



On the analytic commutator for Λ – Ω differential systems

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Received 3 January 2023, appeared 6 July 2023

Communicated by Gabriele Villari

Abstract. In this paper, we give the necessary and sufficient conditions for some Λ – Ω differential systems to have an analytic commutator, use these properties to judge the origin point of the Λ – Ω differential systems to be an isochronous center.

Keywords: analytic commutator, isochronous center, Λ – Ω system, center conditions.

2020 Mathematics Subject Classification: 34C07, 34C05, 34C25, 37G15.


1 Introduction

Consider differential systems of the form

$$\begin{cases} x' = -y + P, \\ y' = x + Q, \end{cases} \quad (1.1)$$

where $P = \sum_{k=2}^{\infty} P_k(x, y)$ and $Q = \sum_{k=2}^{\infty} Q_k(x, y)$ where, P_k and Q_k are homogeneous polynomials in x and y of degree k . If every orbit in a punctured neighbourhood of O is a nontrivial cycle then the origin point $O(0,0)$ is said to be a center. In particular, if every cycle in a punctured neighbourhood of O has the same period then this origin point is said to be an isochronous center. Christian Huygens is credited with being one of the first scholars to study isochronous systems in the XVII century, even before the development of the differential calculus. Huygens investigated the cycloidal pendulum, which has isochronous oscillations in opposition to the monotonicity of the period of the usual pendulum. It is probably the first example of a nonlinear isochrone. For more details see [10, 12]. However, it is far from being completely resolved, beside some specific families of vector fields [4, 7].

By [1], we know that for any analytic system (1.1), the existence of an analytic commutator with linear part $(x, y)^t$ is a necessary and sufficient condition for the origin to be an isochronous center. In [2, 3] Algaba and Reyes have studied a particular case of this family are the plane polynomial systems which have a center focus equilibrium at the origin and whose angular speed is constant. In these systems, the origin is the only finite equilibrium and if

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it is a center, it will be automatically isochronous. These systems, up to a linear change of variable, have the following form:

$$x' = -y + xH(x, y), \quad y' = x + yH(x, y), \quad (1.2)$$

$H(0,0) = 0$. They pointed out that if (1.2) has an analytic commutator then it is in the form of $(U, V)^t = (xK(x, y), yK(x, y))^t$, where K and H are polynomials of the same degree. They also characterize the system $x' = -y + P_s(x, y) + xH(x, y)$, $y' = x + Q_s(x, y) + yH(x, y)$, where $H(x, y)$ is a polynomial with degree greater than or equal to s , if it has a polynomial commutator, then it is in the form of $(U, V)^t = (u_s(x, y) + xK(x, y), v_s(x, y) + yK(x, y))^t$, here P_s, Q_s, u_s, v_s are homogeneous polynomials of degree s .

There are only a few families of polynomial differential systems in which a complete classification of the isochronous centers is known, and almost all of them have polynomial commutator. The quadratic isochronous centers, characterized by Loud [17]. In Pleshkan [18], cubic isochronous centers with homogeneous nonlinear part are settled. In Christopher and Devlin[6], the isochronous centers of the Kukles family are obtained. Commutators of quadratic centers are computed in Sabatini [19]; commutators of cubic systems with homogeneous nonlinear part can be found in Gasull et al. [11]; commutators for the Kukles system can be seen in Volokitin and Ivanov [20]. The first example of a polynomial isochronous center without polynomial commutator is found in Devlin [8].

A center of (1.1) is called a **Weak Center** if the Poincaré–Liapunov first integral can be written as $H = \frac{1}{2}(x^2 + y^2)(1 + h.o.t.)$. By literature [13]-[16] we know that a center of an analytic or polynomial differential system (1.1) is a weak center if and only if it can be written as

$$\begin{cases} x' = -y\bar{\Lambda} + x\Omega, \\ y' = x\bar{\Lambda} + y\Omega, \end{cases} \quad (1.3)$$

where $\bar{\Lambda} = 1 + \Lambda(x, y)$ and $\Omega = \Omega(x, y)$ are analytic functions or polynomials such that $\Lambda(0,0) = \Omega(0,0) = 0$. The class of differential systems (1.3) is called the Λ - Ω system. The weak centers contain the uniform isochronous centers and the holomorphic isochronous centers [13], they also contain the class of centers studied by Alwash and Lloyd [5], but they do not coincide with all classes of isochronous centers [13], because in general weak centers are not isochronous.

In [14, 16] Llibre et al. put forward such conjecture.

Conjecture. The polynomial differential system of degree m

$$\begin{cases} x' = -y(1 + \mu(a_2x - a_1y)) + x((a_1x + a_2y) + \Phi_{m-1}(x, y)), \\ y' = x(1 + \mu(a_2x - a_1y)) + y((a_1x + a_2y) + \Phi_{m-1}(x, y)), \end{cases} \quad (1.4)$$

where $(\mu + m - 2)(a_1^2 + a_2^2) \neq 0$, and $\Phi_{m-1}(x, y)$ is a homogeneous polynomial of degree $m - 1$ has a weak center at the origin if and only if system (1.4) after a linear change of variables $(x, y) \rightarrow (X, Y)$ is invariant under the transformations $(X, Y, t) \rightarrow (-X, Y, -t)$. They have proved the conjecture holds for $m = 2, 3, 4, 5, 6$. And remarked that the only difficulty for proving conjecture for the Λ - Ω systems of degree m with $m > 6$ is the huge number of computations for obtaining the conditions that characterize the centers. In [21, 22] we use a method different from Llibre [14] and more simply, without huge number of computation, get the necessary and sufficient conditions for the origin point of Λ - Ω systems:

$$\begin{cases} x' = -y(1 + \mu(a_2x - a_1y)) + x(v(a_1x + a_2y) + \Psi_{m-1} + \Psi_{2m-1}), \\ y' = x(1 + \mu(a_2x - a_1y)) + y(v(a_1x + a_2y) + \Psi_{m-1} + \Psi_{2m-1}) \end{cases}$$

and

$$\begin{cases} x' = -y(1 + \mu(a_2x - a_1y)) + x(v(a_1x + a_2y) + \Psi_2 + \Psi_n), \\ y' = x(1 + \mu(a_2x - a_1y)) + y(v(a_1x + a_2y) + \Psi_2 + \Psi_n), \end{cases}$$

where $m > 2$, $n \geq 5$ and Ψ_k is a homogeneous polynomial of degree k , to be a center. Of special note is that the function Ω in the above two systems is a polynomial of missing some terms. For the polynomial differential system of higher degree, especially when the polynomial has no missing any term, it is difficult to derive the necessary conditions for the singular point being a center by either the Lyapunov's power series method or Poincaré's successor function method. Although according to Hilbert's finite basis theory, the necessary conditions must be obtained in finite steps, how much this finite number is very difficult to know [9]. To avoid finding this finite number, we will find the central conditions by determining when it has an analytic commutator.

In the following we will discuss the center problem for the Λ - Ω system (1.3) with Ω no missing any terms. Specifically, consider Λ - Ω differential systems:

$$\begin{cases} x' = -y(1 + \mu(a_1y - a_2x)) + x(\lambda(a_1x + a_2y) + H(x, y)), \\ y' = x(1 + \mu(a_1y - a_2x)) + y(\lambda(a_1x + a_2y) + H(x, y)) \end{cases} \quad (1.5)$$

and

$$\begin{cases} x' = -y(1 + \mu(a_1y - a_2x) + \phi_2(x, y)) + x(\lambda(a_1x + a_2y) + \psi_2(x, y) + H(x, y)), \\ y' = x(1 + \mu(a_1y - a_2x) + \phi_2(x, y)) + y(\lambda(a_1x + a_2y) + \psi_2(x, y) + H(x, y)), \end{cases} \quad (1.6)$$

where λ, μ, a_1, a_2 are real numbers such that $\mu(a_1^2 + a_2^2) \neq 0$, $H(x, y) = \sum_{k=2}^{\infty} h_k(x, y)$ or $H(x, y) = \sum_{k=3}^n h_k(x, y)$, $h_k(x, y)$ is a homogeneous polynomial of degree k . We will give the necessary and sufficient conditions for these two families of differential systems to have a polynomial commutator or analytic commutator, apply the obtained results to judge the origin point of their to be a center (isochronous center, weak center).

2 Analytic commutator

As $a_1^2 + a_2^2 \neq 0$, taking $X = a_1x + a_2y$, $Y = a_1y - a_2x$, the system (1.5) becomes

$$\begin{cases} X' = -Y(1 + \mu Y) + X(\lambda X + H(X, Y)), \\ Y' = X(1 + \mu Y) + Y(\lambda X + H(X, Y)). \end{cases}$$

For convenience, let us consider

$$\begin{cases} x' = -y(1 + \mu y) + x(\lambda x + H(x, y)) = P(x, y), \\ y' = x(1 + \mu y) + y(\lambda x + H(x, y)) = Q(x, y), \end{cases} \quad (2.1)$$

where $\mu \neq 0$, $H = \sum_{i=2}^{\infty} h_i(x, y)$, $h_i(x, y)$ is homogeneous polynomials of degree i . By [2], if system (2.1) has an analytic commutator, then either it has the form

$$(U, V)^t = (x + u_2 + xK(x, y), y + v_2 + yK(x, y))^t \quad (2.2)$$

or

$$(U, V)^t = (u_2 + xK(x, y), v_2 + yK(x, y))^t,$$

where $K = \sum_{i=2}^{\infty} k_i(x, y)$, $k_i(x, y)$ is homogeneous polynomials of degree i . In this paper, we are only interested in the center problem for system (2.1), therefore, we will only discuss when does (2.1) have a commutator in the form of (2.2)?

Lemma 2.1. *If $xh_{n-1} + yk_{n-1} = 0$, then*

$$u_2 h_{n-1} - P_2 k_{n-1} - x(h_{n-1x} u_2 + h_{n-1y} v_2 - k_{n-1x} P_2 - k_{n-1y} Q_2) = \mu(n-3)(x^2 + y^2)k_{n-1}, \quad (2.3)$$

where $u_2 = 2\mu xy$, $v_2 = \mu(y^2 - x^2)$, $P_2 = \mu(x^2 - y^2)$, $Q_2 = 2\mu xy$.

Proof. As $xh_{n-1} + yk_{n-1} = 0$, $xh_{n-1x} = -h_{n-1} - yk_{n-1x}$, $xh_{n-1y} = -k_{n-1} - yk_{n-1y}$, thus

$$\begin{aligned} & x(h_{n-1x} u_2 + h_{n-1y} v_2 - k_{n-1x} P_2 - k_{n-1y} Q_2) \\ &= -u_2(h_{n-1} + yk_{n-1x}) - v_2(k_{n-1} + yk_{n-1y}) - k_{n-1x} P_2 - k_{n-1y} Q_2 \\ &= \mu(x^2 + y^2)k_{n-1} - \mu(x^2 + y^2)(xk_{n-1x} + yk_{n-1y}) \\ &= (2-n)\mu(x^2 + y^2)k_{n-1}, u_2 h_{n-1} - P_2 k_{n-1} \\ &= 2\mu xy h_{n-1} - \mu(x^2 - y^2)k_{n-1} = -\mu(x^2 + y^2)k_{n-1}. \end{aligned}$$

Add the above two equations, it follows that equation (2.3) is valid. \square

Lemma 2.2. *For n -th degree homogeneous polynomial functions $h_n(x, y)$ and $k_n(x, y)$, if they satisfy*

$$xh_n + yk_n = 0, \quad (2.4)$$

and

$$x(nh_n + yk_{nx} - xk_{ny}) = (n-3)\mu(x^2 + y^2)k_{n-1}, \quad (n = 3, 4, \dots) \quad (2.5)$$

then

$$h_n = \sum_{j=0}^{n-3} (-1)^{j+1} \mu^{j+1} \lambda_{n-j} C_{n-3}^j x^{n-1-j} y^{j+1}, \quad (n = 3, 4, 5, \dots) \quad (2.6)$$

and

$$k_n = \sum_{j=0}^{n-3} (-1)^j \lambda_{n-j} \mu^j C_{n-3}^j x^{n-j} y^j, \quad (n = 3, 4, 5, \dots), \quad (2.7)$$

where λ_i ($i = 3, 4, \dots$) are real numbers.

Proof. Based on the assumptions, when $n = 3$ we have

$$xh_3 + yk_3 = 0, \quad 3h_3 + yk_{3x} - xk_{3y} = 0.$$

Putting $x = \cos \theta$, $y = \sin \theta$, the above equations become

$$\cos \theta h_3(\cos \theta, \sin \theta) + \sin \theta k_3(\cos \theta, \sin \theta) = 0,$$

$$\frac{dk_3(\cos \theta, \sin \theta)}{d\theta} = -3 \tan \theta k_3(\cos \theta, \sin \theta),$$

solving these equations we deduce that

$$k_3(\cos \theta, \sin \theta) = \lambda_3 \cos^3 \theta.$$

Similarly, when $n = 4$ we obtain

$$\frac{dk_4(\cos \theta, \sin \theta)}{d\theta} = -4 \tan \theta k_4(\cos \theta, \sin \theta) - \mu \sec \theta k_3(\cos \theta, \sin \theta),$$

solving this linear equation we have

$$k_4(\cos \theta, \sin \theta) = \cos^4 \theta (\lambda_4 - \lambda_3 \mu \tan \theta).$$

Suppose that

$$k_n(\cos \theta, \sin \theta) = \cos^n \theta \sum_{j=0}^{n-3} (-1)^j C_{n-3}^j \lambda_{n-j} \mu^j \tan^j \theta. \quad (2.8)$$

Next we will prove that (2.8) is also true when n is replaced by $n + 1$.

In fact, by assuming we obtain

$$\frac{dk_{n+1}(\cos \theta, \sin \theta)}{d\theta} = -(n+1) \tan \theta k_{n+1} - (n-2) \mu \sec \theta k_n(\cos \theta, \sin \theta).$$

Substituting (2.8) into the above yields

$$\frac{dk_{n+1}(\cos \theta, \sin \theta)}{d\theta} = -(n+1) \tan \theta k_{n+1} - (n-2) \cos^{n-1} \theta \left(\sum_{j=0}^{n-3} (-1)^j C_{n-3}^j \lambda_{n-j} \mu^{j+1} \tan^j \theta \right),$$

solving this linear equation we get

$$\begin{aligned} k_{n+1} &= \cos^{n+1} \theta \left(\lambda_{n+1} - (n-2) \int \cos^{-2} \theta \left(\sum_{j=0}^{n-3} (-1)^j C_{n-3}^j \lambda_{n-j} \mu^{j+1} j \tan^j \theta \right) d\theta \right) \\ &= \cos^{n+1} \theta \left(\lambda_{n+1} - (n-2) \sum_{j=0}^{n-3} (-1)^j C_{n-3}^j \lambda_{n-j} \frac{1}{j+1} \mu^{j+1} \tan^{j+1} \theta \right) \\ &= \cos^{n+1} \theta \sum_{j=0}^{n-2} (-1)^j C_{n-2}^j \lambda_{n+1-j} \mu^j \tan^j \theta. \end{aligned}$$

Therefore, by mathematical induction, the relation (2.8) is valid for any $n \geq 3$. So, the relations (2.6) and (2.7) are valid. \square

Theorem 2.3. *The system (2.1) has an analytic commutator in the form of (2.2), if and only if*

$$\lambda = \mu,$$

$$u_2 = 2\mu xy, \quad v_2 = \mu(y^2 - x^2),$$

$$h_2 = -\lambda_2 xy, \quad k_2 = \lambda_2 x^2,$$

$$h_n = \sum_{j=0}^{n-3} (-1)^{j+1} \lambda_{n-j} C_{n-3}^j x^{n-1-j} y^{j+1}, \quad (n = 3, 4, 5, \dots)$$

$$k_n = \sum_{j=0}^{n-3} (-1)^j \lambda_{n-j} C_{n-3}^j x^{n-j} y^j, \quad (n = 3, 4, 5, \dots),$$

where λ_i ($i = 2, 3, 4, \dots$) are real numbers.

Moreover, the origin point of (2.1) is a center and isochronous center.

Proof. By [2], the vector (2.2) is an commutator of system (2.1) if and only if the Lie bracket vanishes, that is,

$$\begin{pmatrix} U_x & U_y \\ V_x & V_y \end{pmatrix} \cdot \begin{pmatrix} P \\ Q \end{pmatrix} - \begin{pmatrix} P_x & P_y \\ Q_x & Q_y \end{pmatrix} \cdot \begin{pmatrix} U \\ V \end{pmatrix} = 0, \quad (2.9)$$

expanding it

$$\begin{aligned} & \left(1 + u_{2x} + \sum_{i=2}^{\infty} (xk_i)_x\right) \left(-y + P_2 + \sum_{i=2}^{\infty} xh_i\right) + \left(u_{2y} + \sum_{i=2}^{\infty} (xk_i)_y\right) \left(x + Q_2 + \sum_{i=2}^{\infty} yh_i\right) \\ &= \left(P_{2x} + \sum_{i=2}^{\infty} (xh_i)_x\right) \left(x + u_2 + \sum_{i=2}^{\infty} xk_i\right) + \left(-1 + P_{2y} + \sum_{i=2}^{\infty} (xh_i)_y\right) \left(y + v_2 + \sum_{i=2}^{\infty} yk_i\right), \end{aligned} \quad (2.10)$$

$$\begin{aligned} & \left(v_{2x} + \sum_{i=2}^{\infty} (yk_i)_x\right) \left(-y + P_2 + \sum_{i=2}^{\infty} xh_i\right) + \left(1 + v_{2y} + \sum_{i=2}^{\infty} (yk_i)_y\right) \left(x + Q_2 + \sum_{i=2}^{\infty} yh_i\right) \\ &= \left(1 + Q_{2x} + \sum_{i=2}^{\infty} (yh_i)_x\right) \left(x + u_2 + \sum_{i=2}^{\infty} xk_i\right) + \left(Q_{2y} + \sum_{i=2}^{\infty} (yh_i)_y\right) \left(y + v_2 + \sum_{i=2}^{\infty} yk_i\right), \end{aligned} \quad (2.11)$$

where $P_2 = \lambda x^2 - \mu y^2$, $Q_2 = (\lambda + \mu)xy$.

From the terms of degree 2 of (2.10) and (2.11) equal to zero follows that

$$\begin{cases} v_2 = P_2 + yu_{2x} - xu_{2y}, \\ u_2 = -Q_2 + xv_{2y} - yv_{2x}. \end{cases} \quad (2.12)$$

Solving (2.12) we get

$$u_2 = (\lambda + \mu)xy, \quad v_2 = \lambda y^2 - \mu x^2. \quad (2.13)$$

By the terms of degree 3 of (2.10) and (2.11) equal to zero we obtain

$$\begin{cases} u_{2x}P_2 + u_{2y}Q_2 - P_{2x}u_2 - P_{2y}v_2 = x(2h_2 + yk_{2x} - xk_{2y}), \\ v_{2x}P_2 + v_{2y}Q_2 - Q_{2x}u_2 - Q_{2y}v_2 = y(2h_2 + yk_{2x} - xk_{2y}). \end{cases} \quad (2.14)$$

The first equation of above multiplied by y minus the second equation multiplied by x , we deduce that

$$P_2(yu_{2x} - xv_{2x}) + Q_2(yu_{2y} - xv_{2y}) = u_2(yP_{2x} - xQ_{2x}) + v_2(yP_{2y} - xQ_{2y}),$$

which yields $\mu(\lambda - \mu) = 0$, in view of $\mu \neq 0$, then $\lambda = \mu$. Therefore,

$$u_2 = 2\mu xy, \quad v_2 = \mu(y^2 - x^2), \quad P_2 = \mu(x^2 - y^2), \quad Q_2 = 2\mu xy. \quad (2.15)$$

Substituting (2.15) into (2.14) which follows that

$$2h_2 + yk_{2x} - xk_{2y} = 0. \quad (2.16)$$

From the terms of degree 4 of equations (2.10) and (2.11) equal to zero follows that

$$u_2h_2 - P_2k_2 = x(3h_3 + h_{2x}u_2 + h_{2y}v_2 - k_{2x}P_2 - k_{2y}Q_2 + yk_{3x} - xk_{3y}), \quad (2.17)$$

$$v_2h_2 - Q_2k_2 = y(3h_3 + h_{2x}u_2 + h_{2y}v_2 - k_{2x}P_2 - k_{2y}Q_2 + yk_{3x} - xk_{3y}). \quad (2.18)$$

Equation (2.17) multiplied by y minus (2.18) multiplied by x which implies that

$$h_2(yu_2 - xv_2) = k_2(yP_2 - xQ_2),$$

substituting (2.15) into the above equation we get

$$xh_2 + yk_2 = 0. \quad (2.19)$$

Solving equations (2.16) and (2.19) we deduce that

$$h_2 = -\lambda_2 xy, \quad k_2 = \lambda_2 x^2, \quad (2.20)$$

where λ_2 is a constant. Substituting (2.20) into (2.17) we get

$$3h_3 + yk_{3x} - xk_{3y} = 0. \quad (2.21)$$

Similarly, by the terms of degree $n + 1$ of equations (2.10) and (2.11) equal to zero we deduce that

$$u_2 h_{n-1} - P_2 k_{n-1} = x(nh_n + h_{n-1}x u_2 + h_{n-1}y v_2 - k_{n-1}x P_2 - k_{n-1}y Q_2 + yk_{nx} - xk_{ny}), \quad (2.22)$$

$$v_2 h_{n-1} - Q_2 k_{n-1} = y(nh_n + h_{n-1}x u_2 + h_{n-1}y v_2 - k_{n-1}x P_2 - k_{n-1}y Q_2 + yk_{nx} - xk_{ny}), \quad (2.23)$$

from these equations follow that

$$xh_{n-1} + yk_{n-1} = 0. \quad (2.24)$$

Using (2.24) and Lemma 2.1 we get

$$u_2 h_{n-1} - P_2 k_{n-1} - x(h_{n-1}x u_2 + h_{n-1}y v_2 - k_{n-1}x P_2 - k_{n-1}y Q_2) = \mu(n-3)(x^2 + y^2)k_{n-1}.$$

Substituting this equation into (2.22) which yields that

$$x(nh_n + yk_{nx} - xk_{ny}) = (n-3)\mu(x^2 + y^2)k_{n-1}. \quad (2.25)$$

Similarly, using the terms of degree $n + 2$ of (2.10) and (2.11) equal to zero we obtain

$$xh_n + yk_n = 0. \quad (2.26)$$

By equations (2.25) and (2.26) and Lemma 2.1 imply that k_n, h_n can be expressed by (2.6) and (2.7).

By [1,2], the origin point of (2.1) is a center and isochronous center.

In summary, the proof is finished. \square

Corollary 2.4. *If in the system (2.1), $H(x, y) = \sum_{i=2}^n h_i(x, y), h_i(x, y)$ ($i = 2, 1, \dots, n$) are homogeneous polynomials of degree i , and it has a polynomial commutator in the form of (2.2), if and only if,*

$$\lambda = \mu; \quad u_2 = 2\mu xy, \quad v_2 = \mu(y^2 - x^2); \quad h_2 = -\lambda_2 xy, \quad k_2 = \lambda_2 x^2; \quad h_j = k_j = 0 \quad (j = 3, 4, \dots, n).$$

Proof. By the proof of Theorem 2.3 we get that

$$\lambda = \mu, \quad u_2 = 2\mu xy, \quad v_2 = \mu(y^2 - x^2), \quad h_2 = -\lambda_2 xy, \quad k_2 = \lambda_2 x^2$$

and

$$x(jh_j + yk_{jx} - xk_{jy}) = (j-3)\mu(x^2 + y^2)k_{j-1} \quad (j = 3, 4, \dots, n, n+1)$$

and

$$xh_j + yk_j = 0, \quad (j = 3, 4, \dots, n, n+1).$$

Taking $j = n + 1$, we get $k_n = 0$ and $h_n = 0$, substituting thus into the above equations with $j = n$ which implies that k_{n-1} and $h_{n-1} = 0$, as so on we can deduce $k_j = 0$ and $h_j = 0$, ($j = 3, 4, \dots, n$). Thus the proof is completed. \square

3 Polynomial commutator

In this section we will discuss when does the system (1.6) with $\lambda = \mu \neq 0$ and $h_3 \neq 0$, have a polynomial commutator in the form

$$\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} x + u_2 + u_3 + xk_3(x, y) \\ y + v_2 + v_3 + yk_3(x, y) \end{pmatrix}, \quad (3.1)$$

where $u_l(x, y) = \sum_{i+j=l} u_{ij}x^i y^j$, $v_l(x, y) = \sum_{i+j=l} v_{ij}x^i y^j$, ($l = 2, 3$); $k_3 = \sum_{i+j=3} k_{ij}x^i y^j$.

Without losing generality, suppose that $\mu = 1$, otherwise taking $X = \mu x$, $Y = \mu y$.

First, let us consider system

$$\begin{cases} x' = -y(1 + y) + P_3(x, y) + x(x + h_3(x, y)) = P(x, y), \\ y' = x(1 + y) + Q_3(x, y) + y(x + h_3(x, y)) = Q(x, y), \end{cases} \quad (3.2)$$

where $P_3 = \sum_{i+j=3} p_{ij}x^i y^j$, $Q_3 = \sum_{i+j=3} q_{ij}x^i y^j$, $h_3 = \sum_{i+j=3} h_{ij}x^i y^j$.

Theorem 3.1. *The system (3.2) with $P_3 \cdot h_3 \neq 0$, has a polynomial commutator in the form of (3.1), if and only if*

$$p_{03}^2 - p_{30}(p_{12} + 2p_{30}) = 0, \quad (3.3)$$

$$p_{21}p_{30} + p_{12}p_{03} + 6p_{30}p_{03} = 0, \quad (3.4)$$

$$Q_3 = p_{03}x^3 - p_{12}x^2y + p_{21}xy^2 - p_{30}y^3, \quad (3.5)$$

$$u_2 = 2xy, \quad v_2 = y^2 - x^2, \quad (3.6)$$

$$u_3 = -(p_{21} + 2p_{03})x^3 - p_{12}x^2y - 3p_{03}xy^2 - p_{30}y^3, \quad (3.7)$$

$$v_3 = (2p_{30} + p_{12})x^3 - p_{21}x^2y + 3p_{30}xy^2 - p_{03}y^3,$$

$$h_3 = (p_{12} + 3p_{30})x(- (p_{12} + 2p_{30})x^2 + (2p_{12} + 3p_{30})y^2) - (p_{21} + 3p_{03})y(p_{12}x^2 + p_{30}y^2), \quad (3.8)$$

$$k_3 = (p_{12} + 2p_{30})x^2((p_{21} + 3p_{03})x - 3(p_{12} + 2p_{30})y) + p_{30}y^2(3(p_{21} + 3p_{03})x - (p_{12} + 3p_{30})y). \quad (3.9)$$

Moreover, the origin point of (3.2) is a center and isochronous center.

Proof. By (2.9), the vector (3.1) is a commutator of (3.2), if and only if

$$\begin{aligned} & (1 + u_{2x} + u_{3x} + (xk_3)_x)(-y + P_2 + P_3 + xh_3) + (u_{2y} + u_{3y} + (xk_3)_y)(x + Q_2 + Q_3 + yh_3) \\ &= (P_{2x} + P_{3x} + (xh_3)_x)(x + u_2 + u_3 + xk_3) \\ &+ (-1 + P_{2y} + P_{3y} + (xh_3)_y)(y + v_2 + v_3 + yk_3), \end{aligned} \quad (3.10)$$

$$\begin{aligned} & (v_{2x} + v_{3x} + (yk_3)_x)(-y + P_2 + P_3 + xh_3) + (1 + v_{2y} + v_{3y} + (yk_3)_y)(x + Q_2 + Q_3 + yh_3) \\ &= (1 + Q_{2x} + Q_{3x} + (yh_3)_x)(x + u_2 + u_3 + xk_3) \\ &+ (Q_{2y} + Q_{3y} + (yh_3)_y)(y + v_2 + v_3 + yk_3), \end{aligned} \quad (3.11)$$

where $P_2 = x^2 - y^2$, $Q_2 = 2xy$.

Similar to the proof of Theorem 2.3, from the terms of degree 2 of equations (3.10) and (3.11) equal to zero follows that (2.12) and (2.13) are valid. By the terms of degree 3 of equations (3.10) and (3.11) equal to zero we deduce that

$$\begin{cases} v_3 + xu_{3y} - yu_{3x} - 2P_3 = 0, \\ u_3 + yv_{3x} - xv_{3y} + 2Q_3 = 0. \end{cases} \quad (3.12)$$

Equating the same power of x and y of (3.12) which yields that

$$u_{12} = u_{30} + p_{21} - q_{30}, \quad u_{21} = u_{03} - p_{12} - q_{03}, \quad (3.13)$$

$$p_{21} - q_{12} + 3(p_{03} - q_{30}) = 0, \quad p_{12} + q_{21} + 3(p_{30} + q_{03}) = 0. \quad (3.14)$$

$$v_{30} = -u_{03} + 2p_{30} + p_{12} + q_{03}, \quad v_{21} = u_{30} + 2q_{30}, \quad (3.15)$$

$$v_{12} = -u_{03} - 2q_{03}, \quad v_{03} = u_{30} + p_{21} - q_{30} + 2p_{03}.$$

By the terms of degree 4 of equations (3.10) and (3.11) equal to zero we get

$$u_{2x}P_3 + u_{2y}Q_3 + u_{3x}P_2 + u_{3y}Q_2 - u_3P_{2x} - v_3P_{2y} - u_2P_{3x} - v_2P_{3y} = x(3h_3 + yk_{3x} - xk_{3y}), \quad (3.16)$$

$$v_{2x}P_3 + v_{2y}Q_3 + v_{3x}P_2 + v_{3y}Q_2 - u_3Q_{2x} - v_3Q_{2y} - u_2Q_{3x} - v_2Q_{3y} = y(3h_3 + yk_{3x} - xk_{3y}). \quad (3.17)$$

Equation (3.16) multiplied by y minus (3.17) multiplied by x which implies that

$$\begin{aligned} & P_3(yu_{2x} - xv_{2x}) + Q_3(yu_{2y} - xv_{2y}) + P_2(yu_{3x} - xv_{3y}) + Q_2(yu_{3x} - xv_{3y}) \\ & = u_3(yP_{2x} - xQ_{2x}) + v_3(yP_{2y} - xQ_{2y}) + u_2(yP_{3x} - xQ_{3x}) + v_2(yP_{3y} - xQ_{3y}). \end{aligned} \quad (3.18)$$

Comparing the coefficients of the same power of x and y on both sides of the equation (3.18) and (3.16) we obtain

$$u_{03} = p_{12} + q_{03} + q_{21}, \quad u_{30} = -p_{21} + q_{30} - 3p_{03}, \quad u_{30} = -3p_{21} - 2q_{30} + 2q_{12},$$

$$u_{03} = q_{03}, \quad 2u_{03} = -6p_{30} - 5p_{12} - 4q_{03} - 5q_{21}, \quad 2u_{30} = 2p_{21} + 3p_{03} - 4q_{12} - 7q_{30},$$

$$v_{30} = 2p_{30} - q_{21}, \quad u_{12} - 2v_{03} = -p_{03}, \quad 2v_{21} - 3u_{30} = 3p_{21} + 6q_{30} - 2q_{12},$$

$$2u_{21} - 3v_{12} - 6u_{03} = -2p_{12} + 3q_{03}, \quad 3v_{12} - 5v_{30} - 4u_{21} = -4p_{30} + 4p_{12} + 5q_{21} - 3q_{03},$$

$$3u_{30} + 4v_{03} - 4v_{21} - 5u_{12} = -3p_{21} + 5p_{03} + 4q_{12}.$$

According to the above equations and (3.13) and (3.15) we get

$$p_{03} - q_{30} = 0, \quad p_{30} + q_{03} = 0, \quad q_{12} - p_{21} = 0, \quad p_{12} + q_{21} = 0,$$

$$u_{30} = -p_{21} - 2p_{03}, \quad u_{21} = -p_{12}, \quad u_{12} = -3p_{03}, \quad u_{03} = -p_{30},$$

$$v_{30} = p_{12} + 2p_{30}, \quad v_{21} = -p_{21}, \quad v_{12} = 3p_{03}, \quad v_{03} = -p_{03}.$$

Consequently, the relations (3.5) and (3.7) are valid.

Using (3.5)–(3.7) and (3.16) we deduce that

$$3h_3 + yk_{3x} - xk_{3y} = 0. \quad (3.19)$$

By this equation we get

$$k_3 = -(h_{21} + 2h_{03})x^3 + 3h_{30}x^2y - 3h_{03}xy^2 + (h_{12} + 2h_{30})y^3. \quad (3.20)$$

From the terms of degree 5 of equations (3.10) and (3.11) equal to zero which follows that

$$u_2h_3 - P_2k_3 + u_{3x}P_3 + u_{3y}Q_3 - P_{3x}u_3 - P_{3y}v_3 = x(h_{3x}u_2 + h_{3y}v_2 - k_{3x}P_2 - k_{3y}Q_2), \quad (3.21)$$

$$v_2h_3 - Q_2k_3 + v_{3x}P_3 + v_{3y}Q_3 - Q_{3x}u_3 - Q_{3y}v_3 = y(h_{3x}u_2 + h_{3y}v_2 - k_{3x}P_2 - k_{3y}Q_2). \quad (3.22)$$

Equation (3.21) multiplied by y minus (3.22) multiplied by x which implies that

$$\begin{aligned} h_3(yu_2 - xv_2) - k_3(yP_2 - xQ_2) + P_3(yu_{3x} - xv_{3x}) + Q_3(yu_{3y} - xv_{3y}) \\ = u_3(yP_{3x} - xQ_{3x}) + v_3(yP_{3y} - xQ_{3y}). \end{aligned} \quad (3.23)$$

Comparing the coefficients of the same power of x and y on both sides of equations (3.21) and (3.23) and using (3.5)–(3.7) and (3.20) we get

$$\begin{aligned} 4h_{03} + h_{21} &= -2p_{21}p_{30} - p_{12}p_{03} - p_{21}p_{12}, \\ 2h_{12} + 5h_{30} &= 2p_{21}p_{03} + 3p_{12}p_{30} + p_{12}^2 + 6p_{03}^2, \\ h_{21} + 5h_{03} &= -3p_{30}p_{03} - p_{12}p_{21} - p_{03}p_{12} - 3p_{30}p_{21}, \\ 3h_{12} + 7h_{30} &= 6p_{03}^2 - 3p_{30}^2 + p_{12}^2 + 2p_{12}p_{30} + 2p_{21}p_{03}, \\ h_{03} &= -3p_{30}p_{03} - p_{30}p_{21}, \\ h_{12} + 2h_{30} &= -3p_{30}^2 - p_{12}p_{30}, \\ h_{30} &= 6p_{30}^2 + 6p_{03}^2 + 5p_{30}p_{12} + 2p_{21}p_{03} + p_{12}^2, \\ h_{12} + 4h_{30} &= 9p_{30}^2 + 12p_{03}^2 + 9p_{30}p_{12} + 4p_{21}p_{03} + 2p_{12}^2. \end{aligned}$$

Simplifying the above equations to obtain

$$h_{30} = 6p_{30}^2 + 6p_{03}^2 + 5p_{30}p_{12} + 2p_{21}p_{03} + p_{12}^2; \quad (3.24)$$

$$h_{21} = 2p_{21}p_{30} - p_{12}p_{03} - p_{21}p_{12} + 12p_{30}p_{03}, \quad (3.25)$$

$$h_{12} = -15p_{30}^2 - 12p_{03}^2 - 11p_{30}p_{12} - 4p_{21}p_{03} - 2p_{12}^2, \quad (3.26)$$

$$h_{03} = -3p_{30}p_{03} - p_{21}p_{30}. \quad (3.27)$$

By the terms of degree 6 of equations (3.10) and (3.11) equal to zero we get

$$2(u_3h_3 - P_3k_3) = x(h_{3x}u_3 + h_{3y}v_3 - k_{3x}P_3 - k_{3y}Q_3), \quad (3.28)$$

$$2(v_3h_3 - Q_3k_3) = y(h_{3x}u_3 + h_{3y}v_3 - k_{3x}P_3 - k_{3y}Q_3). \quad (3.29)$$

Equation (3.28) multiplied by y minus (3.29) multiplied by x we deduce that

$$h_3(yu_3 - xv_3) = k_3(yP_3 - xQ_3). \quad (3.30)$$

Equating the coefficients of the same power of x and y on both sides of the equation (3.28) we obtain

$$\begin{aligned} (p_{21} + 5p_{03})h_{30} - (p_{12} + 3p_{30})h_{21} - 2p_{30}h_{03} &= 0, \\ p_{12}h_{30} + (p_{12} + 2p_{30})h_{12} + (p_{21} + 3p_{03})h_{03} &= 0, \\ 2p_{21}h_{30} + (p_{21} - p_{03})h_{12} + (p_{12} - 3p_{30})h_{03} &= 0, \\ (2p_{12} + 3p_{30})h_{30} + (p_{12} + 2p_{30})h_{12} &= 0, \\ 2p_{03}h_{30} + p_{03}h_{12} + p_{30}h_{03} &= 0. \end{aligned}$$

As $h_3 \neq 0$, the determinant of the coefficient matrix of four equations of the above is equal to zero, that is

$$W_1 = p_{30}(p_{12} + 3p_{30})(p_{21}p_{30} + p_{12}p_{03} + 6p_{30}p_{03}) = 0. \quad (3.31)$$

Equating the coefficients of the same power of x and y on both sides of the equation (3.30) we get

$$\begin{aligned}(p_{12} + 2p_{30})h_{30} + p_{03}h_{21} + 2p_{03}h_{03} &= 0, \\ p_{03}h_{30} - p_{30}h_{21} + 2(p_{12} + p_{30})h_{03} &= 0, \\ 2(p_{12} + p_{30})h_{30} + (p_{12} + 2p_{30})h_{12} - p_{03}h_{03} &= 0, \\ 2p_{03}h_{30} + p_{03}h_{12} + p_{30}h_{03} &= 0.\end{aligned}$$

The determinant of the coefficient matrix of the above equations is equal to zero, that is

$$W_2 = -(p_{30}(p_{12} + 2p_{30}) - p_{03}^2)^2 = 0. \quad (3.32)$$

By (3.31) and (3.32) and $P_3 \cdot h_3 \neq 0$, which implies that $p_{30} \cdot p_{03} \cdot (p_{12} + 2p_{30}) \neq 0$ and the relations (3.3) and (3.4) are valid and

$$h_{12} = -\frac{2p_{12} + 2p_{30}}{p_{12} + 2p_{30}}h_{30}, \quad h_{21} = -\frac{p_{12}p_{30}}{p_{03}(p_{12} + 2p_{30})}h_{30}, \quad h_{03} = -\frac{p_{03}}{p_{12} + 2p_{30}}h_{30}.$$

Using (3.4) and (3.5) and (3.24)–(3.27) we get

$$\begin{aligned}h_{30} &= -(p_{12} + 2p_{30})(p_{12} + 3p_{30}), & h_{21} &= -p_{12}(p_{21} + 3p_{03}), \\ h_{12} &= (p_{12} + 3p_{30})(3p_{30} + 2p_{12}), & h_{03} &= -p_{30}(p_{21} + 3p_{03}), \\ k_{30} &= (p_{21} + 3p_{03})(p_{12} + 2p_{30}), & k_{21} &= -3(p_{12} + 2p_{30})(p_{12} + 3p_{30}), \\ k_{12} &= 3p_{30}(p_{21} + 3p_{03}), & k_{03} &= -p_{30}(p_{12} + 3p_{30}).\end{aligned}$$

Therefore, the functions u_3, v_3, h_3, k_3 are expressed by (3.7)–(3.9), respectively.

By [1] [2], the origin point of (3.2) is a center and isochronous center. \square

Consider Λ - Ω system

$$\begin{cases} x' = -y(1 + y + \phi_2(x, y)) + x(x + \psi_2(x, y) + h_3(x, y)), \\ y' = x(1 + y + \phi_2(x, y)) + y(x + \psi_2(x, y) + h_3(x, y)), \end{cases} \quad (3.33)$$

where $\phi_2 = a_{20}x^2 + a_{11}xy + a_{02}y^2$, $\psi_2 = b_{20}x^2 + b_{11}xy + b_{02}y^2$, a_{ij}, b_{ij} are real numbers.

By Theorem 3.1, taking $P_3 = -y\phi_2 + x\psi_2$, $Q_3 = x\phi_2 + y\psi_2$, which follows the following corollary.

Corollary 3.2. *The system (3.33) has a polynomial commutator in the form of*

$$\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} x + u_2 + u_3 + xk_3(x, y) \\ y + v_2 + v_3 + yk_3(x, y) \end{pmatrix}$$

if and only if

$$a_{20} + a_{02} = 0, \quad b_{20} + b_{02} = 0, \quad a_{20}^2 - b_{20}^2 + b_{20}a_{11} = 0, \quad b_{11}b_{20} - a_{11}a_{20} + 4a_{20}b_{20} = 0.$$

$$u_2 = 2xy, \quad v_2 = y^2 - x^2,$$

$$u_3 = -(b_{11} + a_{20})x^3 + (b_{20} + a_{11})x^2y - 3a_{20}xy^2 - b_{20}y^3,$$

$$v_3 = (b_{20} - a_{11})x^3 - (b_{11} - a_{20})x^2y + 3b_{20}xy^2 - a_{20}y^3,$$

$$h_3 = (2b_{20} - a_{11})((a_{11} - b_{20})x^3 + (b_{20} - 2a_{11})x^2y) + (b_{11} + 2a_{20})((a_{11} + b_{20})x^2y - b_{20}y^3),$$

$$k_3 = (2a_{20} + b_{11})((b_{20} - a_{11})x^3 + 3b_{20}xy^2) - (2b_{20} - a_{11})(3(b_{20} - a_{11})x^2y + b_{20}y^3).$$

Moreover, the origin point of (3.33) is a center and isochronous center and weak center.

4 Examples

In Theorem 2.3, taking $\lambda = \mu = \lambda_i = 1$ ($i = 2, 3, \dots$) which yields the following example.

Example 4.1. Λ - Ω -differential system

$$\begin{cases} x' = -y(1+y) + x(x - xy - x^2y \sum_{j=3}^{\infty} (x-y)^{j-3}), \\ y' = x(1+y) + y(x - xy - x^2y \sum_{j=3}^{\infty} (x-y)^{j-3}) \end{cases} \quad (4.1)$$

has a commutator

$$\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} x + 2xy + x(x^2 + x^3 \sum_{j=3}^{\infty} (x-y)^{j-3}) \\ y + y^2 - x^2 + y(x^2 + x^3 \sum_{j=3}^{\infty} (x-y)^{j-3}) \end{pmatrix}$$

and the origin point of (4.1) is a center and isochronous center and weak center.

In Theorem 3.1 taking $p_{30} = 1$, $p_{21} = -5$, $p_{12} = -1$, $p_{03} = 1$ which implies the following example.

Example 4.2. Differential system

$$\begin{cases} x' = -y(1+y) + x^3 - 5x^2y - xy^2 + y^3 + x(x - 2(x+y)(x^2 - y^2)), \\ y' = x(1+y) + x^3 + x^2y - 5xy^2 - y^3 + y(x - 2(x+y)(x^2 - y^2)) \end{cases} \quad (4.2)$$

has a commutator

$$\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} x + 2xy + 3x^3 + x^2y - 3xy^2 - y^3 - 2x(x+y)^3 \\ y + y^2 - x^2 + x^3 + 5x^2y + 3xy^2 - y^3 - 2y(x+y)^3 \end{pmatrix} \quad (4.3)$$

and the origin point of (4.2) is a center and isochronous center.

In Corollary 3.1 taking $a_{20} = 1$, $a_{11} = 0$, $a_{02} = -1$, $b_{20} = 1$, $b_{11} = -4$, $b_{02} = -1$ we deduce the following example.

Example 4.3. Λ - Ω differential system

$$\begin{cases} x' = -y(1+y+x^2-y^2) + x(x+x^2-y^2-4xy-2(x+y)(x^2-y^2)), \\ y' = x(1+y+x^2-y^2) + y(x+x^2-y^2-4xy-2(x+y)(x^2-y^2)) \end{cases} \quad (4.4)$$

has a commutator (4.3) and the origin point of (4.4) is a center and weak center.

The above three examples have been verified and they are correct.

Acknowledgements

Funding: This study was funded by the National Natural Science Foundation of China (62173292, 12171418).

Conflict of interest: The author declares that she has no conflict of interest.

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