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Controllability results for weakly blowing up reaction-diffusion system*

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Abstract: In this paper, we consider the controllability of a general reaction-diffusion system with homogeneous Dirichlet boundary conditions. We prove the exact controllability to the trajectories and the approximate controllability of the system which contains certain superlinear nonlinearities. The Kakutani fixed point theorem, global Carleman estimates, and the regularity argument of the parabolic system are used.

Keywords: Controllability, Fixed points, Reaction-diffusion systems.

1 Introduction and main results

In this paper, we consider the following reaction-diffusion systems

$$\begin{aligned} u_t &= \Delta u + f_1(u,v) + \chi_\omega f, \\ v_t &= \Delta v + f_2(u,v) + \chi_\omega g, \end{aligned} \qquad (x,t) \in \Omega \times (0,T_0), \tag{1.1}$$

with initial and boundary conditions

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \qquad x \in \Omega, \tag{1.2}$$

$$u(x,t) = v(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T_0), \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^N, 1 \le N < 6$, with the smooth boundary $\partial\Omega, Q_{T_0} = \Omega \times (0,T_0)$, $f_i \in C^1(\mathbb{R}\times\mathbb{R})$ with $f_i(0,0) = 0, i = 1,2$, and f, g are control functions acting on the nonempty open set $\omega \subset \Omega$. χ_ω denotes the characteristic function of the set ω.

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Throughout this paper, the symbols $W^{k,p}(\Omega)$ and $W_p^{2,1}(Q_{T_0}), 1 \le p < \infty, k$ integer, denote the usual Sobolev spaces and

$$\widetilde{W}_{q_N}^{1,1}(Q_{T_0}) = \{\ y \in W_{q_N}^{1,1}(Q_{T_0});\ y \text{ vanishes for } (x,t) \in \partial\Omega \times (0,T_0) \text{ and } t = 0\}.$$

We assume that

(H1) q_N is a positive integer. $q_N \in (2,\infty)$ when $N = 1,2$, and $q_N \in (\frac{N+2}{2}, \frac{2(N+2)}{N-2})$ when $N = 3,4,5$.

With the assumption (H1), we have the following embeddings (see [15], p.61):

$$W_{q_N}^{2,1}(Q_{T_0}) \hookrightarrow C^{\alpha_0,\alpha_0/2}(Q_{T_0}), \quad W^{2(1-1/q_N),q_N}(\Omega) \hookrightarrow L^\infty(\Omega) \text{ with } 0 < \alpha_0 < 2 - \frac{N+2}{q_N}.$$

Due to the system (1.1)–(1.3), we are interested in the generalized solution in the following sense.

Definition 1.1 *(u,v) is said to be a generalized solution of* (1.1)–(1.3) *if it satisfies the following conditions:*
(i) $(u,v) \in (L^{q_N}(0,T_0; W_0^{1,q_N}(\Omega)) \cap W_{q_N}^{2,1}(Q_{T_0}))^2$;
(ii) (u,v) *satisfies* (1.1) a.e. *in* Q_T *and* $(u-u_0, v-v_0) \in (\widetilde{W}_{q_N}^{1,1}(Q_{T_0}))^2$.

Let us consider local existence of the generalized solutions. We can do some deformation for (1.1)–(1.3) as

$$\begin{aligned} &u_t = \Delta u + f_1(u,v) - f_1(0,0) + \chi_\omega f, \\ &v_t = \Delta v + f_2(u,v) - f_2(0,0) + \chi_\omega g, \end{aligned} \qquad (x,t) \in \Omega \times (0,T_0), \tag{1.4}$$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \qquad x \in \Omega, \tag{1.5}$$

$$u(x,t) = v(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T_0). \tag{1.6}$$

Rewrite (1.4)–(1.6) in the following form

$$\begin{aligned} &u_t = \Delta u + F_1(u,v;0,0)u + F_2(u,v;0,0)v + \chi_\omega f, \\ &v_t = \Delta v + F_3(u,v;0,0)u + F_4(u,v;0,0)v + \chi_\omega g, \end{aligned} \qquad (x,t) \in \Omega \times (0,T_0), \tag{1.7}$$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \qquad x \in \Omega, \tag{1.8}$$

$$u(x,t) = v(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T_0), \tag{1.9}$$

where

$$F_1(u,v;0,0) = \int_0^1 \frac{\partial f_1}{\partial u}(\theta u, \theta v) d\theta,$$

$$F_2(u,v;0,0) = \int_0^1 \frac{\partial f_1}{\partial v}(\theta u, \theta v) d\theta,$$

$$F_3(u,v;0,0) = \int_0^1 \frac{\partial f_2}{\partial u}(\theta u, \theta v) d\theta,$$

$$F_4(u,v;0,0) = \int_0^1 \frac{\partial f_2}{\partial v}(\theta u, \theta v) d\theta,$$

and assume that

(H2) $F_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfy

$$\lim_{|(s,\mu)|\to\infty} \frac{|F_i(s,\mu;0,0)|}{\ln^{3/2}(1+|s|+|\mu|)} = 0, \qquad i = 1,2,3,4.$$

Then we have the following local existence result for the solution of (1.1)–(1.3).

Theorem 1.1 *Let* (H1) *and* (H2) *be satisfied. Then for any* $f, g \in L^{q_N}(Q_{T_0})$, $(u_0, v_0) \in (W_0^{1,q_N}(\Omega) \cap W^{2,q_N}(\Omega))^2$, *there exists* $T_1 \in (0, T_0)$ *such that* (1.1)–(1.3) *has a generalized solution* (u, v) *in* Q_{T_1}.

The result above may not be new, but it is difficult to find its proof. For the completeness of the text, we will give the proof in the Appendix at the end of this paper.

Let us now analyze the controllability property. Consider the solution of the problem (without control functions)

$$\begin{aligned} u_t^* &= \Delta u^* + f_1(u^*, v^*), \\ v_t^* &= \Delta v^* + f_2(u^*, v^*), \end{aligned} \qquad (x,t) \in \Omega \times (0,T), \tag{1.10}$$

with the initial and boundary conditions

$$u^*(x,0) = u_0^*(x), \quad v^*(x,0) = v_0^*(x), \qquad x \in \Omega, \tag{1.11}$$

$$u^*(x,t) = v^*(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T). \tag{1.12}$$

Let (u^*, v^*) be an arbitrary bounded trajectory of (1.10)–(1.12) globally defined on $[0,T], T < T^*$, where $T^* \in (0, \infty]$ is the maximal existence time, corresponding to the data $u_0^*, v_0^* \in W_0^{1,q_N}(\Omega) \cap W^{2,q_N}(\Omega)$. Setting $u = \overline{u} - u^*, \quad v = \overline{v} - v^*$, where $(\overline{u}, \overline{v}, f, g)$ satisfies (1.1)–(1.3), we obtain

$$\begin{aligned} u_t &= \Delta u + f_1(u+u^*, v+v^*) - f_1(u^*, v^*) + \chi_\omega f, \\ v_t &= \Delta v + f_2(u+u^*, v+v^*) - f_2(u^*, v^*) + \chi_\omega g, \end{aligned} \qquad (x,t) \in \Omega \times (0,T), \tag{1.13}$$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \qquad x \in \Omega, \tag{1.14}$$

$$u(x,t) = v(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T), \tag{1.15}$$

where $u_0(x) = \overline{u}(x,0) - u_0^*(x), v_0(x) = \overline{v}(x,0) - v_0^*(x)$. Then the system (1.13)–(1.15) can be rewritten as follows:

$$\begin{aligned} u_t &= \Delta u + F_1(u,v;u^*,v^*)u + F_2(u,v;u^*,v^*)v + \chi_\omega f, \\ v_t &= \Delta v + F_3(u,v;u^*,v^*)u + F_4(u,v;u^*,v^*)v + \chi_\omega g, \end{aligned} \qquad (x,t) \in \Omega \times (0,T), \tag{1.16}$$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \qquad x \in \Omega, \tag{1.17}$$

$$u(x,t) = v(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T), \tag{1.18}$$

where

$$F_1(u,v;u^*,v^*) = \int_0^1 \frac{\partial f_1}{\partial u}(\theta u + u^*, \theta v + v^*) d\theta,$$
$$F_2(u,v;u^*,v^*) = \int_0^1 \frac{\partial f_1}{\partial v}(\theta u + u^*, \theta v + v^*) d\theta,$$
$$F_3(u,v;u^*,v^*) = \int_0^1 \frac{\partial f_2}{\partial u}(\theta u + u^*, \theta v + v^*) d\theta,$$
$$F_4(u,v;u^*,v^*) = \int_0^1 \frac{\partial f_2}{\partial v}(\theta u + u^*, \theta v + v^*) d\theta.$$

In this paper, we assume that
(H3) $F_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfy

$$\lim_{|(s,\mu)|\to\infty} \frac{|F_i(s,\mu;s_0,\mu_0)|}{\ln^{3/2}(1+|s|+|\mu|)} = 0, \qquad i = 1,2,3,4,$$

uniformly in $(s_0,\mu_0) \in K \times K$, with $K \subset \mathbb{R}$ being compact.

Definition 1.2 *The system* (1.1)–(1.3) *is said to be exactly controllable to the trajectories at time* $T < T^*$ *if for any initial data* $(u_0,v_0) \in (W_0^{1,q_N}(\Omega) \cap W^{2,q_N}(\Omega))^2$, *there exist control functions* $f,g \in L^{q_N}(Q_T)$ *such that the corresponding solution* (u,v) *of* (1.1)–(1.3) *is also defined on* $[0,T]$ *and satisfies*

$$(u(\cdot,T),v(\cdot,T)) = (u^*(\cdot,T),v^*(\cdot,T)) \quad \text{a.e. in } \Omega. \tag{1.19}$$

Definition 1.3 *The system* (1.1)–(1.3) *is approximately controllable at time* T *if for any* $T > 0$, *initial data* $(u_0,v_0) \in (W_0^{1,q_N}(\Omega) \cap W^{2,q_N}(\Omega))^2$, $u_d, v_d \in L^2(\Omega)$ *and* $\varepsilon > 0$, *there exist* $f,g \in L^{q_N}(Q_T)$ *such that the corresponding solution* (u,v) *of* (1.1)–(1.3) *satisfies*

$$\|u(\cdot,T) - u_d\|_{L^2(\Omega)} \le \varepsilon \ \text{ and } \ \|v(\cdot,T) - v_d\|_{L^2(\Omega)} \le \varepsilon. \tag{1.20}$$

Remark 1.1 *Clearly, the exact controllability to the trajectories of* (1.1)–(1.3) *is equivalent to the exact null controllability of* (1.16)–(1.18). *Therefore, we only need to prove the exact null controllability of* (1.16)–(1.18).

In recent years, the controllabilities of the nonlinear parabolic systems have been studied by many authors (see [3]–[9] and the references therein). For reaction-diffusion systems, Anita and Barbu [3] have considered the local null controllability with $f_i(x,u,v) = \alpha_i a(x)uv$, $i = 1,2$, where the α_i are the positive constants and a is a function in $L^\infty(\Omega)$ such that $a \ge a_0 > 0$ *a.e.* in Ω, where a_0 is a constant. Wang and Zhang [6] extended that result to the systems with only one control force. In [7], F. Ammar Khodja, A. Benabdallah and C. Dupaix obtained local null controllability of a general reaction-diffusion system. There seems to have been relatively little work devoted to the global exact controllability and the approximate controllability of the systems which contains certain superlinear nonlinearities. This is a precisely problem which we consider in this paper. It is worth mentioning that these nonlinearities may lead to the state of the systems blow-up without imposing any control functions (see Section 2). It is well-known that the blow-up phenomena is usually adverse to us in reality. Therefore, we want to prevent the blow-up phenomena happening. Our method is that we introduce some control functions into the systems to change the dynamics of the systems. The advantage of this method not only avoids the occurrence of the blow-up phenomena, but also leads the state to the ideal targets at the given time. In other words, the systems achieve the controllability. Since the control functions act on a subset of the domain where the state work on, the method is realizable in the point view of practice.

Motivated by the article [1], our main results are stated as follows:

Theorem 1.2 *Let* $1 \le N < 6$. *Suppose that* (H1) *and* (H3) *hold. Then the system* (1.1)–(1.3) *is exactly controllable to the trajectories at time* T.

As a consequence of Theorem 1.2, we have the approximate controllability result.

Theorem 1.3 *Let* $1 \le N < 6$. *Suppose that* (H1) *and* (H3) *hold. Then the system* (1.1)–(1.3) *is approximately controllable at time* T.

Our results rest on a generalized fixed point theorem of Kakutani (see [10], p.7) which has been used in a variety of areas in differential equations and control theory (for instance see [1], [4]).

Theorem 1.4 (Kakutani) *Let* K *be a compact convex subset of a Banach space* X *and let* $T : K \to 2^X$ *be an upper semicontinuous mapping with convex values* $T(x)$ *such that* $T(x) \subset K$, $\forall x \in K$. *Then there is at least one* $x \in K$ *such that* $x \in T(x)$.

The rest of this paper is organized as follows. In Section 2, we give some blow-up and global existence results to a special case of (1.1)–(1.3), which point out that blow-up may occur. The exact controllability and the approximate controllability results are proved in the Section 3 and the Section 4 respectively. In Appendix at the end of the paper, we give the proof of Theorem 1.1 for the sake of completeness.

2 Blow-up and global existence of solutions for (1.1)–(1.3) in the absence of control functions

In this section, we will give an example to show that the solutions of (1.1)–(1.3) may occur blow-up phenomena, provided that F_i ($i = 1, 2, 3, 4$) satisfy (H3). Our crucial theorem is the following result to be proved later.

Theorem 2.1 *Take* $f_i(u,v) = (1+u+v)\ln^{\gamma_i}(1+u+v),\ i = 1,2$ *with* $1 < \gamma_i < 3/2$ *in* (1.1). *Then* (H3) *holds and the solutions of* (1.1)–(1.3) *blow-up in the absence of control functions with large nonnegative initial data.*

In order to prove Theorem 2.1, we merely consider the classical solutions of the following weakly coupled reaction-diffusion system

$$\begin{aligned} u_t &= \Delta u + (1+u+v)^{p_1}\ln^{q_1}(1+u+v), \\ v_t &= \Delta v + (1+u+v)^{p_2}\ln^{q_2}(1+u+v), \end{aligned} \qquad (x,t) \in \Omega \times (0,T), \tag{2.1}$$

with initial and boundary conditions

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \qquad x \in \Omega, \tag{2.2}$$

$$u(x,t) = v(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T), \tag{2.3}$$

where $u_0, v_0 \in C^{2,\alpha_0}(\overline{\Omega})(0 < \alpha_0 < 1)$ are nonnegative functions and constants $p_i, q_i \geq 0, i = 1,2$.

For the local existence of a classical solution for (2.1)–(2.3) we refer to Chapter 12 in [16]. Many authors have considered the global existence and blow-up of solutions for some reaction-diffusion systems (see e.g. [13][14]). As far as we know, there are no the similar results for the reaction functions in (2.1). Therefore, we prove the global existence and the blow-up of the solutions for the system (2.1)–(2.3) first.

Let us begin with a single parabolic equation.

$$u_t - Lu = f(u), \qquad (x,t) \in \Omega \times (0,T), \tag{2.4}$$

$$u(x,0) = \phi(x), \qquad x \in \Omega, \tag{2.5}$$

$$u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T), \tag{2.6}$$

where

$$Lu := \sum_{i,j=1}^{n} (a_{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u,$$

$(a_{ij}(x))$ is a uniformly positive definite matrix, the coefficients of L are sufficiently smooth in $\overline{\Omega} \times [0,T), \phi$ is a Hölder continuous function in $\overline{\Omega}$, and $f(u)$ is Lipschitz continuous in $\mathbb{R}$. The following blow-up results are basic and well known (see [12]):

Lemma 2.1 *If* $f(u) \geq 0,\ f'(u) > 0$ *for* $u > 0,\ f$ *is convex with* $\int^\infty du/f(u) < \infty$ *and* $\phi \geq 0,\ \int_\Omega \phi\ dx$ *is sufficiently large, then there is a* $T_1^* > 0$ *such that the solution of* (2.4)–(2.6) *exists in* $Q_{T_1^*}$, *but does not exist in* $Q_{T_1^*+\varepsilon}$ *for any* $\varepsilon > 0$.

Remark 2.1 *We note that* $f(u) = (1+u)\ln^p(1+u)$ *with* $p > 1$ *satisfies the requirements in Lemma* 2.1, *and the solution of* (2.4)–(2.6) *may blow-up in a finite time.*

Since the reaction functions in (2.1) are quasi-monotone increasing, we have the following existence-comparison theorem.

Lemma 2.2 *Let* $(\widetilde{u}, \widetilde{v})$ *and* $(\widehat{u}, \widehat{v})$ *be a pair of upper and lower solutions* ([13]) *of equations* (2.1)–(2.3) *in* Q_T *such that* $(\widehat{u}, \widehat{v}) \leq (\widetilde{u}, \widetilde{v})$ *in* Q_T. *Then the problem* (2.1)–(2.3) *has a unique solution* (u, v) *and*

$$(\widehat{u}, \widehat{v}) \leq (u, v) \leq (\widetilde{u}, \widetilde{v}) \quad in \quad Q_T.$$

This lemma is a particular case of Theorem 2.2 in [13]. Therefore, we omit the proof here. Now, we are in a position to show the following global existence result for (2.1)–(2.3).

Theorem 2.2 *Assume* $0 < p_i + q_i \leq 1, i = 1, 2$. *Then the solution of* (2.1)–(2.3) *is nonnegative and global.*

Proof. If $0 < p_i + q_i < 1$, we put $U(t) = V(t) = Ce^t - \frac{1}{2}$ with $C \geq \max\{\sup_\Omega u_0(x) + 1, \sup_\Omega v_0(x) + 1, 2^{\frac{p_i+q_i}{1-(p_i+q_i)}}\}$. Then, we can consider $(U(t), V(t))$ and $(0, 0)$ as a pair of upper and lower solutions of problem (2.1)–(2.3). $U(t)$ and $V(t)$ satisfy

$$\begin{aligned}
U_t(t) \geq& \Delta U(t) + (1 + U(t) + V(t))^{p_1+q_1} \\
\geq& \Delta U(t) + (1 + U(t) + V(t))^{p_1} \ln^{q_1}(1 + U(t) + V(t)), \\
V_t(t) \geq& \Delta V(t) + (1 + U(t) + V(t))^{p_2+q_2} \\
\geq& \Delta V(t) + (1 + U(t) + V(t))^{p_2} \ln^{q_2}(1 + U(t) + V(t)),
\end{aligned}$$

with $p_i + q_i < 1, i = 1, 2$. Moreover, $(U(t), V(t)) \geq (0, 0)$ on the parabolic boundary. Thus, we know from Lemma 2.2 that problem (2.1)–(2.3) has a unique solution (u, v) and

$$0 \leq u(x, t) \leq U(t), \quad 0 \leq v(x, t) \leq V(t), \ (x, t) \in Q_T.$$

If $p_i + q_i = 1$, the solution of the following linear system

$$\begin{aligned}
&u_t = \Delta u + (1 + u + v), && \\
&v_t = \Delta v + (1 + u + v), && (x, t) \in \Omega \times (0, T), \\
&u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), && x \in \Omega, \\
&u(x, t) = v(x, t) = 0, && (x, t) \in \partial\Omega \times (0, T),
\end{aligned}$$

could be regarded as a upper solution of (2.1)–(2.3), since

$$\begin{aligned}
u_t =& \Delta u + (1 + u + v)^{p_1} \ln^{q_1}(1 + u + v) \\
&\leq \Delta u + (1 + u + v)^{p_1+q_1} \leq \Delta u + (1 + u + v), \\
v_t =& \Delta v + (1 + u + v)^{p_2} \ln^{q_2}(1 + u + v) \\
&\leq \Delta v + (1 + u + v)^{p_2+q_2} \leq \Delta v + (1 + u + v).
\end{aligned}$$

We can get the conclusion as the solution of linear system is global. □

Corresponding to Lemma 2.1 for the single equation case, we have the following blow-up result for (2.1)–(2.3).

Theorem 2.3 *Assume*
(i) $p_i \geq 1,\ q_i > 1, \quad i = 1, 2,$
(ii) $\int_\Omega u_0\, dx$ *or* $\int_\Omega v_0\, dx$ *is large enough.*
Then the solutions of (2.1)–(2.3) *blow-up in a finite time.*

Proof. By the conditions, we can get

$$u_t = \Delta u + (1 + u + v)^{p_1} \ln^{q_1}(1 + u + v) \geq \Delta u + (1 + u) \ln^{q_1}(1 + u).$$

Using Lemma 2.1 and Remark 2.1, we get immediately that u blow-up in a finite time. This implies the same conclusion about v as well. □

By Theorem 2.3, we can obtian Theorem 2.1.

Proof of Theorem 2.1. We consider the nonnegative solutions of (2.1)–(2.3) with nonnegative initial data. Obviously, the condition (H3) holds by virtue of $\gamma_i < \frac{3}{2}$. If we choose one of the initial data large enough and use $\gamma_i > 1$, then all the conditions of Theorem 2.3 are satisfied. Then the blow-up phenomena will occur in the absence of control functions. □

3 Proof of the exact controllability result

In this section, we are devoted to prove Theorem 1.2. The proof is based on the null controllability of the linear parabolic system and the Kakutani fixed point theorem.

3.1 Observability estimate

For $R > 0$, we set

$$K_R = \{(y, z) \in (L^\infty(Q_T))^2; \|y\|_{L^\infty(Q_T)} + \|z\|_{L^\infty(Q_T)} \leq R\}.$$

Let $(y, z) \in K_R$. Consider the linearized version of (1.16)–(1.18):

$$\begin{aligned} &u_t = \Delta u + a(x,t)u + b(x,t)v + \chi_\omega f, \\ &v_t = \Delta v + c(x,t)u + d(x,t)v + \chi_\omega g, \end{aligned} \qquad (x,t) \in \Omega \times (0,T), \tag{3.1}$$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \qquad x \in \Omega, \tag{3.2}$$

$$u(x,t) = v(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T), \tag{3.3}$$

with $a(x,t) = F_1(y,z;u^*,v^*)$, $b(x,t) = F_2(y,z;u^*,v^*)$, $c(x,t) = F_3(y,z;u^*,v^*)$ and $d(x,t) = F_4(y,z;u^*,v^*)$, and its adjoint problem:

$$\begin{aligned} &-\psi_t = \Delta\psi + a(x,t)\psi + c(x,t)\zeta, \\ &-\zeta_t = \Delta\zeta + b(x,t)\psi + d(x,t)\zeta, \end{aligned} \qquad (x,t) \in \Omega \times (0,T), \tag{3.4}$$

$$\psi(x,T) = \psi_0(x), \quad \zeta(x,T) = \zeta_0(x), \qquad x \in \Omega, \tag{3.5}$$

$$\psi(x,t) = \zeta(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T). \tag{3.6}$$

For the linear system (3.1)–(3.3), the following result is well known.

Lemma 3.1 (see [15], p.616) *Let $a, b, c, d \in L^\infty(Q_T)$. For any $f, g \in L^{q_N}(Q_T), (u_0, v_0) \in (W_0^{1,q_N}(\Omega) \cap W^{2,q_N}(\Omega))^2$, system* (3.1)–(3.3) *has a unique solution $(u, v) \in (L^{q_N}(0, T; W_0^{1,q_N}(\Omega)) \cap W^{2,1}_{q_N}(Q_T))^2$ and moreover*

$$\|(u, v)\|_{W^{2,1}_{q_N}(Q_T)} \leq C(\|(u_0, v_0)\|_{W^{2(1-1/q_N),q_N}(\Omega)} + \|(f, g)\|_{L^{q_N}(Q_T)}).$$

From (H3) we can see that for each $\eta' > 0$, there exits $C_{\eta'} \geq 1$, such that

$$|F_i(s, \mu; s_0, \mu_0)| \leq C_{\eta'} + \eta' \ln^{3/2}(1 + |s| + |\mu|) \leq 2(C_{\eta'} + \eta'^{2/3} \ln(1 + |s| + |\mu|))^{3/2},$$

for any $(s, \mu) \in \mathbb{R} \times \mathbb{R}$ and $(s_0, \mu_0) \in K \times K$, with $K \subset \mathbb{R}$. Replacing $(2\eta')^{2/3}$ by η, we obtain that

$$\|a(x,t)\|^{2/3}_{L^\infty(Q_T)},\ \|b(x,t)\|^{2/3}_{L^\infty(Q_T)},\ \|c(x,t)\|^{2/3}_{L^\infty(Q_T)},\ \|d(x,t)\|^{2/3}_{L^\infty(Q_T)} \leq C_\eta + \eta \ln(1 + R). \tag{3.7}$$

By Lemma 3.1, we have that for any $(y, z) \in K_R$, (3.1)–(3.3) possesses one solution $(u, v) \in (L^{q_N}(0, T; W_0^{1,q_N}(\Omega)) \cap W^{2,1}_{q_N}(Q_T))^2$.

Following [11], let us introduce some notations. Let $\omega' \Subset \omega$ be a subdomain of ω and let β be a function in $C^2(\overline{\Omega})$ such that

$$\min\{|\nabla\beta(x)|, x \in \overline{\Omega \setminus \omega'}\} > 0 \quad \text{and} \quad \frac{\partial \beta}{\partial \nu} \leq 0 \text{ on } \partial\Omega,$$

where ν denotes the outward unit normal to $\partial\Omega$. Moreover, we can always assume that β satisfies

$$\min\{\beta(x), x \in \overline{\Omega}\} \geq \max\left\{\frac{3}{4}\|\beta\|_{L^\infty(\Omega)}, \ln 3\right\},$$

and set

$$\rho(x,t) := \frac{e^{\lambda\beta(x)}}{t(T-t)},\ (x,t) \in Q_T, \quad \alpha(x,t) := \tau \frac{e^{\frac{4}{3}\lambda\|\beta\|_{L^\infty(\Omega)}} - e^{\lambda\beta(x)}}{t(T-t)},\ (x,t) \in Q_T, \tag{3.8}$$

where $\lambda > 0$ and $\tau > 0$ are appropriate positive constants. The following results hold:

Theorem 3.1 (see [2] and [11] p.288). *There exist $\lambda_0 > 0, \tau_0 > 0$ and a positive constant C such that for any $\lambda \geq \lambda_0$, $\tau \geq \tau_0$ and $s \geq -3$, the inequality*

$$\begin{aligned} &\iint_{Q_T} \left(\frac{1}{\lambda}|z_t|^2 + \frac{1}{\lambda}|D_x^2 z|^2 + \lambda\tau^2\rho^2|\nabla z|^2 + \lambda^4\tau^4\rho^4 z^2\right) \rho^{2s-1} e^{-2\alpha} dxdt \\ &\leq C\left(\tau \iint_{Q_T} |z_t \pm \Delta z|^2 \rho^{2s} e^{-2\alpha} dxdt + \lambda^4\tau^4 \int_0^T \int_{\omega'} z^2 \rho^{2s+3} e^{-2\alpha} dxdt\right) \end{aligned} \tag{3.9}$$

holds for any function $z(x,t)$ satisfying homogeneous Dirichlet condition and the right-hand side of (3.9) *is finite. Moreover, the constant τ_0 is of the form $\tau_0 = c_0(\Omega, \omega')(T + T^2)$, and the constants C, c_0 and λ_0 only depend on Ω and ω'.*

Lemma 3.2 (see [7]) *Let $\lambda_0 > 1$, C being the constant given in Theorem* 3.1. *Then for any $\lambda \geq \lambda_0$, $\tau \geq \tau_1 = \frac{T^2}{4}\left(\frac{4C}{\lambda^4}\right)^{1/3} \|(a,b,c,d)\|^{2/3}_{L^\infty(Q_T)}$ and $s \geq -3$, the solution (ψ, ζ) of* (3.4)–(3.6) *satisfies:*

$$I(s,\psi) + I(s,\zeta) \leq C\lambda^4 \int_0^T \int_{\omega'} (\psi^2 + \zeta^2)\delta^{2s+3} e^{-2\alpha} dxdt, \tag{3.10}$$

where $\delta = \tau\rho$ and $I(s,z) = \iint_{Q_T} \left(\frac{1}{\lambda}|z_t|^2 + \frac{1}{\lambda}|\Delta z|^2 + \lambda\delta^2|\nabla z|^2 + \lambda^4\delta^4 z^2\right)\delta^{2s-1}e^{-2\alpha}dxdt.$

From the results above, we can obtain the observability estimate as follows:

Lemma 3.3 *Under the assumptions of Lemma* 3.2, *the solution (ψ, ζ) of* (3.4)–(3.6) *satisfies:*

$$\|(\psi,\zeta)(0)\|^2_{L^2(\Omega)} \leq C_T \int_0^T \int_\omega e^{-2\alpha}(\psi^2 + \zeta^2) dxdt, \tag{3.11}$$

with $C_T = \exp\left\{C\left(1 + \frac{1}{T} + (1 + \|(a,b,c,d)\|_\infty)T + \|(a,b,c,d)\|^{2/3}_\infty\right)\right\}$, *where* $\|(a,b,c,d)\|_\infty = (\|a\|^2_{L^\infty(Q_T)} + \|b\|^2_{L^\infty(Q_T)} + \|c\|^2_{L^\infty(Q_T)} + \|d\|^2_{L^\infty(Q_T)})^{1/2}$.

Proof. By the definition of function I and Lemma 3.2,

$$\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_\Omega e^{-2\alpha}(\psi^2 + \zeta^2)dxdt \leq \frac{1}{\lambda^4}\left(I(-\frac{3}{2},\psi) + I(-\frac{3}{2},\zeta)\right) \leq C\int_0^T \int_\omega e^{-2\alpha}(\psi^2 + \zeta^2)dxdt.$$

As $e^{-2\alpha(x,t)} \geq e^{-\frac{C_\tau}{T^2}}$ on $\left(\frac{T}{4}, \frac{3T}{4}\right) \times \Omega$,we get

$$\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_\Omega (\psi^2 + \zeta^2)dxdt \leq Ce^{\frac{C_\tau}{T^2}} \int_0^T \int_\omega e^{-2\alpha}(\psi^2 + \zeta^2)dxdt. \tag{3.12}$$

Recall that (ψ, ζ) satisfies (3.4) and take $m = 3\|(a,b,c,d)\|_{L^\infty(Q_T)}$,

$$\begin{aligned}
&\frac{d}{dt}\left(e^{-2m(T-t)}(\|\psi(t)\|^2_{L^2(\Omega)} + \|\zeta(t)\|^2_{L^2(\Omega)})\right) \\
&= 2e^{-2m(T-t)}(m\|\psi(t)\|^2_{L^2(\Omega)} + m\|\zeta(t)\|^2_{L^2(\Omega)} + \int_\Omega (\psi\psi_t + \zeta\zeta_t)dx) \\
&= 2e^{-2m(T-t)}(m\|\psi(t)\|^2_{L^2(\Omega)} + m\|\zeta(t)\|^2_{L^2(\Omega)} + \int_\Omega (|\nabla\psi|^2 + |\nabla\zeta|^2)dx \\
&\quad - \int_\Omega a\psi^2 dx - \int_\Omega d\zeta^2 dx - \int_\Omega (b+c)\psi\zeta dx \qquad (3.13)\\
&\geq 2e^{-2m(T-t)}\Big[(m - \|a\|_{L^\infty(Q_T)} - \frac{1}{2}(\|b\|_{L^\infty(Q_T)} + \|c\|_{L^\infty(Q_T)})\int_\Omega \psi^2 dx \\
&\quad +(m - \|d\|_{L^\infty(Q_T)} - \frac{1}{2}(\|b\|_{L^\infty(Q_T)} + \|c\|_{L^\infty(Q_T)})\int_\Omega \zeta^2 dx\Big] \\
&\geq 0.
\end{aligned}$$

We obtain that the function $e^{-2m(T-t)}(\|\psi(t)\|^2_{L^2(\Omega)}+\|\zeta(t)\|^2_{L^2(\Omega)})$ is increasing in t. Then by the monotonicity and the mean value theorem of integral, we can get

$$\begin{aligned}\frac{T}{2}e^{-2mT}(\|\psi(0)\|^2_{L^2(\Omega)}+\|\zeta(0)\|^2_{L^2(\Omega)}) &\leq \int_{\frac{T}{4}}^{\frac{3T}{4}}\int_\Omega e^{-2m(T-t)}(\psi^2+\zeta^2)dxdt \\ &\leq e^{-\frac{mT}{2}}\int_{\frac{T}{4}}^{\frac{3T}{4}}\int_\Omega(\psi^2+\zeta^2)dxdt.\end{aligned} \tag{3.14}$$

Let $\tau = C\|(a,b,c,d)\|^{2/3}_{L^\infty(Q_T)}T^2$ with C sufficiently large. Combining (3.12) and (3.14), we obtain (3.11). □

Our crucial lemma is the following:

Lemma 3.4 *Let* $1 \leq N < 6$. *Suppose that* (H1) *and* (H3) *hold. Then the system* (3.1)–(3.3) *is null-controllable, that is, for any* $(u_0, v_0) \in (W_0^{1,q_N}(\Omega) \cap W^{2,q_N}(\Omega))^2$ *and* $T > 0$, *there exist* $f, g \in L^{q_N}(Q_T)$ *such that the associated solution* $(u, v) \in (L^{q_N}(0,T; W_0^{1,q_N}(\Omega)) \cap W^{2,1}_{q_N}(Q_T))^2$ *with*

$$(u,v)(T) = (0,0) \quad \text{a.e. in } \Omega. \tag{3.15}$$

Moreover,

$$\|\chi_\omega f\|^2_{L^{q_N}(Q_T)} + \|\chi_\omega g\|^2_{L^{q_N}(Q_T)} \leq C_T\|(u_0,v_0)\|^2_{L^2(\Omega)}, \tag{3.16}$$

where C_T *is defined in Lemma* 3.3.

Proof. For any given $(y,z) \in K_R$ and any $\varepsilon > 0$, we consider the following optimal control problem

$$(\mathrm{P}_\varepsilon)\quad \text{Minimize}\left\{\frac{1}{2}\int_0^T\int_\omega e^{2\alpha}(f^2+g^2)dxdt + \frac{1}{2\varepsilon}\int_\Omega[u^2(x,T)+v^2(x,T)]dx\right\},$$

where $(f,g) \in L^2(Q_T)$ and (u,v) is the solution of (3.1)–(3.3) corresponding to (f,g). The existence of a pair of solution $(f_\varepsilon, g_\varepsilon)$ to the problem (P_ε) follows from a standard argument. By the Pontryagin maximum principle, we have

$$f_\varepsilon = \chi_\omega e^{-2\alpha}\psi_\varepsilon, \quad g_\varepsilon = \chi_\omega e^{-2\alpha}\zeta_\varepsilon, \tag{3.17}$$

where $(\psi_\varepsilon, \zeta_\varepsilon)$ is the solution of

$$\begin{cases} -\psi_t = \Delta\psi + a(x,t)\psi + c(x,t)\zeta, \\ -\zeta_t = \Delta\zeta + b(x,t)\psi + d(x,t)\zeta, \end{cases} \qquad (x,t) \in \Omega\times(0,T), \tag{3.18}$$

$$\psi(x,T) = -\frac{1}{\varepsilon}u_\varepsilon(x,T), \quad \zeta(x,T) = -\frac{1}{\varepsilon}v_\varepsilon(x,T), \qquad x \in \Omega, \tag{3.19}$$

$$\psi(x,t) = \zeta(x,t) = 0, \qquad (x,t) \in \partial\Omega\times(0,T). \tag{3.20}$$

Multiplying the first equation in (3.18) by u_ε and the second equation by v_ε and integrating on Q_T yield

$$\iint_{Q_T} [(\psi_\varepsilon)_t + \Delta\psi_\varepsilon + a(x,t)\psi_\varepsilon + c(x,t)\zeta_\varepsilon]u_\varepsilon dxdt \\ + \iint_{Q_T} [(\zeta_\varepsilon)_t + \Delta\zeta_\varepsilon + b(x,t)\psi_\varepsilon + d(x,t)\zeta_\varepsilon]v_\varepsilon dxdt = 0.$$

Using (3.1)–(3.3) and integrating by parts, it follows that

$$\iint_{Q_T} (\psi_\varepsilon\chi_\omega f_\varepsilon + \zeta_\varepsilon\chi_\omega g_\varepsilon)dxdt \\ = \int_\Omega [(\psi_\varepsilon u_\varepsilon)(T) + (\zeta_\varepsilon v_\varepsilon)(T)]dx - \int_\Omega [(\psi_\varepsilon u_\varepsilon)(0) + (\zeta_\varepsilon v_\varepsilon)(0)]dx.$$

By virtue of (3.17) and (3.19), we have

$$\frac{1}{2}\iint_{Q_T} \chi_\omega e^{2\alpha}(f_\varepsilon^2 + g_\varepsilon^2)dxdt + \frac{1}{2\varepsilon}\int_\Omega (u_\varepsilon^2(x,T) + v_\varepsilon^2(x,T))dx \\ \leq \|(\psi_\varepsilon,\zeta_\varepsilon)(x,0)\|_{L^2(\Omega)}\|(u_0,v_0)\|_{L^2(\Omega)}. \tag{3.21}$$

Using Lemma 3.3 gives

$$\|(\psi_\varepsilon,\zeta_\varepsilon)(x,0)\|^2_{L^2(\Omega)}\|(u_0,v_0)\|_{L^2(\Omega)} \leq C_T \iint_{Q_T} \chi_\omega e^{-2\alpha}(\psi_\varepsilon^2 + \zeta_\varepsilon^2)dxdt\|(u_0,v_0)\|_{L^2(\Omega)}, \tag{3.22}$$

By (3.17) and (3.21), we can get

$$\iint_{Q_T} \chi_\omega e^{-2\alpha}(\psi_\varepsilon^2 + \zeta_\varepsilon^2)dxdt \leq 2\|(\psi_\varepsilon,\zeta_\varepsilon)(x,0)\|_{L^2(\Omega)}\|(u_0,v_0)\|_{L^2(\Omega)}. \tag{3.23}$$

Combining (3.21)–(3.23), we obtain

$$\frac{1}{2}\int_0^T\int_\omega e^{-2\alpha}(\psi_\varepsilon^2 + \zeta_\varepsilon^2)dxdt + \frac{1}{2\varepsilon}\|(u_\varepsilon,v_\varepsilon)(x,T)\|^2_{L^2(\Omega)} \leq C_T\|(u_0,v_0)\|^2_{L^2(\Omega)}. \tag{3.24}$$

Next, we will show that our control functions are in $L^{q_N}(Q_T)$. We introduce $\varphi_\varepsilon = e^{-2\alpha}\psi_\varepsilon$. By (3.18)–(3.20), we have

$$\begin{aligned} &(\varphi_\varepsilon)_t + \Delta(\varphi_\varepsilon) = G_\varepsilon(x,t) && (x,t)\in\Omega\times(0,T), \\ &\varphi_\varepsilon(x,T) = 0, && x\in\Omega, \\ &\varphi_\varepsilon(x,t) = 0, && (x,t)\in\partial\Omega\times(0,T), \end{aligned}$$

with $G_\varepsilon(x,t) = -c(x,t)e^{-2\alpha}\zeta_\varepsilon - 4(\nabla\alpha)(e^{-2\alpha}\nabla\psi_\varepsilon) + [\Delta(e^{-2\alpha}) + (e^{-2\alpha})_t - a(x,t)e^{-2\alpha}]\psi_\varepsilon$. By parabolic regularity, we have

$$\|\varphi_\varepsilon\|_{W_2^{2,1}(Q_T)} \leq C\|G_\varepsilon(x,t)\|_{L^2(Q_T)}.$$

On the other hand, setting

$$I_1 = \iint_{Q_T} c^2(x,t)e^{-4\alpha}\zeta_\varepsilon^2 dxdt,$$

we have, using (3.10) in Lemma 3.2,

$$\begin{aligned} I_1 &= \iint_{Q_T} (c^2(x,t)e^{-2\alpha})(e^{-2\alpha}\zeta_\varepsilon^2)dxdt \le \|c^2(x,t)e^{-2\alpha}\|_{L^\infty(Q_T)} \iint_{Q_T} e^{-2\alpha}\zeta_\varepsilon^2 dxdt \\ &\le C_T \int_0^T \int_\omega e^{-2\alpha}(\psi_\varepsilon^2 + \zeta_\varepsilon^2)dxdt. \end{aligned}$$

In the same way, setting

$$\begin{aligned} I_2 &= \iint_{Q_T} |(\nabla\alpha)(e^{-2\alpha}\nabla\psi_\varepsilon)|^2 dxdt, \\ I_3 &= \iint_{Q_T} |[\Delta(e^{-2\alpha}) + (e^{-2\alpha})_t - a(x,t)e^{-2\alpha}]\psi_\varepsilon|^2 dxdt, \end{aligned}$$

we can obtain by the same kind of computations that

$$I_2, I_3 \le C_T \int_0^T \int_\omega e^{-2\alpha}(\psi_\varepsilon^2 + \zeta_\varepsilon^2)dxdt.$$

It follows from these inequalities that

$$\|\varphi_\varepsilon\|^2_{W_2^{2,1}(Q_T)} \le C_T \int_0^T \int_\omega e^{-2\alpha}(\psi_\varepsilon^2 + \zeta_\varepsilon^2)dxdt.$$

Set $\vartheta_\varepsilon = e^{-2\alpha}\zeta_\varepsilon$ and repeat the above process. We can conclude that

$$\|\vartheta_\varepsilon\|^2_{W_2^{2,1}(Q_T)} \le C_T \int_0^T \int_\omega e^{-2\alpha}(\psi_\varepsilon^2 + \zeta_\varepsilon^2)dxdt.$$

Now, by the embedding $W_2^{2,1}(Q_T) \hookrightarrow L^{q_N}(Q_T)$ and (3.24), we have

$$\begin{aligned} \|f_\varepsilon\|^2_{L^{q_N}(Q_T)} + \|g_\varepsilon\|^2_{L^{q_N}(Q_T)} &= \|\chi_\omega\varphi_\varepsilon\|^2_{L^{q_N}(Q_T)} + \|\chi_\omega\vartheta_\varepsilon\|^2_{L^{q_N}(Q_T)} \\ &\le C_T \int_0^T \int_\omega e^{-2\alpha}(\psi_\varepsilon^2 + \zeta_\varepsilon^2)dxdt \le C_T\|(u_0, v_0)\|^2_{L^2(\Omega)}. \end{aligned} \tag{3.25}$$

From (3.25) and Lemma 3.1, it follows, at least for a subsequence, that for $\varepsilon \to 0$

$$\begin{aligned} &(f_\varepsilon, g_\varepsilon) \rightharpoonup (f, g) \text{ weakly in } (L^{q_N}(Q_T))^2, \\ &(u_\varepsilon, v_\varepsilon) \rightharpoonup (u, v) \text{ weakly in } (L^{q_N}(0,T; W_0^{1,q_N}(\Omega)) \cap W_{q_N}^{2,1}(Q_T))^2, \end{aligned}$$

and (u, v, f, g) satisfy (3.1)–(3.3). By (3.24), we have $(u, v)(T) = 0$. Moreover, (3.16) holds. This completes the proof of Lemma 3.4. □

3.2 Proof of Theorem 1.2

For each $(y,z) \in K_R$, we define a map $\Phi : K_R(\subset L^2(Q_T)^2) \to 2^{L^2(Q_T)^2}$ by

$$\begin{aligned}\Phi(y,z) = \{(u,v) \in (L^{q_N}(0,T;W_0^{1,q_N}(\Omega)) \cap W_{q_N}^{2,1}(Q_T))^2;\ \exists\ f,g \in L^{q_N}(Q_T) \\ \text{satisfying (3.16) such that } (u,v) \text{ is the solution to the system} \\ \text{(3.1)–(3.3) corresponding to } (y,z), f, g \text{ and } (u,v)(T) = 0 \text{ a.e. in } \Omega\}.\end{aligned}$$

It is readily seen that $\Phi(y,z)$ is nonempty, closed and convex in $L^2(Q_T)$. Then we prove $\Phi(K_R) \subset K_R$ with sufficiently large $R > 0$. First of all, we show that

$$\|(u,v)\|^2_{L^\infty(Q_T)} \le C_T \|(u_0,v_0)\|^2_{L^\infty(\Omega)}. \tag{3.26}$$

Exactly as in [4], we can obtain

$$\begin{aligned}\|(u,v)(t)\|_{L^\infty(\Omega)} \le & C\Big(\|(u_0,v_0)\|_{L^\infty(\Omega)} + T^{-\frac{N+2}{2q_N}+1}(\|\chi_\omega f\|_{L^{q_N}(Q_T)} + \|\chi_\omega g\|_{L^{q_N}(Q_T)}) \\ & + (1 + \|(a,b,c,d)\|_{L^\infty(Q_T)}) \int_0^t \|(u,v)(\tau)\|_{L^\infty(\Omega)} d\tau\Big),\end{aligned}$$

and by Gronwall's inequality,

$$\begin{aligned}\|(u,v)\|_{L^\infty(Q_T)} \le & C e^{C(1+\|(a,b,c,d)\|_{L^\infty(Q_T)})T} \cdot \\ & \Big(\|(u_0,v_0)\|_{L^\infty(\Omega)} + T^{-\frac{N+2}{2q_N}+1}(\|\chi_\omega f\|_{L^{q_N}(Q_T)} + \|\chi_\omega g\|_{L^{q_N}(Q_T)})\Big).\end{aligned} \tag{3.27}$$

Then by (3.27) and Lemma 3.4, we can get (3.26). Substituting (3.7) into (3.26), we have

$$\begin{aligned}\|(u,v)\|^2_{L^\infty(Q_T)} \le & \exp\Big\{C\Big(1 + \frac{1}{T} + (1 + [C_\eta + \eta \ln(1+R)]^{3/2})T \\ & + C_\eta + \eta \ln(1+R)\Big)\Big\} \|(u_0,v_0)\|^2_{L^\infty(\Omega)}.\end{aligned}$$

Choosing $T := T(R,\eta) = [C_\eta + \eta \ln(1+R)]^{-1}$, we get for R sufficiently large

$$\begin{aligned}\|(u,v)\|^2_{L^\infty(Q_T)} \le & \exp\{C(1 + \frac{1}{T})\} \|(u_0,v_0)\|^2_{L^\infty(\Omega)} \\ = & \exp\{C(1 + C_\eta + \eta \ln(1+R)\} \|(u_0,v_0)\|^2_{L^\infty(\Omega)} \\ \le & (1+R)^{\eta C} \exp\{C(1+C_\eta)\} \|(u_0,v_0)\|^2_{L^\infty(\Omega)}.\end{aligned}$$

Taking $\eta = \frac{1}{2C}$ yields

$$\|(u,v)\|^2_{L^\infty(Q_T)} \le C(1+R)^{1/2} \|(u_0,v_0)\|^2_{L^\infty(\Omega)}.$$

For R sufficiently large,

$$\|(u,v)\|_{L^\infty(Q_T)} \le R.$$

It follows that $\Phi(K_R) \subset K_R$.

Moreover, by the parabolic regularity and (3.16) we have

$$\|(u,v)\|_{W^{2,1}_{q_N}(Q_T)} \le C_T(\|(u_0,v_0)\|_{L^2(\Omega)}.$$

By the embedding $W^{2,1}_{q_N}(Q_T) \hookrightarrow C^{\alpha_0,\alpha_0/2}(Q_T)$ $(0 < \alpha_0 < 1)$ and the Arzelà-Ascoli theorem, $\Phi(K_R)$ is a relatively compact subset of $(L^2(Q_T))^2$.

Note also that Φ is upper semicontinuous in $(L^2(Q_T))^2$. Indeed, let $(y_n,z_n) \in K_R, (y_n,z_n) \to (y,z)$ in K_R, and $(u_n,v_n) \in \Phi(y_n,z_n), (u_n,v_n) \to (u,v)$ in $(L^2(Q_T))^2$. We want to show $(u,v) \in \Phi(y,z)$. By Lemma 3.1 and Lemma 3.4 it follows (selecting a subsequence if necessary) that

$$\begin{aligned}
(f_n,g_n) &\rightharpoonup (f,g) \text{ weakly in } (L^{q_N}(Q_T))^2,\\
(u_n,v_n) &\rightharpoonup (\tilde{u},\tilde{v}) \text{ strongly in} (C(Q_T))^2,\\
&\qquad\qquad \text{weakly in } (L^{q_N}(0,T;W_0^{1,q_N}(\Omega)) \cap W^{2,1}_{q_N}(Q_T))^2.
\end{aligned}$$

Then we obtain $(u,v) = (\tilde{u},\tilde{v}) \in (L^{q_N}(0,T;W_0^{1,q_N}(\Omega)) \cap W^{2,1}_{q_N}(Q_T))^2$. Thus letting n tend to $+\infty$ in the system

$$\begin{aligned}
&\left.\begin{aligned}(u_n)_t &= \Delta u_n + F_1(y_n,z_n;u^*,v^*)u_n + F_2(y_n,z_n;u^*,v^*)v_n + \chi_\omega f_n,\\
(v_n)_t &= \Delta v_n + F_3(y_n,z_n;u^*,v^*)u_n + F_4(y_n,z_n;u^*,v^*)v_n + \chi_\omega g_n,\end{aligned}\right\} && (x,t) \in \Omega \times (0,T),\\
&u_n(x,0) = u_0(x), \quad v_n(x,0) = v_0(x), && x \in \Omega,\\
&u_n(x,t) = v_n(x,t) = 0, && (x,t) \in \partial\Omega \times (0,T),
\end{aligned}$$

we conclude that (u,v,f,g) satisfy (3.1)–(3.3), (3.15) and (3.16), i.e., $(u,v) \in \Phi(y,z)$. Therefore, Φ is upper semicontinuous in $(L^2(Q_T))^2$.

Then applying the Kakutani fixed point theorem we infer that there is at least one $(y,z) \in K_R$ such that $(y,z) \in \Phi(y,z)$. Hence, our assertion is proved for any $T = T(R,\eta)$. For any $T > T(R,\eta)$, clearly, we can choose control defined on $(0,T(R,\eta))$ which gives a solution satisfying (3.15) at $T = T(R,\eta)$. We then extend the solution from 0 to the whole interval $(0,T)$. □

4 Proof of the approximate controllability result

In this section we will prove Theorem 1.3. Let $T > 0$ and $u_0, v_0 \in W_0^{1,q_N}(\Omega) \cap W^{2,q_N}(\Omega)$ be given. Observe that we only need to consider the final data $u_d, v_d \in C_0^3(\Omega)$, since this space is dense in $L^2(\Omega)$.

For the given time T, there exists $\delta > 0$, which is small enough and only depends on $\Omega, f_1, f_2, u_d, v_d$ and ε, such that the following auxiliary system:

$$\begin{aligned}
&h_t = \Delta h + f_1(h,\xi), && \\
&\xi_t = \Delta \xi + f_2(h,\xi), && (x,t) \in \Omega \times (T-\delta, T), \\
&h(x,T-\delta) = u_d(x), \quad \xi(x,T-\delta) = v_d(x), && x \in \Omega, \\
&h(x,t) = \xi(x,t) = 0, && (x,t) \in \partial\Omega \times (T-\delta, T),
\end{aligned}$$

possesses a solution $(h,\xi) \in (C(\overline{\Omega \times (T-\delta,T)}) \cap C^{2,1}(\overline{\Omega \times (T-\delta,T)}))^2$ (see [16]) satisfying $\|h(\cdot,T) - u_d\|_{L^2(\Omega)} \le \varepsilon$, $\|\xi(\cdot,T) - v_d\|_{L^2(\Omega)} \le \varepsilon$.

We consider the trajectory $u^* \equiv v^* \equiv 0$ on $[0, T-\delta]$ with $u_0 = v_0 = 0$ and $f = g = 0$. By Theorem 1.2, we can find $(\widetilde{f}_1, \widetilde{g}_1) \in L^{q_N}(\Omega \times (0, T-\delta))$ so that

$$u(x,T-\delta) = 0, \ v(x,T-\delta) = 0 \ \text{ a.e. in } \Omega.$$

On $[T-\delta, T]$, we take the solution of auxiliary system as the trajectory. Using Theorem 1.2 again, there exists $(\widetilde{f}_2, \widetilde{g}_2) \in L^{q_N}(\Omega \times (T-\delta, T))$ such that the solution of the system

$$\begin{aligned}
&u_t = \Delta u + f_1(u,v) + \chi_\omega \widetilde{f}_2, && \\
&v_t = \Delta v + f_2(u,v) + \chi_\omega \widetilde{g}_2, && (x,t) \in \Omega \times (T-\delta, T), \\
&u(x,T-\delta) = 0, \quad v(x,T-\delta) = 0, && x \in \Omega, \\
&u(x,t) = v(x,t) = 0, && (x,t) \in \partial\Omega \times (T-\delta, T)
\end{aligned}$$

satisfies

$$u(x,T) = h(x,T), \ v(x,T) = \xi(x,T) \ \text{ a.e. in } \Omega.$$

Finally, we define

$$(f(x,t), \ g(x,t)) = \begin{cases} (\widetilde{f}_1(x,t), \ \widetilde{g}_1(x,t)) & (x,t) \in \Omega \times [0, T-\delta], \\ (\widetilde{f}_2(x,t), \ \widetilde{g}_2(x,t)) & (x,t) \in \Omega \times (T-\delta, T]. \end{cases}$$

Obviously, (1.20) holds. This completes the proof of Theorem 1.3. □

5 Appendix.

In this part, we will give the proof of Theorem 1.1. By deformation, we only need to prove local existence of (1.7)–(1.9) combining with Schauder fixed point theorem.

For $R' > 0, T_1 < T_0$, we set

$$K_{R',T_1} = \{(y,z) \in (L^\infty(Q_{T_1}))^2; \|y\|_{L^\infty(Q_{T_1})} + \|z\|_{L^\infty(Q_{T_1})} \le R'\},$$

where R', T_1 will be determined in later. The set K_{R',T_1} is a bounded closed convex subset of $(L^\infty(Q_{T_1}))^2$. Let $(y,z) \in K_{R',T_1}$. We consider the linearized version of (1.7)–(1.9) in Q_{T_1}.

$$\begin{aligned} &u_t = \Delta u + a'(x,t)u + b'(x,t)v + \chi_\omega f, \\ &v_t = \Delta v + c'(x,t)u + d'(x,t)v + \chi_\omega g, \end{aligned} \qquad (x,t) \in \Omega \times (0,T_1), \tag{5.1}$$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \qquad x \in \Omega, \tag{5.2}$$

$$u(x,t) = v(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T_1), \tag{5.3}$$

with $a'(x,t) = F_1(y,z;0,0)$, $b'(x,t) = F_2(y,z;0,0)$, $c'(x,t) = F_3(y,z;0,0)$ and $d'(x,t) = F_4(y,z;0,0)$. From (H2) we can see for each $\eta_0 > 0$, there exits $C(\eta_0) \geq 1$, such that

$$\begin{aligned} &\|a'(x,t)\|^{2/3}_{L^\infty(Q_{T_1})}, \ \|b'(x,t)\|^{2/3}_{L^\infty(Q_{T_1})}, \ \|c'(x,t)\|^{2/3}_{L^\infty(Q_{T_1})}, \ \|d'(x,t)\|^{2/3}_{L^\infty(Q_{T_1})} \\ &\leq C_{\eta_0} + \eta_0 \ln(1+R'). \end{aligned} \tag{5.4}$$

By Lemma 3.1, we have that for any $(y,z) \in K_{R',T_1}$, (5.1)–(5.3) possesses one solution $(u,v) \in (L^{q_N}(0,T_1;W_0^{1,q_N}(\Omega)) \cap W^{2,1}_{q_N}(Q_{T_1}))^2$. Denote $\Phi' : K_{R',T_1} \to (L^\infty(Q_{T_1}))^2$ by $\Phi'(y,z) = (u,v)$. Then, by Sobolev embedding theorem, $\Phi'(K_{R',T_1})$ is a relatively compact subset of $(L^\infty(Q_{T_1}))^2$.

Note that Φ' is continuous in $(L^\infty(Q_{T_1}))^2$. Indeed, let $(y_n,z_n) \in K_{R',T_1}$, $(y_n,z_n) \to (y,z)$ in K_{R',T_1} and $(u_n,v_n) = \Phi'(y_n,z_n)$, $(u_n,v_n) \to (u,v)$ in $(L^\infty(Q_{T_1}))^2$. We only need to prove $\Phi'(y,z) = (u,v)$. We have $(u_n,v_n) \in (L^{q_N}(0,T_1;W_0^{1,q_N}(\Omega)) \cap W^{2,1}_{q_N}(Q_{T_1}))^2$. By Sobolev embedding theorem and Lemma 3.1, it follows (selecting a subsequence if necessary) that

$$\begin{aligned} (u_n,v_n) \to (\tilde{u},\tilde{v}) \text{ strongly in} (C(Q_{T_1}))^2, \\ \text{and weakly in} (L^{q_N}(0,T_1;W_0^{1,q_N}(\Omega)) \cap W^{2,1}_{q_N}(Q_{T_1}))^2. \end{aligned}$$

Then we obtain $(u,v) = (\tilde{u},\tilde{v}) \in (L^{q_N}(0,T_1;W_0^{1,q_N}(\Omega)) \cap W^{2,1}_{q_N}(Q_{T_1}))^2$. Thus letting n tend to $+\infty$ in the system

$$\begin{aligned} &(u_n)_t = \Delta u_n + F_1(y_n,z_n;0,0)u_n + F_2(y_n,z_n;0,0)v_n + \chi_\omega f, \\ &(v_n)_t = \Delta v_n + F_3(y_n,z_n;0,0)u_n + F_4(y_n,z_n;0,0)v_n + \chi_\omega g, \end{aligned} \qquad (x,t) \in \Omega \times (0,T_1),$$

$$u_n(x,0) = u_0(x), \quad v_n(x,0) = v_0(x), \qquad x \in \Omega,$$

$$u_n(x,t) = v_n(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T_1),$$

we conclude that (u,v,f,g) satisfy (5.1)–(5.3), i.e., $\Phi'(y,z) = (u,v)$. Therefore, Φ' is continuous in $(L^\infty(Q_{T_1}))^2$.

Next we will choose suitable R' and T_1 to have that $\Phi'(K_{R',T_1}) \subset K_{R',T_1}$. For any fixed $f,g \in L^{q_N}(Q_{T_0})$ and $(u_0,v_0) \in (W_0^{1,q_N}(\Omega) \cap W^{2,q_N}(\Omega))^2$, there exists $C_1(T_0) > 0$ such that

$$\|f\|_{L^{q_N}(Q_{T_0})} + \|g\|_{L^{q_N}(Q_{T_0})} \leq C_1(T_0)\|(u_0,v_0)\|_{L^\infty(\Omega)}.$$

Exactly same as in [4], we can obtain

$$\begin{aligned}\|(u,v)\|_{L^\infty(Q_{T_1})} \le{}& C\exp\{C(1+\|(a',b',c',d')\|_{L^\infty(Q_{T_1})})T_1\}\cdot \\ & \left(\|(u_0,v_0)\|_{L^\infty(\Omega)} + T_1^{-\frac{N+2}{2q_N}+1}(\|f\|_{L^{q_N}(Q_{T_1})} + \|g\|_{L^{q_N}(Q_{T_1})})\right) \\ \le{}& \exp\{C_2(1+\frac{1}{T_1} + (1+\|(a',b',c',d')\|_{L^\infty(Q_{T_1})})T_1)\}\|(u_0,v_0)\|_{L^\infty(\Omega)}.\end{aligned} \tag{5.5}$$

Substituting (5.4) into (5.5), we have

$$\|(u,v)\|_{L^\infty(Q_{T_1})} \le \exp\{C_3(1+\frac{1}{T_1} + (1+[C_{\eta_0}+\eta_0\ln(1+R')]^{3/2})T_1)\}\|(u_0,v_0)\|_{L^\infty(\Omega)}. \tag{5.6}$$

Choosing $\eta_0 = \frac{1}{12C_3}$ and $T_1 = [C_{\eta_0} + \eta_0\ln(1+R')]^{-1}$, we see that, for R' large enough,

$$\begin{aligned}\|(u,v)\|^2_{L^\infty(Q_{T_1})} &\le \exp\{6C_3(1+C_{\eta_0}+\eta_0\ln(1+R'))\}\|(u_0,v_0)\|^2_{L^\infty(\Omega)} \\ &\le \exp\{6C_3(1+C_{\eta_0})\}(1+R')^{1/2}\|(u_0,v_0)\|^2_{L^\infty(\Omega)},\end{aligned}$$

and then

$$\|(u,v)\|^2_{L^\infty(Q_{T_1})} \le \frac{1}{4}R'^2.$$

It follows that $\Phi'(K_{R',T_1}) \subset K_{R',T_1}$. Then by Schauder fixed point theorem, we can conclude that the system (1.7)–(1.9) has a local solution $(u,v) \in (L^\infty(Q_{T_1}))^2$. By classical parabolic regularity, the solution also satisfies $(u,v) \in (L^{q_N}(0,T_1;W_0^{1,q_N}(\Omega)) \cap W^{2,1}_{q_N}(Q_{T_1}))^2$. □

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