Indeed, the problem of proving (1.2) is the main purpose of the overdetermined boundary value problem, which is an interesting problem while many authors used different methods to obtain a huge amount of literature. For  $F(\xi) = |\xi|$  and  $p = 2$ , the pioneering work [31] by Serrin proved that if problem (P) admits a solution  $u \in C^2(\Omega)$  then necessarily the domain  $\Omega$  is a ball by the moving planes method which is now well known to us. Following the method of Serrin, there are many works discussed vary types of overdetermined problems, e.g., [25, 30] for the exterior domain, [2, 23] for the ring-shape domain and also [28, 35] for lowly regularity assumption of  $\partial\Omega$ . For the same problem in [31], a totally different method to obtain the same result was discovered by Weinberger [37] whose proof is the first successful attempt to use an

associated P-function. In the spirit of Weinberger [37], using the P-function, integral identity and Alexandrov theorem (see [1] or [24]), lots of results of the overdetermined problems have obtained, we refer to [17, 20-22, 36]. There are also other alternative methods to discuss the overdetermined problems, for example, [6-11, 38].

The aim of this paper is to prove (1.2) is true for general  $F$  and  $1 < p < +\infty$ . Recently, for  $p = 2$ , G. Wang and C. Xia successfully proved (1.2) in [36]. Motivated by [36], we can also prove (1.2) for general  $1 < p < +\infty$ . The method in this paper is similar to [17] and [36], where the constant mean curvature of any level set of  $u$  is obtain by using the Pohozaev identity, the maximum principle on a suitable P-function and the relationship between the operator  $Q$  and the mean curvature of any level set of  $u$ .

Before we present our main result, let us give out the definition of the weak solution. A measurable function  $u$  is called a weak solution to problem (P) if  $u \in W_0^{1,p}(\Omega)$  and

$$\int_{\Omega} F^{p-1}(\nabla u) F_{\xi}(\nabla u) \cdot \nabla v \, dx = \int_{\Omega} v \, dx, \quad \forall v \in W_0^{1,p}(\Omega), \quad (1.3)$$

together with the condition  $F(\nabla u) = c$  on  $\partial\Omega$ . It was observed in [4] or [34] that for any weak solution of (1.3),  $u \in C^{1,\alpha}(\bar{\Omega})$  for some  $0 < \alpha < 1$ . Hence the condition  $F(\nabla u) = c$  on  $\partial\Omega$  is well defined.

The main result of this paper is as follows.

**Theorem 1.1** Let  $F : \mathbb{R}^n \rightarrow [0, +\infty)$  be a convex function of class  $C^4(\mathbb{R}^n \setminus \{0\})$ , which is even and positively homogeneous of degree 1. Assume further that  $F(\xi) > 0$  for any  $\xi \neq 0$ , and  $\text{Hess}(F^p)$  is positive definite in  $\mathbb{R}^n \setminus \{0\}$  for  $1 < p < +\infty$ . If the overdetermined boundary value problem (P) has a weak solution in a connected bounded domain  $\Omega \subset \mathbb{R}^n$  with  $\partial\Omega \in C^4$ . Then up to translation and scaling,  $\partial\Omega$  is of Wulff shape.

**Remark 1.1** As  $F(\xi) = |\xi|$ , Theorem 1.1 is the same as that if we take  $A(t) = t^{p-2}$  in [17]. Roughly speaking, Theorem 1.1 is the anisotropic version of partially results of [17]. Furthermore, if  $p = 2$ , then Theorem 1.1 is just the Theorem 1 of [36].

**Remark 1.2** From the assumption in Theorem 1.1, it is easy to see that  $F^p$  is strictly convex in  $\mathbb{R}^n$ , which will be used to prove the Pohozaev identity in Lemma 3.2.

**Remark 1.3** Since  $\text{Hess}(F^p)$  is positive definite in  $\mathbb{R}^n \setminus \{0\}$ , one can deduce that  $Q$  is a uniformly elliptic operator in any compact subsets of  $\Omega \setminus \mathcal{C}$ , where  $\mathcal{C} = \{x \in \Omega \mid \nabla u = 0\}$ . By virtue of  $F \in C^4(\mathbb{R}^n \setminus \{0\})$ , we have by the classical elliptic regularity theory that the weak solution  $u$  in fact belongs to  $C^3(\Omega \setminus \mathcal{C})$ , which implies that  $u_{ijk}$  is well defined in  $\Omega \setminus \mathcal{C}$  (see Lemma 3.1).

The outline of the paper is as follows. In Section 2, we give some preliminaries. In Section 3, the key result about the P-function which plays an important role to prove Theorem 1.1 is obtained. In Section 4, the main result of this paper is proved. Section 5 is our acknowledgements.

## 2 Some preliminaries

In what follows, we recall some useful results of the function  $F$  introduced in Section 1, the relationship between the Wulff shape of  $F$  and the F-mean curvature, the relationship between the operator  $Q$  and the F-mean curvature.

Firstly, we give some properties of the 1-homogeneous function  $F$ .

**Proposition 2.1.** (see [36]) Let  $F : \mathbb{R}^n \rightarrow [0, +\infty)$  be a 1-homogeneous function, then the following holds:

- (i)  $\sum_{i=1}^n F_{\xi_i}(\xi)\xi_i = F(\xi)$ ;
- (ii)  $\sum_{j=1}^n F_{\xi_i\xi_j}(\xi)\xi_j = 0$ , for any  $i = 1, 2, \dots, n$ ;
- (iii)  $F^*(F_\xi(\xi)) = 1$  and  $F(F_\xi^*(\xi)) = 1$ , for any  $\xi \neq 0$ .

We denote by  $H_F$  the F-mean curvature or anisotropic mean curvature. Further details about  $H_F$ , we refer to [12-14, 26, 36, 39].

Now we give a result concerning the the relationship between the Wulff shape of  $F$  and  $H_F$ , which shows that the Wulff shape can be characterized as a compact connected hypersurface with constant F-mean curvature.

**Proposition 2.2** (see [24]) Let  $X : M \rightarrow \mathbb{R}^n$  be an embedded compact hypersurface without boundary in the Euclidean space. If  $H_F(M)$  is constant, then up to translations and scaling,  $M$  is of Wulff shape.

**Remark 2.1** Proposition 2.2 is the anisotropic version of Alexandrov theorem in [1].

Let  $\Omega$  be a domain in  $\mathbb{R}^n$  and  $u \in C^2(\bar{\Omega} \setminus \{x \in \mathbb{R}^n \mid \nabla u = 0\})$ . We denote by  $S_t$  a level set of  $u$ , that is

$$S_t = \{x \in \bar{\Omega} \mid u(x) = t\}$$

and assume that  $S_t$  is smooth.

For simplicity of notation, in the following we use  $F_{\xi_i}(\nabla u) = F_i$ ,  $F_{\xi_i\xi_j}(\nabla u) = F_{ij}$ , and the Roman indices follow the summation convention.

For the relationship between the operator  $Q$  and the F-mean curvature of a level set, we have

**Lemma 2.1** Let  $H_F(S_t)$  be the F-mean curvature of the level set  $S_t$ . We have

$$Qu = F^{p-1}F_{ij}u_{ij} + (p-1)F^{p-2}F_iF_ju_{ij} = F^{p-1}H_F(S_t) + (p-1)F^{p-2}\frac{\partial^2 u}{\partial \nu_F^2} \quad (2.1)$$

for any  $x$  with  $u(x) = t$ , where  $\nu_F$  is the normal of  $\partial\Omega$  with respect to  $F$ , that is,  $\nu_F = F_\xi(\nu) = F_\xi(\nabla u)$ .

**Proof.** The proof is similar to that of Theorem 3 in [36], the detail proof is omitted.

To investigate the overdetermined boundary value problem  $(P)$ , we start in a natural way to study the symmetry of solution to problem  $(P)$  when  $\partial\Omega$  is the Wulff shape of  $F$ .

**Lemma 2.2** Let  $\Omega = \{x \in \mathbb{R}^n \mid F^*(x) < 1\}$ , i.e.,  $\partial\Omega$  is the Wulff shape of  $F$ , and

$$u(x) = \frac{(p-1)}{p} \left(\frac{1}{n}\right)^{1/(p-1)} (1 - (F^*(x))^{p/(p-1)}), \quad \forall x \in \Omega.$$

Then  $u$  satisfies problem (P) and  $c = \left(\frac{1}{n}\right)^{1/(p-1)}$ , the level sets  $S_t(u)$  of  $u$  is the Wulff shape of  $F$ .

**Proof.** The solution to problem (P.1)–(P.2) can be found by minimizing the functional

$$J(v) = \int_{\Omega} \frac{1}{p} F^p(\nabla v) - v \, dx. \quad (2.2)$$

Since  $F^p$  is strictly convex, the minimizer  $u$  of the functional  $J$  on  $W_0^{1,p}(\Omega)$  is unique. We denote by  $u^\sharp$  the convex symmetrization of  $u$ , by the Pólya-Szegő inequality (see [3]), we have

$$J(u) \geq J(u^\sharp).$$

Notice that the minimizer  $u$  is unique, then the following holds,

$$u(x) = u(F^*(x)). \quad (2.3)$$

So we need only consider functions of the form

$$v(x) = v(F^*(x)) = v(r), \quad (2.4)$$

where  $r = F^*(x)$ . In view of Proposition 2.1 (iii) and (2.4), we have

$$\begin{aligned} J(v) &= \int_0^1 n\omega_n \left( \frac{1}{p} F^p(v'(r)\nabla F^*(x)) - v(r) \right) r^{n-1} \, dr \\ &= \int_0^1 n\omega_n \left( \frac{1}{p} (v'(r))^p (F(\nabla F^*(x)))^p - v(r) \right) r^{n-1} \, dr \\ &= \int_0^1 n\omega_n \left( \frac{1}{p} (v'(r))^p - v(r) \right) r^{n-1} \, dr. \end{aligned} \quad (2.5)$$

The corresponding Euler equation of the one-dimensional problem (2.5) is

$$- (p(v')^{p-1} r^{n-1})' - p r^{n-1} = 0.$$

We immediately have

$$u(x) = \frac{(p-1)}{p} \left( \frac{1}{n} \right)^{1/(p-1)} \left( 1 - (F^*(x))^{p/(p-1)} \right).$$

Next, we only need to check that  $F(\nabla u) = c$  on  $\partial\Omega$ . Indeed, since Proposition 2.1 (iii) and  $F$  is positively homogeneous of degree 1, we have

$$\begin{aligned} F(\nabla u) &= F \left( \left( \frac{1}{n} \right)^{1/(p-1)} (F^*(x))^{1/(p-1)} \nabla F^*(x) \right) \\ &= \left( \frac{1}{n} \right)^{1/(p-1)} (F^*(x))^{1/(p-1)} F(\nabla F^*(x)) \\ &= \left( \frac{1}{n} \right)^{1/(p-1)} := c \end{aligned}$$

for any  $x \in \partial\Omega$ .

Furthermore, it is easy to find that for  $x \in S_t(u)$ ,

$$t = \frac{(p-1)}{p} \left(\frac{1}{n}\right)^{1/(p-1)} \left(1 - (F^*(x))^{p/(p-1)}\right),$$

which implies that  $F^*(x) = h(t)$  for  $x \in S_t(u)$ . This means that  $F^*(x)$  is constant on  $S_t(u)$ , then the level sets  $S_t(u)$  of  $u$  is the Wulff shape of  $F$ .

This completes the proof.

### 3 P-function

In this section, we consider the P-function defined by

$$P(x) := \frac{2(p-1)}{p} F^p(\nabla u) + \frac{2}{n} u(x), \quad x \in \Omega. \quad (3.1)$$

The motivation for studying the P-function defined by (3.1) comes from an investigation of the analogous one dimensional problem  $Qu = -1$  in  $\Omega \subset \mathbb{R}$ , that is

$$(F^{p-1}(u')F'(u'))' + 1 = 0. \quad (3.2)$$

Multiplying (3.2) by  $2u'$ , we have

$$2(p-1)F^{p-2}(u')F'(u')u''F'(u')u' + 2F^{p-1}(u')u''F''(u')u' + 2u' = 0 \quad (3.3)$$

By Proposition 2.1 (i)(ii), we have  $F'(u')u' = F(u')$  and  $F''(u')u' = 0$ , then

$$2(p-1)F^{p-1}(u')dF(u') + 2du = 0. \quad (3.4)$$

After integration of (3.4), we have

$$\frac{2(p-1)}{p} F^p(u') + 2u \equiv \text{constant}.$$

Following the arguments about P-function presented in [17, 20, 27, 32, 36], we can obtain the following result.

**Theorem 3.1** Let  $u$  be a weak solution to problem  $(P)$ . Then we have

$$P(x) \equiv \frac{2(p-1)}{p} c^p, \quad \forall x \in \Omega. \quad (3.5)$$

In order to complete the proof of Theorem 3.1, we need to prove two importance Lemmas: a maximum principle for P-function (see Lemma 3.1 below) and a Pohozaev-type integral identity of P-function (see Lemma 3.2 below).

**Lemma 3.1** Let  $u$  be a weak solution to the overdetermined boundary value problem  $(P)$ . Then  $P(x)$  attains its maximum on  $\partial\Omega$ . Moreover, if  $P(x)$  is not constant in  $\Omega$  and  $P(x)$  attains its maximum in a point  $\tilde{x} \in \Omega$ , then necessarily  $\nabla u(\tilde{x}) = 0$ .

**Proof .** Notice that  $P \in C^3(\Omega \setminus \mathcal{C})$  (see Remark 1.3), where  $\mathcal{C} = \{x \in \Omega \mid \nabla u = 0\}$ , the following calculations are all taken in  $\Omega \setminus \mathcal{C}$ .

Let

$$a_{ij}(\nabla u) = F^{p-1}F_{ij} + (p-1)F^{p-2}F_iF_j. \quad (3.6)$$

By some long but straightforward computations, one may obtain an elliptic inequality of second order

$$a_{ij}P_{ij} + L_iP_i \geq 0, \quad \text{in } \Omega \setminus \mathcal{C}, \quad (3.7)$$

where  $P_i = \partial P / \partial x_i$  and  $L_iP_i$  denote the terms with  $P_i$  (see (3.21)-(3.26) below).

Rewriting (3.7) as

$$F^{p-2}(FF_{ij} + (p-1)F_iF_j)P_{ij} + F^{p-2}F^{2-p}L_iP_i \geq 0, \quad \text{in } \Omega \setminus \mathcal{C}. \quad (3.8)$$

Set

$$\bar{a}_{ij} = FF_{ij} + (p-1)F_iF_j, \quad \text{in } \Omega \setminus \mathcal{C}, \quad (3.9)$$

$$\bar{L}_i = F^{2-p}L_i, \quad \text{in } \Omega \setminus \mathcal{C}. \quad (3.10)$$

Since  $F > 0$  in  $\Omega \setminus \mathcal{C}$ , by (3.9) and (3.10), we can deduce from (3.8) that

$$\bar{a}_{ij}P_{ij} + \bar{L}_iP_i \geq 0, \quad \text{in } \Omega \setminus \mathcal{C}. \quad (3.11)$$

Notice that  $\frac{1}{2}F^2$  is 2-homogeneous, we have

$$\left(\frac{1}{2}F^2\right)_{ij} = FF_{ij} + F_iF_j$$

is 0-homogeneous. Moreover,  $F_i$  is 0-homogeneous. Hence, we have

$$\bar{a}_{ij} = FF_{ij} + (p-1)F_iF_j = FF_{ij} + F_iF_j + (p-2)F_iF_j$$

is also 0-homogeneous, i.e., we can view  $\bar{a}_{ij}$  as a function on the compact set  $S^{n-1}$ . Since  $Hess(\frac{1}{p}F^p) = (a_{ij})$  is positive definite in  $\mathbb{R}^n \setminus \{0\}$  and  $\lambda(a_{ij}) = F^{p-2}\lambda(\bar{a}_{ij})$ , we know that  $(\bar{a}_{ij})$  is also positive definite in  $\mathbb{R}^n \setminus \{0\}$ . By the 0-homogeneous and positive definite of  $(\bar{a}_{ij})$ , it must have a uniformly positive lower(upper) bounds for all its eigenvalues. Then, there exist  $\bar{\lambda}, \bar{\Lambda} > 0$  such that

$$\bar{\lambda}|\zeta|^2 \leq \bar{a}_{ij}(\xi)\zeta_i\zeta_j \leq \bar{\Lambda}|\zeta|^2, \quad \text{for any } \xi \neq 0, \zeta \in \mathbb{R}^n.$$

Using a standard convolution argument, we can find a family of  $\{\bar{a}_{ij}^\epsilon\}$  in  $C^\infty(\mathbb{R}^n)$  such that

$$\bar{a}_{ij}^\epsilon \rightarrow \bar{a}_{ij} \text{ in } C_{loc}^1(\mathbb{R}^n \setminus \{0\}) \text{ as } \epsilon \rightarrow 0, \quad (3.12)$$

$$\frac{\lambda}{2}|\zeta|^2 \leq \bar{a}_{ij}^\epsilon(\xi)\zeta_i\zeta_j \leq \frac{3\Lambda}{2}|\zeta|^2, \quad \text{for any } \xi, \zeta \in \mathbb{R}^n. \quad (3.13)$$

Since  $u \in C^{1,\alpha}(\bar{\Omega})$ , we can choose a sequence  $\{u^\epsilon\}$  in  $C^\infty(\bar{\Omega})$  such that

$$u^\epsilon \rightarrow u \text{ in } C^{1,\alpha}(\bar{\Omega}) \text{ as } \epsilon \rightarrow 0. \quad (3.14)$$

Define

$$\bar{g}(x) = \begin{cases} \bar{a}_{ij}(\nabla u)P_{ij} + \bar{L}_iP_i, & \text{in } \Omega \setminus \mathcal{C}, \\ 0, & \text{in } \mathcal{C}. \end{cases}$$

Hence,  $\bar{g} \in C_{loc}(\bar{\Omega} \setminus \mathcal{C})$  for any  $p \geq 1$  and  $\bar{g} \geq 0$  in  $\Omega$ . It is easy to see that there exist vector-valued functions  $\{\bar{L}^\epsilon\} \subset C^0(\bar{\Omega}, \mathbb{R}^n)$  and  $\{\bar{g}^\epsilon\} \subset C^\infty(\Omega)$  such that

$$\bar{L}^\epsilon \rightarrow \bar{L} \text{ uniformly in any compact sets in } \Omega \setminus \mathcal{C}, \quad (3.15)$$

$$\bar{g}^\epsilon \rightarrow \bar{g} \text{ in } C_{loc}(\bar{\Omega} \setminus \mathcal{C}). \quad (3.16)$$

Consider now the solution  $P^\epsilon$  to

$$\begin{cases} \bar{a}_{ij}^\epsilon(\nabla u^\epsilon)P_{ij}^\epsilon + \bar{L}_i^\epsilon P_i^\epsilon = \bar{g}^\epsilon(x) \geq 0, & \text{in } \Omega, \\ P^\epsilon = \frac{2(p-1)}{p}c^p, & \text{on } \partial\Omega. \end{cases}$$

By the uniformly ellipticity (3.13) and regularity theory, the above approximate problem admits a solution  $P^\epsilon \in C^\infty(\bar{\Omega})$ . In view of this, the maximum principle implies that  $P^\epsilon$  attains its maximum on  $\partial\Omega$ , i.e.,

$$\max_{\bar{\Omega}} P^\epsilon(x) = \max_{\partial\Omega} P^\epsilon(x) = \max_{\bar{\Omega} \setminus U} P^\epsilon(x), \quad (3.17)$$

where  $U$  is any neighborhood of  $\mathcal{C}$ .

On the other hand, the  $L^p$  regularity theory shows that  $P^\epsilon$  is uniformly bounded in  $W_{loc}^{2,p}(\Omega \setminus \mathcal{C})$ . Hence, by the convergence in (3.12), (3.14)–(3.16) and the compact embedding theory, there exists a subsequence of  $\{P^\epsilon\}$  such that

$$P^\epsilon \rightarrow P \text{ in } C_{loc}^1(\bar{\Omega} \setminus \mathcal{C}).$$

Therefore, by taking  $\epsilon \rightarrow 0$  in (3.17), we obtain

$$\max_{\bar{\Omega} \setminus \mathcal{C}} P(x) = \max_{\partial\Omega} P(x). \quad (3.18)$$



Now, we show that  $P$  attains its maximum over  $\bar{\Omega}$  on  $\partial\Omega$ , that is

$$\max_{\bar{\Omega}} P(x) = \max_{\partial\Omega} P(x). \quad (3.19)$$

Suppose that there exists some point  $x_0 \in \Omega$  such that  $\max_{\bar{\Omega}} P(x) = P(x_0) > \max_{\partial\Omega} P(x)$ , by (3.18) we know that  $x_0$  must belong to the interior of  $\mathcal{C}$ . However, the interior of  $\mathcal{C}$  is empty which follows directly from equation  $Qu = -1$ . Thus we prove (3.19). Moreover, if  $P(x)$  is not constant in  $\Omega$  and  $P(x)$  attains its maximum in a point  $\tilde{x} \in \Omega$  then necessarily  $\nabla u(\tilde{x}) = 0$ . This completes the proof of Lemma 3.1.

The remaining part of this proof is need only to prove the elliptic inequality (3.7). We first calculate the derivatives up to the second order of  $P$ ,

$$P_i = 2(p-1)F^{p-1}F_k u_{ki} + \frac{2}{n}u_i, \quad (3.20)$$

$$P_{ij} = 2(p-1)^2 F^{p-2} F_l F_k u_{lj} u_{ki} + 2(p-1)F^{p-1} F_{kl} u_{lj} u_{ki} + 2(p-1)F^{p-1} F_k u_{kij} + \frac{2}{n}u_{ij}. \quad (3.21)$$

It follows from Proposition 2.1 (i) and (3.20) that

$$F_k u_{ki} = \frac{P_i}{2(p-1)F^{p-1}} - \frac{1}{n(p-1)F^{p-1}}u_i \quad (3.22)$$

and

$$F_i F_k u_{ki} = \frac{F_i}{2(p-1)F^{p-1}}P_i - \frac{1}{n(p-1)F^{p-2}}. \quad (3.23)$$

By (3.6),  $Qu = -1$  can be written as

$$a_{ij}u_{ij} = (F^{p-1}F_{ij} + (p-1)F^{p-2}F_i F_j) u_{ij} = -1. \quad (3.24)$$

From (3.23) and (3.24), we have

$$F^{p-1}F_{ij}u_{ij} = -\frac{F_i}{2F}P_i + \frac{1}{n} - 1. \quad (3.25)$$

By successive differentiation of (3.24) with respect to  $x_k$ , we obtain

$$a_{ij}u_{ijk} + 2(p-1)F^{p-2}F_{il}F_j u_{lk}u_{ij} + (p-1)F^{p-2}F_l F_{ij} u_{lk}u_{ij} + F^{p-1}F_{ijl}u_{lk}u_{ij} + (p-1)(p-2)F^{p-3}F_l F_i F_j u_{lk}u_{ij} = 0. \quad (3.26)$$

From Proposition 2.1 (ii), we have

$$F_{ij}u_j = 0 \quad (3.27)$$

for any  $i$ . Taking derivative of (3.27) with respect to  $x_i$  and summing, we obtain

$$F_{ij}u_{ij} + F_{ijl}u_{li}u_j = 0. \quad (3.28)$$

From (3.21), (3.24) and (3.26), we deduce that

$$\begin{aligned} a_{ij}P_{ij} &= a_{ij} \left( 2(p-1)^2 F^{p-2} F_l F_k u_{lj} u_{ki} + 2(p-1) F^{p-1} F_{kl} u_{lj} u_{ki} + 2(p-1) F^{p-1} F_k u_{kij} + \frac{2}{n} u_{ij} \right) \\ &= (F^{p-1} F_{ij} + (p-1) F^{p-2} F_i F_j) 2(p-1)^2 F^{p-2} F_l F_k u_{lj} u_{ki} \\ &\quad + (F^{p-1} F_{ij} + (p-1) F^{p-2} F_i F_j) 2(p-1) F^{p-1} F_{kl} u_{lj} u_{ki} \\ &\quad + 2(p-1) F^{p-1} F_k a_{ij} u_{kij} + \frac{2}{n} a_{ij} u_{ij} \\ &= 2(p-1)^3 F^{p-2} F^{p-2} F_i F_j F_l F_k u_{lj} u_{ki} + 2(p-1)^2 F^{p-2} F^{p-1} F_{ij} F_l F_k u_{lj} u_{ki} \\ &\quad + 2(p-1)^2 F^{p-2} F^{p-1} F_i F_j F_{kl} u_{lj} u_{ki} + 2(p-1) F^{p-1} F^{p-1} F_{ij} F_{kl} u_{lj} u_{ki} \\ &\quad + 2(p-1) F^{p-1} F_k (-2(p-1) F^{p-2} F_{il} F_j u_{lk} u_{ij} - (p-1) F^{p-2} F_l F_{ij} u_{lk} u_{ij} \\ &\quad - F^{p-1} F_{ijl} u_{lk} u_{ij} - (p-1)(p-2) F^{p-3} F_l F_i F_j u_{lk} u_{ij}) - \frac{2}{n} \\ &= 2(p-1) F^{p-1} F^{p-1} F_{ij} F_{lk} u_{lj} u_{ki} + 2(p-1)^2 F^{p-2} F^{p-2} F_i F_j F_l F_k u_{lk} u_{ij} \\ &\quad - 2(p-1)^2 F^{p-1} F^{p-2} F_k F_l F_{ij} u_{lk} u_{ij} - 2(p-1) F^{p-1} F^{p-1} F_{ijl} F_k u_{lk} u_{ij} - \frac{2}{n}. \end{aligned} \quad (3.29)$$

Denote

$$\begin{aligned} I_1 &= 2(p-1) F^{p-1} F^{p-1} F_{ij} F_{lk} u_{lj} u_{ki}, \\ I_2 &= 2(p-1)^2 F^{p-2} F^{p-2} F_i F_j F_l F_k u_{lk} u_{ij}, \\ I_3 &= -2(p-1)^2 F^{p-1} F^{p-2} F_k F_l F_{ij} u_{lk} u_{ij}, \\ I_4 &= -2(p-1) F^{p-1} F^{p-1} F_{ijl} F_k u_{lk} u_{ij}. \end{aligned}$$

Then, (3.29) is rewritten as

$$a_{ij}P_{ij} = I_1 + I_2 + I_3 + I_4 - \frac{2}{n}. \quad (3.30)$$

The term  $I_2$  can be computed as follows,

$$\begin{aligned} I_2 &= 2(p-1)^2 F^{p-2} F^{p-2} \left( \frac{F_i}{2(p-1) F^{p-1}} P_i - \frac{1}{n(p-1) F^{p-2}} \right)^2 \\ &= 2 \left( \frac{F_i}{2F} P_i - \frac{1}{n} \right)^2 = \frac{2}{n^2} + \text{terms of } P_i, \end{aligned} \quad (3.31)$$

where we used (3.23).

The term  $I_3$  can be computed as follows,

$$\begin{aligned} I_3 &= -2(p-1)^2 F^{p-2} \left( -\frac{F_i}{2F} P_i + \frac{1}{n} - 1 \right) \left( \frac{F_i}{2(p-1)F^{p-1}} P_i - \frac{1}{n(p-1)F^{p-2}} \right) \\ &= (p-1) \frac{2}{n} \left( \frac{1}{n} - 1 \right) + \text{terms of } P_i, \end{aligned} \quad (3.32)$$

where we have used (3.23) and (3.25).

The term  $I_4$  can be computed as follows,

$$\begin{aligned} I_4 &= -2(p-1)F^{p-1}F^{p-1}F_{ijl} \left( \frac{P_l}{2(p-1)F^{p-1}} - \frac{1}{n(p-1)F^{p-1}}u_l \right) u_{ij} \\ &= -F^{p-2}F_lF_{jil}u_lu_{ij}P_l + \frac{2}{n}F^{p-1}F_{jil}u_lu_{ij} \\ &= F^{p-2}F_lF_{li}u_lP_l - \frac{2}{n}F^{p-1}F_{li}u_{li} \\ &= F_l \left( -\frac{F_i}{2F^2}P_i + \frac{1}{nF} - \frac{1}{F} \right) P_l - \frac{2}{n} \left( -\frac{F_i}{2F}P_i + \frac{1}{n} - 1 \right) \\ &= -\frac{2}{n} \left( \frac{1}{n} - 1 \right) + \text{terms of } P_i, \end{aligned} \quad (3.33)$$

where we have used Proposition 2.1 (i), (3.22), (3.28) and (3.25).

The term  $I_1$  can be computed as follows,

$$\begin{aligned} I_1 &= 2(p-1) \left( a_{ij} - (p-1)F^{p-2}F_iF_j \right) u_{lj} \left( a_{lk} - (p-1)F^{p-2}F_lF_k \right) u_{ki} \\ &= 2(p-1)a_{ij}u_{lj}a_{lk}u_{ki} + 2(p-1)^3F^{p-2}F^{p-2}F_iF_ku_{ki}F_lF_ju_{lj} - 4(p-1)^2F^{p-2}F_iu_{ki}F_ju_{lj}a_{lk} \\ &= 2(p-1)a_{ij}u_{lj}a_{lk}u_{ki} + 2(p-1)^3(F^{p-2})^2 \left( \frac{F_i}{2(p-1)F^{p-1}}P_i - \frac{1}{n(p-1)F^{p-2}} \right)^2 \\ &\quad - 4(p-1)^2F^{p-2} \left( \frac{P_l}{2(p-1)F^{p-1}} - \frac{1}{n(p-1)F^{p-1}}u_l \right) \\ &\quad \cdot \left( \frac{P_k}{2(p-1)F^{p-1}} - \frac{1}{n(p-1)F^{p-1}}u_k \right) (F^{p-1}F_{lk} + (p-1)F^{p-2}F_lF_k) \\ &= 2(p-1)a_{ij}u_{lj}a_{lk}u_{ki} + (p-1)\frac{2}{n^2} + \text{terms of } P_i \\ &\quad - 4 \left( \frac{1}{4F}F_{kl}P_lP_k - \frac{1}{4nF}F_{kl}u_kP_l + \frac{1}{n^2F}F_{kl}u_ku_l + (p-1)\frac{1}{4F^2}F_kF_lP_lP_k \right. \\ &\quad \left. - (p-1)\frac{1}{4nF^2}F_ku_kF_lP_l + (p-1)\frac{1}{n^2F^2}F_ku_kF_lu_l \right) \\ &= 2(p-1)a_{ij}u_{lj}a_{lk}u_{ki} + (p-1)\frac{2}{n^2} - (p-1)\frac{4}{n^2} + \text{terms of } P_i \\ &= 2(p-1)a_{ij}u_{lj}a_{lk}u_{ki} - (p-1)\frac{2}{n^2} + \text{terms of } P_i, \end{aligned} \quad (3.34)$$

where we have used Proposition 2.1 (i) (ii) and (3.22)-(3.24) .

Combining (3.30) with (3.31)-(3.34), we obtain

$$a_{ij}P_{ij} = 2(p-1)a_{ij}u_{lj}a_{lk}u_{ki} + \text{terms of } P_i - (p-1)\frac{2}{n^2} + \frac{2}{n^2} + (p-1)\frac{2}{n}\left(\frac{1}{n}-1\right) - \frac{2}{n}\left(\frac{1}{n}-1\right) - \frac{2}{n}. \quad (3.35)$$

Notice that

$$-(p-1)\frac{2}{n^2} + \frac{2}{n^2} + (p-1)\frac{2}{n}\left(\frac{1}{n}-1\right) - \frac{2}{n}\left(\frac{1}{n}-1\right) - \frac{2}{n} = -(p-1)\frac{2}{n},$$

then (3.35) yields

$$a_{ij}P_{ij} + L_iP_i = (p-1)\left(2a_{ij}u_{lj}a_{lk}u_{ki} - \frac{2}{n}\right), \quad (3.36)$$

where  $L_iP_i$  denote the terms of  $P_i$  in (3.35).

Now since  $(a_{ij})$  is a symmetric and positive definite matrix in  $\Omega \setminus \mathcal{C}$ , for a fixed point  $x$ , we can choose coordinates around  $x$  such that

$$a_{ij}(x) = \lambda_i\delta_{ij},$$

with  $\lambda_i > 0$  for any  $i$ . Thus (3.24) is rewritten as

$$\lambda_i u_{ii} = -1. \quad (3.37)$$

From (3.36) and (3.37), we obtain

$$\begin{aligned} a_{ij}P_{ij} + L_iP_i &= (p-1)\left(2\lambda_i\lambda_j u_{ij}^2 - \frac{2}{n}\right) \\ &\geq (p-1)(2\lambda_i^2 u_{ii}^2 - \frac{2}{n}) \\ &\geq (p-1)\frac{2}{n}(\lambda_i^2 u_{ii}^2 - 1) = 0, \end{aligned} \quad (3.38)$$

which proves that (3.7) holds.

**Lemma 3.2** Let  $u$  be a weak solution to the overdetermined boundary value problem (P). Then  $P(x)$  satisfies the following identity

$$\int_{\Omega} P(x) dx = \frac{2(p-1)}{p} c^p |\Omega|, \quad (3.39)$$

where  $|\Omega|$  is the  $n$ -dimensional volume of  $\Omega$ .

**Proof.** For a weak solution  $u$ , which actually belongs to  $C^{1,\alpha}(\Omega)$  (see [4]). We first prove that the following integral identity

$$\int_{\partial\Omega} F^p(\nabla u)\langle x, \nu \rangle d\sigma = \int_{\Omega} \frac{(p-n)}{p-1} F^p(\nabla u) - \frac{p}{p-1} \langle x, \nabla u \rangle dx \quad (3.40)$$

holds for  $u \in C^1(\Omega)$ . In order to obtain (3.40), we need a Pohozaev-type integral identity in [16] (see also [29] for  $u \in C^2$ ).

Indeed, since  $u \in C^1(\Omega)$  and  $\frac{1}{p}F^p(\nabla u)$  is strictly convex (see Remark 1.2), choose  $h = x$  and  $a = 1$  in Theorem 2 [16], by Theorem 2 [16] we have following calculate

$$\begin{aligned} & \int_{\partial\Omega} \frac{1}{p} F^p(\nabla u) \langle x, \nu \rangle d\sigma - \int_{\partial\Omega} F^{p-1}(\nabla u) F_\xi(\nabla u) \nabla u \langle x, \nu \rangle d\sigma \\ &= \int_{\Omega} n \frac{1}{p} F^p(\nabla u) dx - \int_{\Omega} \nabla u F^{p-1}(\nabla u) F_\xi(\nabla u) dx \\ &\quad - \int_{\Omega} F^{p-1}(\nabla u) F_\xi(\nabla u) \nabla u dx + \int_{\Omega} u dx + \int_{\Omega} \langle x, \nabla u \rangle dx \\ &= \int_{\Omega} \frac{(n-p)}{p} F^p(\nabla u) dx + \int_{\Omega} \langle x, \nabla u \rangle dx, \end{aligned} \tag{3.41}$$

where we have used Proposition 2.1 (i) and one of the following identities

$$\int_{\partial\Omega} \langle x, \nu \rangle d\sigma = n|\Omega|, \tag{3.42}$$

$$\int_{\Omega} \langle x, \nabla u \rangle dx = -n \int_{\Omega} u dx, \tag{3.43}$$

$$\int_{\Omega} F^p(\nabla u) dx = \int_{\Omega} u dx, \tag{3.44}$$

which obtained from Green formula or integration by part.

Multiplying (3.41) by  $\frac{p}{1-p}$ , we obtain (3.40).

It follows from (P.3), (3.40) and (3.42)-(3.44) that

$$\begin{aligned} c^p n |\Omega| &= \int_{\Omega} \frac{(p-n)}{p-1} F^p(\nabla u) + \frac{np}{p-1} u dx \\ &= \int_{\Omega} n F^p(\nabla u) + \frac{p}{p-1} u dx + \int_{\Omega} \left( \frac{(p-n)}{p-1} - n \right) F^p(\nabla u) + \frac{(n-1)p}{p-1} u dx \\ &= \int_{\Omega} n F^p(\nabla u) + \frac{p}{p-1} u dx = \frac{np}{2(p-1)} \int_{\Omega} \frac{2(p-1)}{p} F^p(\nabla u) + \frac{2}{n} u dx \\ &= \frac{np}{2(p-1)} \int_{\Omega} P(x) dx, \end{aligned} \tag{3.45}$$

which yields (3.39), i.e.,

$$\int_{\Omega} P(x) = \frac{2(p-1)}{p} c^p |\Omega|.$$

**Proof of Theorem 3.1.** From Lemma 3.1 and Lemma 3.2, we immediately obtain Theorem 3.1.

## 4 Proof of Theorem 1.1

We first claim that  $\nu = \frac{\nabla u}{|\nabla u|}$  is well-defined on the open set  $U := \{x \in \Omega \mid 0 < u(x) < \max_{\bar{\Omega}} u\}$ , which provided that  $\nabla u$  vanish only at points where  $u$  attains its maximum in  $\Omega$  and  $u > 0$  in  $\Omega$ . Indeed, if  $\nabla u(x_0) = 0$ , then  $F(\nabla u(x_0)) = 0$ , by (3.2), we have  $u(x_0) = \frac{n(p-1)}{p}c^p$  and  $\max_{\bar{\Omega}} u(x) = \max_{\bar{\Omega}} \frac{n(p-1)}{p}(c^p - F^p) = \frac{n(p-1)}{p}c^p$ , so  $u(x_0) = \max_{\bar{\Omega}} u(x)$ . Moreover, if  $u(x_0) = \inf_{\bar{\Omega}} u(x) \leq 0$ , by (3.2), we have  $F(\nabla u(x_0)) \geq c > 0$  which yields that  $\nabla u(x_0) \neq 0$ , a contradiction, so  $u > 0$  in  $\Omega$ .

Let  $\nu_F := F_\xi(\nu) = F_\xi(\nabla u)$  on  $U$ , we deduce that

$$\frac{\partial u}{\partial \nu_F} = \nabla u F_\xi(\nabla u) = F(\nabla u) = \left( c^p - \frac{p}{n(p-1)} u \right)^{1/p} := g(u) \quad (4.1)$$

and

$$\frac{\partial^2 u}{\partial \nu_F^2} = \nabla F(\nabla u) F_\xi(\nabla u) = F_i(\nabla u) F_j(\nabla u) u_{ij}. \quad (4.2)$$

On one hand, (4.1) yields that

$$\frac{\partial}{\partial \nu_F} \left( \frac{\partial u}{\partial \nu_F} \right)^2 = \frac{\partial}{\partial \nu_F} g^2(u) = 2g(u)g'(u) \frac{\partial u}{\partial \nu_F}, \quad (4.3)$$

and on the other hand, we have

$$\frac{\partial}{\partial \nu_F} \left( \frac{\partial u}{\partial \nu_F} \right)^2 = 2 \frac{\partial u}{\partial \nu_F} \frac{\partial^2 u}{\partial \nu_F^2}. \quad (4.4)$$

From (4.2)-(4.4), we obtain

$$F_i(\nabla u) F_j(\nabla u) u_{ij} = g(u)g'(u). \quad (4.5)$$

We denote by  $H_F(S_t)$  the F-mean curvature of the level set  $S_t$ ,  $t \in (0, T)$ ,  $T = \max_{\bar{\Omega}} u$ . So (4.1), (4.5) and Lemma 2.1 lead to

$$\begin{aligned} H_F(S_t) &= \frac{1}{F^{p-1}(\nabla u)} \left( Qu - (p-1)F^{p-2}(\nabla u) \frac{\partial^2 u}{\partial \nu_F^2} \right) \\ &= \frac{1}{g^{p-1}(u)} (-1 - (p-1)g^{p-2}(u)g(u)g'(u)) := h(u). \end{aligned} \quad (4.6)$$

The above equality just shows that every level set of  $u$  at height  $t$  between zero and  $T$  is a hypersurface of constant F-mean curvature. By Proposition 2.2, each connected component of it must be of Wulff shape, up to translations.

By the same argument in [17] or [36], we may prove that  $S_t$  is simply connected for any  $t \in (0, T)$ . Indeed, if  $\Gamma_{\check{t}}$  and  $\tilde{\Gamma}_{\check{t}}$  are two connected components of a particular level set  $S_{\check{t}}$  ( $\check{t} \in (0, T)$ ), then one of them must be enclosed in another and both them are of Wulff shape with the same "radius". That is,  $S_{\check{t}}$  contain two nested Wulff shapes of equal "radius", a contradiction. Hence,  $S_{\check{t}}$  has only one component, i.e., is simply connected.

Therefore,  $S_t$  is of Wulff shape for any  $t \in (0, T)$ , and  $\partial\Omega = S_0$  is also of Wulff shape. This completes the proof of Theorem 1.1.

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## References

- [1] A. D. Aleksandrov, Uniqueness theorems for surfaces in the large V, *Vestnik Leningrad Univ.* 13(1958), 5-8.
- [2] G. Alessandrini, A symmetry theorem for condensers, *Math. Methods Appl. Sci.* 15(1992), 315-320.
- [3] A. Alvino, V. Ferone, G. Trombetti, P. L. Lions, Convex symmetrization and applications, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 14(1997), 275-293.
- [4] M. Belloni, V. Ferone, B. Kawohl, Isoperimetric inequalities, Wulff shape and related questions for strongly nonlinear elliptic operators, *Z. Angew. Math. Phys.* 54(2003), 771-783.
- [5] M. Belloni, B. Kawohl, The pseudo-p-Laplace eigenvalue problem and viscosity solutions as  $p \rightarrow \infty$ , *ESAIM Control Optim. Calc. Var.* 10(2004), 25-52.
- [6] B. Brandolini, C. Nitsch, P. Salani, C. Trombetti, On the stability of the Serrin problem, *J. Differ. Equ.* 245(6)(2008), 1566-1583.
- [7] B. Brandolini, C. Nitsch, P. Salani, C. Trombetti, Serrin type overdetermined problems: an alternative proof, *Arch. Ration. Mech. Anal.* 190(2008), 267-280.
- [8] B. Brandolini, C. Nitsch, P. Salani, C. Trombetti, Stability of radial symmetry for a Monge-Ampère overdetermined problem, *Annali di Matematica.* 188(2009), 445-453.
- [9] F. Brock, A. Henrot, A symmetry result for an overdetermined elliptic problem using continuous rearrangement and domain derivative, *Rend. Circ. Mat. Palermo.* 51(2002), 375-390.
- [10] M. Choulli, A. Henrot, Use of the domain derivative to prove symmetry results in partial differential equations, *Math. Nachr.* 192(1998), 91-103.
- [11] A. Cianchi, P. Salani, Overdetermined anisotropic elliptic problems, *Math. Ann.* 345(2009), 859-881.
- [12] U. Clarenz, Enclosure theorems for extremals of elliptic parametric functionals, *Calc. Var. Partial Differ. Equ.* 15(2002), 313-324.
- [13] U. Clarenz, The Wulff-shape minimizes an anisotropic Willmore functional, *Interfaces and Free Boundaries* 6(2004), 351-359.
- [14] U. Clarenz, H. Von Der Mosel, On surfaces of prescribed F-mean curvature, *Pac. J. Math.* 213(2004), 15-36.

- [15] B. Dacorogna, C. E. Pfister, Wulff theorem and best constant in Sobolev inequality, *J. Math. Pures Appl.* 71(1992), 97-118.
- [16] M. Degiovanni, A. Musesti, M. Squassina, On the regularity of solution in the Pucci-Serrin identity, *Calc. Var.* 18(2003), 317-334.
- [17] A. Farina, B. Kawohl, Remarks on an overdetermined boundary value problem, *Calc. Var.* 31(2008), 351-357.
- [18] V. Ferone, B. Kawohl, Remarks on a Finsler-Laplacian, *Proc. Am. Math. Soc.* 137(2009), 247-253.
- [19] I. Fonseca, S. Müller, A uniqueness proof for the Wulff theorem, *Proc. Roy. Soc. Edinburgh Sect. A* 119(1991), 125-136.
- [20] I. Fragala, F. Gazzola, B. Kawohl, Overdetermined boundary value problems with possibly degenerate ellipticity: a geometric approach, *Math. Zeitschr.* 254(2006), 117-132.
- [21] N. Garofalo, J. L. Lewis, A symmetry result related to some overdetermined boundary value problems, *Am. J. Math.* 111(1989), 9-33.
- [22] N. Garofalo, E. Sartori, Symmetry in exterior boundary value problems for quasilinear elliptic equations via blow-up and a priori estimates, *Adv. Diff. Eq.* 4(1999), 137-161.
- [23] A. Greco, Radial symmetry and uniqueness for an overdetermined problem, *Math. Methods Appl. Sci.* 24(2001), 103-115.
- [24] Y. J. He, H. Z. Li, H. Ma, J. Q. Ge, Compact embedded hypersurfaces with constant higher order anisotropic mean curvature, *Indiana Univ. Math. J.* 58(2009), 853-868.
- [25] O. Mendez, W. Reichel, Electrostatic characterization of spheres, *Forum Math.* 12(2000), 223-245.
- [26] B. Palmer, Stability of the Wulff shape, *Proc. Amer. Math. Soc.* 126(1998), 3661-3667.
- [27] L. E. Payne, G. A. Philippin, Some maximum principles for nonlinear elliptic equations in divergence form with applications to capillary surfaces and to surfaces of constant mean curvature, *Nonlinear Anal.* 3(1979), 193-211.
- [28] J. Prajapat, Serrin's result for domains with a corner or cusp, *Duke Math. J.* 91(1998), 29-31.
- [29] P. Pucci, J. Serrin, A general variational identity, *Indiana Univ. Math. J.* 35(3)(1986), 681-703.
- [30] W. Reichel, Radial symmetry for elliptic boundary value problems on exterior domains, *Arch. Ration. Mech. Anal.* 137(1997), 381-394.
- [31] J. Serrin, A symmetry problem in potential theory, *Arch. Ration. Mech. Anal.* 43(1971), 304-318.



- [32] R. Sperb, Maximum Principles and Applications, Mathematics in Science and Engineering, Vol. 157. Academic Press, New York/London. 1981.
- [33] J. Taylor, Crystalline variational problems, Bull. Am. Math. Soc. 84(1978), 568-588.
- [34] P. Tolksdorf, Regularity for a more general class of quasilinear elliptic equations, J. Diff. Equ. 51(1984), 126-150.
- [35] A. L. Vogel, Symmetry and regularity for general regions having solutions to certain overdetermined boundary value problems, Atti Sem. Mat. Fis. Univ. Modena. 40(1992), 443-484.
- [36] G. Wang, C. Xia, A characterization of the Wulff shape by an overdetermined anisotropic PDE, Arch. Ration. Mech. Anal. 199(1)(2011), 99-115..
- [37] H. Weinberger, Remark on the preceding paper of Serrin, Arch. Ration. Mech. Anal. 43(1971), 319-320.
- [38] N. B. Willms, M. Chamberlandt, G. M. L. Gladwell, A duality theorem for an overdetermined eigenvalue problem, Z. Angew. Math. Phys. 46(1995), 623-629.
- [39] S. Winklmann, A note on the stability of the Wulff shape, Arch. Math. 87(2006), 272-279.

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