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On oscillation of solutions of scalar delay differential equation in critical case

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Abstract. In this paper we study the oscillation problem for the known scalar delay differential equation. We assume that the coefficients of this equation have an oscillatory behaviour with an amplitude of oscillation tending to zero at infinity. The asymptotic formulae for the solutions of the considered equation in the so-called critical case are constructed. We give the conditions for existence of oscillatory or nonoscillatory solutions in terms of certain numerical quantities. The obtained results are illustrated by a number of examples.

Keywords: asymptotic integration, delay differential equation, oscillation problem, center manifold, Levinson's theorem.

2020 Mathematics Subject Classification: 34K06, 34K11, 34K19, 34K25, 34C29.

1 Problem statement

In this paper, we construct the asymptotics as $t \to \infty$ for solutions of the following scalar differential equation with variable delay:

$$\dot{x} = -a(t)x(t-\tau(t)), \qquad t \geq t_0 > 0. \tag{1.1}$$

Here $a(t)$ and $\tau(t)$ are real-valued and continuous functions on $[t_0, \infty)$. Further we will impose some additional restrictions on these functions.

One of the main questions usually considered for Eq. (1.1) concerns the oscillation problem of its solutions. Choose $h > 0$ such that $0 \leq \tau(t) \leq h$ for $t \geq T \geq t_0$. By a solution of (1.1) for $t \geq T$, we mean a function $x(t)$ which is continuous on $[T-h, \infty)$, differentiable on $[T, \infty)$ and satisfies (1.1) for $t \geq T$ (by the derivative at $t = T$, we mean the right-hand side derivative). Such a solution $x(t)$ of Eq. (1.1) is said to be *oscillatory* if it has arbitrarily large zeroes. Otherwise, it is called *nonoscillatory*. Evidently, $x(t)$ is nonoscillatory if it is eventually *positive* or eventually *negative*.

The oscillation problem for Eq. (1.1) was studied by many authors. The systematic study of equation (1.1) was started by A. D. Myshkis in [23] (see also [24]). Among the works

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dealing with the oscillation problem we note the results obtained in [14, 15, 17, 21, 34], the series of papers by J. Diblík et al. [7–9, 11, 13], M. Pituk et al. [29, 31, 32], K. M. Chudinov [3–5]. In some of the mentioned papers the oscillation problem is solved by constructing the asymptotic formulae for solutions. The asymptotic properties of solutions of Eq. (1.1) are also studied in [10, 12, 18, 30]. Of course, the mentioned list of papers is not exhaustive due to the enormous amount of studies devoted to the analysis of dynamics of solutions to Eq. (1.1). More references concerning this topic can be found in the lists of cited literature in the mentioned papers. We also note paper [33] that contains the extensive review of works on this subject. Below we give two well-known criteria on oscillation of solutions to Eq. (1.1). In particular, this will allow us to refine the formulation of the problem considered in this paper.

The first of the announced results refers to the equation (1.1) with a constant delay $\tau(t) \equiv \tau$ provided that $a(t) > 0$ as $t \geq t_0$. Let us introduce the following notation. We will denote by $\ln_m t$, where $m \geq 1$, the expression, defined by the formula $\ln_m t = \ln(\ln_{m-1} t)$ and $\ln_0 t = t$. The following theorem holds [9].

Theorem 1.1.

A. *Let us assume that* $a(t) \leq a_m(t)$ *for* $t \to \infty$ *and an integer* $m \geq 0$*, where*

$$a_m(t) = \frac{1}{e\tau} + \frac{\tau}{8et^2} + \frac{\tau}{8e(t\ln t)^2} + \frac{\tau}{8e(t\ln t\ln_2 t)^2} + \dots + \frac{\tau}{8e(t\ln t\ln_2 t\dots\ln_m t)^2}.$$

Then there exists a positive solution $x = x(t)$ *of* (1.1)*. Moreover,*

$$x(t) < e^{-t/\tau}\sqrt{t\ln t\ln_2 t\dots\ln_m t}$$

as $t \to \infty$.

B. *Let us assume that*

$$a(t) > a_{m-2}(t) + \frac{\theta\tau}{8e(t\ln t\ln_2 t\dots\ln_{m-1} t)^2}$$

if $t \to \infty$*, an integer* $m \geq 2$ *and a constant* $\theta > 1$*. Then all the solutions of* (1.1) *oscillate.*

In [13], the authors generalize certain results of Theorem 1.1 to the case of Eq. (1.1) with variable delay $\tau(t)$. One more result on the oscillation of solutions of Eq. (1.1) we would like to point out is due to Koplatadze and Chanturiya [21].

Theorem 1.2. *If* $a(t) \geq 0$, $\tau(t) \geq 0$, $t - \tau(t) \to +\infty$ *as* $t \to +\infty$ *and*

$$\liminf_{t\to+\infty} \int_{t-\tau(t)}^{t} a(s)ds > \frac{1}{e},$$

then all solutions of Eq. (1.1) *oscillate. Conversely, if there exists* $t_0 \geq 0$ *such that*

$$\int_{t-\tau(t)}^{t} a(s)ds \leq \frac{1}{e}.$$

for $t \geq t_0$ *then Eq.* (1.1) *has a nonoscillatory solution.*

The development of the ideas concerning the improvement of the results of Theorem 1.2 may be found, e.g., in [34].

The most difficult situation in the oscillation problem occurs in the so called critical case [15] when

$$\lim_{t\to+\infty} a(t) = \frac{1}{e\tau}, \qquad \lim_{t\to+\infty} \tau(t) = \tau > 0. \tag{1.2}$$

It is known that in this case equation (1.1) may have oscillatory solutions although the «limit equation»

$$\dot{x} = -\frac{1}{e\tau}x(t-\tau), \qquad \tau > 0$$

has positive solution $x(t) = e^{-t/\tau}$. To obtain any general results in this situation is a challenging task. It is necessary to take into account some additional properties of the functions $a(t)$ and $\tau(t)$, in particular, the rate of their tending to limit values in (1.2) and the character of this tending.

In our paper we consider Eq. (1.1) provided that the functions $a(t)$ and $\tau(t)$ have the following asymptotic expansions as $t \to \infty$:

$$a(t) = \frac{1}{e} + a_1(t)t^{-\rho} + a_2(t)t^{-2\rho} + \cdots + a_{k+1}(t)t^{-(k+1)\rho} + O(t^{-(k+2)\rho}), \tag{1.3}$$

$$\tau(t) = 1 + q_1(t)t^{-\rho} + q_2(t)t^{-2\rho} + \cdots + q_{k+1}(t)t^{-(k+1)\rho} + O(t^{-(k+2)\rho}), \tag{1.4}$$

where $\rho > 0$ and $k \in \mathbb{N}$ is chosen such that

$$(k+1)\rho > 1. \tag{1.5}$$

Functions $a_j(t)$, $q_j(t)$, $j = 1, \ldots, k+1$, are finite trigonometric polynomials. Since functions $a(t)$ and $\tau(t)$, in general, oscillate around the limit values, Theorem 1.1 and Theorem 1.2, as well as some other similar results, fail in this case. In this paper we construct the asymptotics as $t \to \infty$ for solutions of Eq. (1.1). The obtained asymptotic formulae will allow us to solve the oscillation problem for Eq. (1.1) in terms of certain numerical quantities that include the information about the coefficients $a_j(t)$, $q_j(t)$ of expansions (1.3) and (1.4) with account of the values of parameter ρ.

This paper is organized as follows. In Section 2 we describe the asymptotic integration method that we use throughout the paper to get the asymptotic formulae for solutions of Eq. (1.1). Asymptotic representations for solutions are constructed in Section 3. In the final section of the paper we summarize the obtained results and indicate the conditions for existence of oscillatory (nonoscillatory) solutions of Eq. (1.1). Moreover, we also give some examples in this section.

2 Description of the asymptotic integration method

In (1.1), we make the change of variable

$$x(t) = e^{-t}y(t), \tag{2.1}$$

to get

$$\dot{y} = y(t) - a(t)e^{\tau(t)}y(t-\tau(t)). \tag{2.2}$$

After some trivial manipulations with the right-hand side of Eq. (2.2) we rewrite it in the form of the functional differential equation

$$\dot{y} = B_0 y_t + G(t, y_t), \tag{2.3}$$

where $y_t(\theta) = y(t+\theta)$ $(-h \le \theta \le 0)$ denotes the element of the space $C_h \equiv C([-h,0],\mathbb{C})$ consisting of all continuous functions defined on $[-h,0]$ and acting to $\mathbb{C}$. We choose the delay $h > 0$ such that the inequalities $0 \le \tau(t) \le h$ hold $t \ge t_0$. The norm in C_h is introduced in the standard way:

$$\|\varphi\|_{C_h} = \sup_{-h\le\theta\le 0} |\varphi(\theta)|. \tag{2.4}$$

Further, B_0 is a bounded linear functional acting from C_h to $\mathbb{C}$ that is defined by the formula

$$B_0\varphi(\theta) = \varphi(0) - \varphi(-1), \qquad \varphi(\theta) \in C_h. \tag{2.5}$$

Finally, functional $G(t,\varphi(\theta))$, acting from C_h to $\mathbb{C}$, has the form

$$G(t,\varphi(\theta)) = \varphi(-1) - a(t)e^{\tau(t)}\varphi(-\tau(t)). \tag{2.6}$$

The asymptotic integration method that we apply in this work to study the dynamics of Eq. (2.3) was suggested by the author in [26,27]. In these papers Eq. (2.3) is considered as a perturbation of the linear autonomous equation

$$\dot{y} = B_0 y_t. \tag{2.7}$$

The main assumption concerning the unperturbed Eq. (2.7) is the following. The characteristic equation should have the finite number of roots (with account of their multiplicities) with zero real parts and all other roots should have negative real parts. Linear bounded functional $G(t,\varphi(\theta))$ is, in some sense, a «small» perturbation consisting of two terms. The first term is a functional that oscillatorily tends to zero as $t \to \infty$ for each $\varphi(\theta)$. The second term is an absolutely integrable on $[t_0,\infty)$ in a certain sense functional, i.e., its values as functions of t belong to $L_1[t_0,\infty)$. Here and in what follows we write that scalar function, vector-function or matrix $F(t)$ belongs to $L_1[t_0,\infty)$, if the integral

$$\int_{t_0}^{\infty} |F(t)|dt,$$

where $|\cdot|$ is an absolute value or certain vector or matrix norm, is finite.

Proposition 2.1. *The characteristic equation*

$$\Delta(\lambda) = 0, \qquad \Delta(\lambda) = \lambda - 1 + e^{-\lambda}, \tag{2.8}$$

constructed for the unperturbed equation (2.7) *with functional* (2.5), *has roots* $\lambda_{1,2} = 0$ *(i.e., zero root of multiplicity two) and all the other roots have negative real parts.*

Proof. It is obvious that $\Delta(0) = \Delta'(0) = 0$ and $\Delta''(0) = 1 \ne 0$. Hence, $\lambda = 0$ is a root of characteristic equation (2.8) with multiplicity two. Note that this equation does not have any other real roots λ. Since $\Delta'(\lambda) = 1 - e^{-\lambda}$, the function $\Delta(\lambda)$ decreases monotonically in the interval $(-\infty,0)$ and increases monotonically in the interval $(0,+\infty)$. At the point $\lambda = 0$ this function has global minimum $\Delta(0) = 0$. Consequently, $\Delta(\lambda) > 0$ for all $\lambda \ne 0$.

Suppose that equation (2.8) has complex root $\lambda = \alpha + i\beta$, where $\alpha,\beta \in \mathbb{R}$ and $\beta > 0$. By equating the real and the imaginary parts in (2.8), we obtain

$$\begin{cases} \alpha - 1 + e^{-\alpha}\cos\beta = 0, \\ \beta - e^{-\alpha}\sin\beta = 0. \end{cases}$$

Due to the well-known inequality, it follows that

$$e^{\alpha} = \frac{\sin \beta}{\beta} < 1.$$

Hence, $\alpha < 0$ and all complex roots have negative real parts. □

Verification of the fact that functional $G(t, \varphi(\theta))$ is a small perturbation is not actually trivial due to the presence of the variable delay $\tau(t)$. The corresponding problems are discussed in paper [27]. It turns out that in this case the choice of the space C_h as the phase space for Eq. (2.3) is not appropriate. We should act in another manner. We remind that function $\varphi \in C_h$ is called Lipschitz continuous if there is a positive constant K (Lipschitz constant) such that

$$|\varphi(\theta_1) - \varphi(\theta_2)| \le K\,|\theta_1 - \theta_2|, \quad -h \le \theta_1, \theta_2 \le 0. \tag{2.9}$$

Note that constant K in (2.9) depends on function $\varphi(\theta)$. Let us introduce the following notation.

Definition 2.2. Denote by LC_h the subspace of C_h consisting of all Lipschitz continuous functions and equipped with the norm

$$\|\varphi\|_{LC_h} = \max\left(\|\varphi\|_{C_h}, K_{\varphi}\right), \tag{2.10}$$

where $K_{\varphi} = \inf K$ and infimum is taken over all constants K for which inequality (2.9) holds. Symbol $\|\varphi\|_{C_h}$ stands for norm (2.4).

We remark that with norm (2.10) the space LC_h is a Banach space. Let $y_t(\theta)$ be the solution of Eq. (2.3) with initial value $y_T = \varphi$, where $\varphi \in C_h$ and $T \ge t_0$. Then, due to continuity property of functions $a(t)$, $\tau(t)$ and the form of functional $G(t, \varphi)$, defined by (2.6), solution $y_t(\theta)$ belongs to the space LC_h for $t \ge T + h$. Therefore, the dynamics of Eq. (2.3) is defined by the behaviour of solutions in LC_h. We can now easily check that $G(t, \varphi)$, as the functional acting from LC_h, is a small perturbation. Since, due to (1.3) and (1.4), the asymptotic formula $a(t)e^{\tau(t)} = 1 + O(t^{-\rho})$ holds as $t \to \infty$, we have

$$G(t, \varphi(\theta)) = \varphi(-1) - \varphi(-\tau(t)) + O(t^{-\rho})\varphi(-\tau(t)).$$

Thus, for each $\varphi \in LC_h$ due to (1.4) with account of (2.10) we conclude that

$$\begin{aligned} |G(t, \varphi(\theta))| &\le |\varphi(-1) - \varphi(-\tau(t))| + O(t^{-\rho})|\varphi(-\tau(t))| \\ &\le K_{\varphi} O(t^{-\rho}) + O(t^{-\rho})\|\varphi\|_{C_h} \le O(t^{-\rho})\|\varphi\|_{LC_h} \end{aligned} \tag{2.11}$$

This proves the «smallness» of the functional $G(t, \varphi(\theta))$ as $t \to \infty$. The oscillatory decreasing character of $G(t, \varphi(\theta))$ as the function of t for each $\varphi \in LC_h$ follows from (2.6) and the corresponding properties of the functions $a_j(t)$, $q_j(t)$ in (1.3), (1.4). In what follows we will give a slightly different representation for the functional $G(t, \varphi(\theta))$. The presence of oscillatory decreasing coefficients in this representation will play an essential role for the implementation of the asymptotic integration method. We now turn to the description of this method.

The asymptotic integration method we apply in this paper is based on the existence for sufficiently large t of the positively invariant manifold in space LC_h that attracts (at the exponential rate) all the trajectories of Eq. (2.3). The dynamics of solutions of Eq. (2.3), lying in this manifold, is described by the two-dimensional linear ordinary differential system. Thus, the fundamental solutions of this system define the main parts of the asymptotic formulae for solutions of Eq. (2.3). We will now describe this method in details. First we need to decompose space C_h into direct sum of two certain subspaces.

It is known that linear autonomous equation (2.7) generates in C_h for $t \geq 0$ a strongly continuous semigroup $T(t)\colon C_h \to C_h$. The solution operator $T(t)$ of Eq. (2.7) is defined as follows: $T(t)\varphi = y_t^{\varphi}(\theta)$, where $\varphi \in C_h$ and $y_t^{\varphi}(\theta)$ is a unique solution of Eq. (2.7) with initial value $y_0^{\varphi}(\theta) = \varphi$. The infinitesimal generator A of this semigroup is defined by $A\varphi = \varphi'(\theta)$, where $\varphi \in D(A)$. The domain of A

$$D(A) = \{\varphi \in C_h \mid \varphi'(\theta) \in C_h, \ \ \varphi'(0) = B_0\varphi\}$$

is dense in C_h. Suppose that B_0 has Riesz representation

$$B_0\varphi = \int_{-h}^{0} d\eta(\theta)\varphi(\theta),$$

where $\eta(\theta)$ is the scalar function of bounded variation on $[-h, 0]$. We can associate with Eq. (2.7) the transposed equation

$$\dot{y}_* = -\int_{-h}^{0} y_*(t-\theta)d\eta(\theta), \qquad t \leq 0, \tag{2.12}$$

where $y_*(t)$ is complex scalar function. The phase space for Eq. (2.12) is $C_h' \equiv C\big([0,h], \mathbb{C}\big)$. For $\psi \in C_h'$ and $\varphi \in C_h$ we define the bilinear form

$$\big(\psi(\xi), \varphi(\theta)\big) = \psi(0)\varphi(0) - \int_{-h}^{0}\int_{0}^{\theta} \psi(\xi-\theta)d\eta(\theta)\varphi(\xi)d\xi. \tag{2.13}$$

Let

$$\Lambda = \{\lambda_1, \lambda_2\},$$

where $\lambda_1 = \lambda_2 = 0$ are the roots of characteristic equation (2.8) from Proposition 2.1. We now decompose C_h into a direct sum

$$C_h = P_\Lambda \oplus Q_\Lambda. \tag{2.14}$$

Here P_Λ is a linear span of generalized eigenfunctions of operator A corresponding to the eigenvalues from Λ and Q_Λ is certain complementary subspace of C_h such that $T(t)Q_\Lambda \subseteq Q_\Lambda$. Let $\Phi(\theta)$ be two-dimensional row-vector whose entries are the generalized eigenfunctions $\varphi_1(\theta), \varphi_2(\theta)$ of operator A corresponding to the eigenvalues from Λ. Thus, the entries of $\Phi(\theta)$ form the basis of P_Λ. Moreover, let $\Psi(\xi)$ be two-dimensional column-vector whose entries $\psi_1(\xi), \psi_2(\xi)$ form the basis of the generalized eigenspace P_Λ^T of the transposed equation (2.12) associated with Λ. We can choose vectors $\Phi(\theta)$ and $\Psi(\xi)$ such that

$$\big(\Psi(\xi), \Phi(\theta)\big) = \big\{\big(\psi_i(\xi), \varphi_j(\theta)\big)\big\}_{1\leq i,j\leq 2} = I. \tag{2.15}$$

Since $\Phi(\theta)$ is the basis of P_Λ and $AP_\Lambda \subseteq P_\Lambda$, there exists (2×2)-matrix D, whose spectrum is Λ, such that $A\Phi(\theta) = \Phi(\theta)D$. From the definition of A, we deduce that

$$\Phi(\theta) = \Phi(0)e^{D\theta}, \qquad T(t)\Phi(\theta) = \Phi(\theta)e^{Dt} = \Phi(0)e^{D(t+\theta)},$$

where $-h \leq \theta \leq 0$ and $t \geq 0$. Analogously, for column-vector $\Psi(\xi)$ we have

$$\Psi(\xi) = e^{-D\xi}\Psi(0), \tag{2.16}$$

where $0 \le \xi \le h$. Vectors $\Phi(0)$ and $\Psi(0)$ are chosen in the following way. Since the entries of row-vector $\Phi(\theta)$ are the generalized eigenfunctions of A, they should belong to $D(A)$. This implies that

$$\Phi'(0) = \Phi(0)D = B_0\Phi = \int_{-h}^{0} d\eta(\theta)\Phi(0)e^{D\theta}.$$

The same reasoning, using (2.12) and (2.16), yields

$$\Psi'(0) = -D\Psi(0) = -\int_{-h}^{0} e^{D\theta}\Psi(0)d\eta(\theta).$$

Finally, the subspaces P_Λ and Q_Λ from decomposition (2.14) may be defined as follows:

$$\begin{aligned} P_\Lambda &= \{\varphi \in C_h \mid \varphi(\theta) = \Phi(\theta)u,\ u \in \mathbb{C}^2\}, \\ Q_\Lambda &= \{\varphi \in C_h \mid (\Psi, \varphi) = 0\}. \end{aligned} \tag{2.17}$$

Here and in what follows symbol $\mathbb{C}^2$ stands for the space of two-dimensional complex column-vectors.

An easy computation yields the following formulae for vectors $\Phi(\theta)$, $\Psi(\xi)$ and matrix D for Eq. (2.7) with functional (2.5):

$$\Phi(\theta) = \begin{pmatrix} 1 & \theta \end{pmatrix}, \quad \Psi(\xi) = \begin{pmatrix} \frac{2}{3} - 2\xi \\ 2 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{2.18}$$

To calculate vectors $\Phi(\theta)$ and $\Psi(\xi)$ we also used condition (2.15). We are now in a position to define the central notion of the proposed method — the notion of critical manifold for Eq. (2.3).

Definition 2.3. Two-dimensional linear space $\mathcal{W}(t) \subset LC_h \subset C_h$ is said to be critical (or center-like) manifold of Eq. (2.3) for $t \ge t_* \ge t_0$ if the following conditions hold:

1. There exists two-dimensional row-vector $H(t,\theta)$, whose entries are continuous in $t \ge t_*$ and belong to LC_h and also subspace Q_Λ as functions of $\theta \in [-h, 0]$ for all $t \ge t_*$. Moreover, $\|H(t,\cdot)\|_{LC_h} \to 0$ as $t \to \infty$, where

$$\|H(t,\cdot)\|_{LC_h} = \|\,|H(t,\cdot)|\,\|_{LC_h}.$$

   Here $|\cdot|$ denotes some vector norm in the space of two-dimensional row-vectors.

2. The space $\mathcal{W}(t)$ for $t \ge t_*$ is defined by the formula

$$\mathcal{W}(t) = \left\{\varphi(\theta) \in LC_h \mid \varphi(\theta) = \Phi(\theta)u + H(t,\theta)u,\ u \in \mathbb{C}^2\right\}. \tag{2.19}$$

3. The space $\mathcal{W}(t)$ is positively invariant for trajectories of Eq. (2.3) for $t \ge t_*$, i.e., if $y_T \in \mathcal{W}(T)$, $T \ge t_*$, then $y_t \in \mathcal{W}(t)$ for $t \ge T$.

The following existence theorem holds (see [27]).

Theorem 2.4. *For sufficiently large t there exists a critical manifold $\mathcal{W}(t)$ of Eq. (2.3) in LC_h.*

Due to the positive invariance of $\mathcal{W}(t)$, the trajectories lying in this manifold for sufficiently large t are described by the formula

$$y_t(\theta) = \Phi(\theta)u(t) + H(t,\theta)u(t), \qquad t \geq T, \quad u(t) \in \mathbb{C}^2.$$

It can be shown (see, e.g., [19,20]), that the vector function $u(t)$ in the above expression satisfies the following ordinary differential system:

$$\dot{u} = \big[D + \Psi(0)G\big(t, \Phi(\theta) + H(t,\theta)\big)\big]u, \qquad t \geq T. \tag{2.20}$$

This system will be referred to as a system on critical manifold. An important property of manifold $\mathcal{W}(t)$ is that it is attractive for all trajectories of Eq. (2.3) (see [27]).

Theorem 2.5. *Suppose that $y(t)$ is a solution of Eq. (2.2), defined for $t \geq T \geq t_0$. Then there exists sufficiently large $t_* \geq T$ such that the following asymptotic formula holds for $t \geq t_*$:*

$$y_t(\theta) = \Phi(\theta)u_H(t) + H(t,\theta)u_H(t) + O(e^{-\beta t}), \qquad t \to \infty.$$

Here $u_H(t)$ ($t \geq t_*$) is a certain solution of Eq. (2.20) and $\beta > 0$ is a certain real number.

Suppose that $u^{(1)}(t), u^{(2)}(t)$ are the fundamental solutions of a system on critical manifold (2.20) and $y(t)$ is an arbitrary solution of Eq. (2.2) defined for $t \geq T$. By Theorem 2.5, this solution has the following asymptotic representation as $t \to \infty$:

$$y(t) = y_t(0) = \big(\Phi(0) + H(t,0)\big)\big(c_1 u^{(1)}(t) + c_2 u^{(2)}(t)\big) + O\big(e^{-\beta t}\big), \qquad t \to \infty, \tag{2.21}$$

where c_1, c_2 are arbitrary complex constants and $\beta > 0$ is a certain real number. Therefore, to solve the oscillation problem for Eq. (2.2) (evidently, for initial Eq. (1.1) as well) we need to construct the asymptotics for the fundamental solutions $u^{(1)}(t)$, $u^{(2)}(t)$ of system (2.20) that define the dynamics of all solutions of Eq. (2.2) due to (2.21). Unfortunately, having determined the type of solutions $u^{(1)}(t)$ and $u^{(2)}(t)$ (oscillatory or nonoscillatory), we cannot answer the question whether all the solutions of Eq. (1.1) are of the same type. This follows from the fact that due to (2.21) if $c_1 = c_2 = 0$ the dynamics of solutions of Eq. (1.1) is defined by the remainder term, whose form is unclear. Thus, in this paper we only give an answer concerning the existence of oscillatory or nonoscillatory solutions.

Now we need to clarify how to construct the row-vector $H(t,\theta)$ needed for system on critical manifold (2.20) and how to obtain the asymptotics for the fundamental matrix of this system. It is shown in [26,27] that vector $H(t,\theta)$ is a solution, in certain week sense, of the following problem:

$$\Phi(\theta)\Psi(0)G\big(t, \Phi(\theta) + H(t,\theta)\big) + H(t,\theta)\left(D + \Psi(0)G\big(t, \Phi(\theta) + H(t,\theta)\big)\right) + \frac{\partial H}{\partial t}$$
$$= \begin{cases} \dfrac{\partial H}{\partial \theta}, & -h \leq \theta < 0, \\ B_0 H + G\big(t, \Phi(\theta) + H(t,\theta)\big), & \theta = 0. \end{cases} \tag{2.22}$$

We can solve this problem approximately. Namely, due to the form of the functional $G(t, \varphi(\theta))$ that is defined by formula (2.6) and taking into account asymptotic representations (1.3), (1.4) we can satisfy problem (2.22) with the row-vector

$$\hat{H}(t,\theta) = H_1(t,\theta)t^{-\rho} + H_2(t,\theta)t^{-2\rho} + \cdots + H_k(t,\theta)t^{-k\rho} \tag{2.23}$$

up to the term $\hat{R}(t,\theta)$ such that $\|\hat{R}(t,\cdot)\|_{LC_h} \in L_1[t_0,\infty)$. Here $k \in \mathbb{N}$ is defined according to (1.3), (1.4) with account of (1.5) and the entries of two-dimensional row-vectors $H_j(t,\theta)$, $j = 1,\dots,k$ are trigonometric polynomials in t whose coefficients are infinitely differentiable in $\theta \in [-h,0]$. Thus, the row-vectors $H_j(t,\theta)$ has the form

$$H_j(t,\theta) = \sum_s \beta_s^{(j)}(\theta) e^{\mathrm{i}\omega_s t}, \tag{2.24}$$

where the row-vectors $\beta_s^{(j)}(\theta)$ are infinitely differentiable in $\theta \in [-h,0]$. We also note that the entries of these row-vectors belong to the subspace Q_Λ. It appears that the problem of finding the vectors $H_j(t,\theta)$ is reduced to solving certain functional boundary problems for linear ordinary differential systems. Namely, we substitute (2.23) for $H(t,\theta)$ in (2.22) and collect terms corresponding to factors $t^{-j\rho}$, $j = 1,\dots,k$. We then seek the solutions of the obtained equations in form (2.24). Substituting the latter in the mentioned equations and matching the coefficients of the corresponding exponentials, we get the functional boundary problems for linear ordinary differential systems. It is proved in[26] that each of these problems is uniquely solvable.

Row-vector $\hat{H}(t,\theta)$ is an approximation, in a certain sense, for vector $H(t,\theta)$ that describes manifold $\mathcal{W}(t)$ according to formula (2.19). To be precise the following approximation theorem holds.

Theorem 2.6. *Suppose that $\mathcal{W}(t)$ is a critical manifold of Eq. (2.3) which exists for sufficiently large t according to Theorem 2.4. Then there exists a sufficiently large t_* such that for $t \ge t_*$ row-vector $H(t,\theta)$ from (2.19) admits the following representation:*

$$H(t,\theta) = \hat{H}(t,\theta) + Z(t,\theta), \qquad t \ge t_* \ge t_0, \quad -\tau \le \theta \le 0. \tag{2.25}$$

Here the row-vector $\hat{H}(t,\theta)$ is defined by formula (2.23) and satisfies Eq. (2.22) up to the term $\hat{R}(t,\theta)$ such that $\|\hat{R}(t,\cdot)\|_{LC_h} \in L_1[t_0,\infty)$. Moreover, $Z(t,\theta)$ is a certain row-vector such that $\|Z(t,\cdot)\|_{LC_h} \to 0$ as $t \to \infty$ and $\|Z(t,\cdot)\|_{LC_h} \in L_1[t_*,\infty)$.

According to (1.3), (1.4), (2.6) with account of formula (2.23), describing the approximate solution of problem (2.22), it can be shown that row-vector $Z(t,\theta)$ in (2.25) has the following asymptotic estimate as $t \to \infty$:

$$\|Z(t,\cdot)\|_{LC_h} = O\Big(\frac{d}{dt}(t^{-\rho})\Big) + O\big(t^{-(k+1)\rho}\big) = O\big(t^{-(\rho+1)}\big) + O\big(t^{-(k+1)\rho}\big). \tag{2.26}$$

The asymptotic integration of system (2.20) is carried out as follows. Due to (1.3), (1.4), (2.6), (2.23) this system in the considered case has the following form:

$$\dot{u} = \Big[D + A_1(t)t^{-\rho} + A_2(t)t^{-2\rho} + \cdots + A_{k+1}(t)t^{-(k+1)\rho} + R(t)\Big]u, \qquad u \in \mathbb{C}^2. \tag{2.27}$$

Here matrix D is defined in (2.18), natural number k is chosen according to (1.5) and $A_1(t),\dots$, $A_{k+1}(t)$ are (2×2)-matrices, whose entries are trigonometric polynomials, i.e., matrices having the form

$$A_j(t) = \sum_s \psi_s^{(j)} e^{\mathrm{i}\omega_s t},$$

where $\psi_s^{(j)}$ are constant complex matrices and ω_s are real numbers. Finally, $R(t)$ is a certain (2×2)-matrix that belongs to $L_1[t_*,\infty)$. It follows from (1.3), (1.4), (2.11) and (2.26) that this matrix has the following asymptotic estimate:

$$R(t) = O\big(t^{-(k+2)\rho}\big) + O\big(t^{-(2\rho+1)}\big), \qquad t \to \infty. \tag{2.28}$$

The main difficulty in the asymptotic integration of system (2.27) as $t \to \infty$ is that its coefficients have an oscillatory behaviour. Therefore, on the first step we utilize in (2.27) the averaging change of variable that makes it possible to exclude the oscillating coefficients from the main part of the system. The following theorem holds (see [25]).

Theorem 2.7. *For sufficiently large t, system (2.27) by the change of variable*

$$u = \Big[I + Y_1(t)t^{-\rho} + Y_2(t)t^{-2\rho} + \cdots + Y_{k+1}(t)t^{-(k+1)\rho}\Big]u_1 \tag{2.29}$$

can be reduced to its averaged form

$$\dot{u}_1 = \Big[D + A_1 t^{-\rho} + A_2 t^{-2\rho} + \cdots + A_{k+1} t^{-(k+1)\rho} + R_1(t)\Big]u_1 \tag{2.30}$$

with constant matrices $A_1, \dots, A_k$ and with matrix $R_1(t)$ from $L_1[t_*, \infty)$. In (2.29), I is the identity matrix and the entries of matrices $Y_1(t), \dots, Y_k(t)$ are trigonometric polynomials having zero mean value.

As a rule, to construct the asymptotics for solutions of (2.30) we need to compute only a few constant matrices. Hence, we give the explicit formulas only for matrices A_1 and A_2. We have

$$A_1 = \mathrm{M}\big[A_1(t)\big], \tag{2.31}$$
$$A_2 = \mathrm{M}\big[A_2(t) + A_1(t)Y_1(t)\big]. \tag{2.32}$$

Here symbol $\mathrm{M}\big[F(t)\big]$ denotes the mean value of the matrix $F(t)$ whose entries are trigonometric polynomials:

$$\mathrm{M}\big[F(t)\big] = \lim_{T\to\infty} \frac{1}{T}\int_0^T F(t)dt.$$

Matrix $Y_1(t)$ in (2.32) is the solution of matrix differential equation

$$\dot{Y}_1 - DY_1 + Y_1 D = A_1(t) - A_1 \tag{2.33}$$

with zero mean value. Finally, matrix $R_1(t)$ in (2.30) has the following form:

$$R_1(t) = \rho Y_1(t)t^{-(\rho+1)} + O\big(t^{-(2\rho+1)}\big) + O\big(t^{-(k+2)\rho}\big), \qquad t \to \infty. \tag{2.34}$$

Here we give the explicit formula for the first term in (2.34) since its form will be necessary for further transformation of system (2.30).

The subsequent transformations of the averaged system (2.30) aim to bring it to the form

$$\dot{u}_2 = \big[A_0 + V(t)\big]t^{-\alpha}u_2 + R_2(t)u_2, \tag{2.35}$$

where $\alpha > 0$ is a certain number, A_0 is a constant matrix, matrix $V(t)$ tends to zero matrix as $t \to \infty$ and $R_2(t) \in L_1[t_*, \infty)$. The following lemma holds (see, for instance, [1,6,16]).

Lemma 2.8 (diagonalization of variable matrices). *Suppose that all eigenvalues of the matrix A_0 are distinct. Moreover, suppose that matrix $V(t) \to 0$ as $t \to \infty$ and $V'(t) \in L_1[t_*, \infty)$. Then for sufficiently large t there exists a nonsingular matrix $C(t)$ such that*

(i) *the columns of this matrix are the eigenvectors of the matrix $A_0 + V(t)$ and $C(t) \to C_0$ as $t \to \infty$. The columns of the constant matrix C_0 are the eigenvectors of the matrix A_0;*

(ii) *the derivative* $C'(t) \in L_1[t_*, \infty)$*;*

(iii) *it brings the matrix* $A_0 + V(t)$ *to diagonal form, i.e.,*

$$C^{-1}(t)\big[A_0 + V(t)\big]C(t) = \hat{\Lambda}(t),$$

where $\hat{\Lambda}(t) = \mathrm{diag}\big(\hat{\lambda}_1(t), \hat{\lambda}_2(t)\big)$ *and* $\hat{\lambda}_1(t), \hat{\lambda}_2(t)$ *are the eigenvalues of the matrix* $A_0 + V(t)$*.*

In (2.35), we make the change of variable

$$u_2(t) = C(t)u_3(t),$$

where $C(t)$ is the matrix from Lemma 2.8. This change of variable brings system (2.35) to what is called L-diagonal form:

$$\dot{u}_3 = \big[\Lambda(t) + R_3(t)\big]u_3, \tag{2.36}$$

where $\Lambda(t) = \mathrm{diag}\big(\lambda_1(t), \lambda_2(t)\big)$, $\lambda_j(t) = \hat{\lambda}_j(t)t^{-\alpha}$ $(j = 1, 2)$ and

$$R_3(t) = -C^{-1}(t)\dot{C}(t) + C^{-1}(t)R_2(t)C(t).$$

The properties (i) and (ii) of the matrix $C(t)$ imply that matrix $R_3(t)$ belongs to $L_1[t_*, \infty)$.

To construct the asymptotics for solutions of L-diagonal system (2.36) as $t \to \infty$ the well-known Theorem of Levinson can be used. Suppose that the following dichotomy condition holds for the entries of the matrix $\Lambda(t)$: either the inequality

$$\int_{t_1}^{t_2} \mathrm{Re}\big(\lambda_i(s) - \lambda_j(s)\big)ds \le K_1, \qquad t_2 \ge t_1 \ge t_*, \tag{2.37}$$

or the inequality

$$\int_{t_1}^{t_2} \mathrm{Re}\big(\lambda_i(s) - \lambda_j(s)\big)ds \ge K_2, \qquad t_2 \ge t_1 \ge t_*, \tag{2.38}$$

is valid for each pair of indices (i, j), where K_1, K_2 are some constants. What follows is Levinson's fundamental theorem (see, e.g., [6, 16, 22]).

Theorem 2.9 (**Levinson**)**.** *Let the dichotomy condition* (2.37), (2.38) *be satisfied. Then the fundamental matrix of L-diagonal system* (2.36) *has the following asymptotics as* $t \to \infty$*:*

$$U(t) = \big(I + o(1)\big)\exp\left\{\int_{t^*}^{t} \Lambda(s)ds\right\}.$$

We note that for the problem considered in this paper the dichotomy condition (2.37), (2.38) is always satisfied since quantities $\mathrm{Re}\big(\lambda_i(t) - \lambda_j(t)\big)$ do not change their signs for sufficiently large t. This follows from the fact that system (2.35) comes from the averaged system (2.30), whose coefficients in the main part do not oscillate and the utilized transformations do not change this property.

3 Construction of asymptotic formulae

In this section we obtain the asymptotic formulae for solutions of Eq. (2.2) as $t \to \infty$. The asymptotics for solutions of the initial Eq. (1.1) can be easily constructed by applying the change of variable (2.1) and, therefore, we will not write it here. First, we get another one representation for functional $G(t, \varphi(\theta))$ in (2.3) that is defined by formula (2.6). By applying Taylor's formula for $a(t)e^{\tau(t)}$ as $t \to \infty$ with account of (1.3), (1.4), we obtain

$$a(t)e^{\tau(t)} = 1 + p_1(t)t^{-\rho} + p_2(t)t^{-2\rho} + \cdots + p_{k+1}(t)t^{-(k+1)\rho} + O(t^{-(k+2)\rho}). \tag{3.1}$$

Here $p_1(t), \ldots, p_{k+1}(t)$ are certain trigonometric polynomials and, in particular,

$$p_1(t) = ea_1(t) + q_1(t), \qquad p_2(t) = ea_2(t) + q_2(t) + \frac{q_1^2(t)}{2} + ea_1(t)q_1(t), \tag{3.2}$$

where $a_i(t), q_i(t)$, $i = 1, 2$, are functions from asymptotic expansions (1.3), (1.4) for coefficients of the initial equation (1.1). For the sequel we need the expressions for the functions $p_1(t)$ and $q_1(t)$ in the form of the trigonometric polynomials:

$$p_1(t) = \sum_{j=-N}^{N} p_1^{(j)} e^{\mathrm{i}\omega_j t}, \qquad q_1(t) = \sum_{j=-N}^{N} q_1^{(j)} e^{\mathrm{i}\omega_j t}, \tag{3.3}$$

where $p_1^{(j)}, q_1^{(j)}$ are, in general, certain complex numbers, ω_j are real numbers and, moreover,

$$p_1^{(-j)} = \bar{p}_1^{(j)}, \quad q_1^{(-j)} = \bar{q}_1^{(j)}, \quad \omega_{-j} = -\omega_j \ (\omega_l \neq \omega_l,\ l \neq m), \quad j = 1, \ldots, N. \tag{3.4}$$

here notation $\bar{a}$ stands for complex conjugate of a. Hence, we have

$$\mathrm{M}\big[p_1(t)\big] = p_1^{(0)}, \quad \mathrm{M}\big[q_1(t)\big] = q_1^{(0)}. \tag{3.5}$$

By using Taylor's formula for $\varphi(-\tau(t))$ as $t \to \infty$, and taking into account (3.1), we finally obtain the following representation for functional $G(t, \varphi(\theta))$:

$$\begin{aligned} G(t, \varphi(\theta)) = {} & \big[q_1(t)\varphi'(-1) - p_1(t)\varphi(-1)\big]t^{-\rho} \\ & + \left[p_1(t)q_1(t)\varphi'(-1) + q_2(t)\varphi'(-1) - p_2(t)\varphi(-1) - \frac{q_1^2(t)}{2}\varphi''(-1)\right]t^{-2\rho} \\ & + O\big(t^{-3\rho}\big). \end{aligned} \tag{3.6}$$

Although the functional $G(t, \varphi(\theta))$ is defined only for elements from C_h, in what follows it will be applied to infinitely differentiable functions and this makes possible to use form (3.6). We proceed now to the problem of construction of the asymptotic formulae for solutions of Eq. (2.2) as $t \to \infty$.

We write system on critical manifold (2.20) in form (2.27). To get the asymptotics for the fundamental solutions of this system we need the explicit formulae for matrices $A_1(t)$ and $A_2(t)$. We use (3.6) and also formula (2.18) to obtain

$$A_1(t) = \Psi(0)\big[q_1(t)\Phi'(-1) - p_1(t)\Phi(-1)\big] = \frac{2}{3}\begin{pmatrix} -p_1(t) & q_1(t) + p_1(t) \\ -3p_1(t) & 3\big(q_1(t) + p_1(t)\big) \end{pmatrix} \tag{3.7}$$

and

$$\begin{aligned} A_2(t) &= \Psi(0)\left[p_1(t)q_1(t)\Phi'(-1) + q_2(t)\Phi'(-1) - p_2(t)\Phi(-1) - \frac{q_1^2(t)}{2}\Phi''(-1)\right] \\ &\quad + \Psi(0)\left[q_1(t)\frac{\partial H_1}{\partial\theta}(t,\theta)\Big|_{\theta=-1} - p_1(t)H_1(t,-1)\right] \\ &= \frac{2}{3}\begin{pmatrix} -p_2(t) & p_1(t)q_1(t)+q_2(t)+p_2(t) \\ -3p_2(t) & 3\big(p_1(t)q_1(t)+q_2(t)+p_2(t)\big)\end{pmatrix} \\ &\quad + \frac{2}{3}\begin{pmatrix} q_1(t)\frac{\partial h_{11}}{\partial\theta}(t,\theta)\Big|_{\theta=-1} - p_1(t)h_{11}(t,-1) & q_1(t)\frac{\partial h_{12}}{\partial\theta}(t,\theta)\Big|_{\theta=-1} - p_1(t)h_{12}(t,-1) \\ 3\Big(q_1(t)\frac{\partial h_{11}}{\partial\theta}(t,\theta)\Big|_{\theta=-1} - p_1(t)h_{11}(t,-1)\Big) & 3\Big(q_1(t)\frac{\partial h_{12}}{\partial\theta}(t,\theta)\Big|_{\theta=-1} - p_1(t)h_{12}(t,-1)\Big)\end{pmatrix}. \end{aligned} \tag{3.8}$$

Here

$$H_1(t,\theta) = \big(h_{11}(t,\theta) \quad h_{12}(t,\theta)\big) \tag{3.9}$$

is a row-vector from representation (2.23) for row-vector $\hat{H}(t,\theta)$ that is an approximation of $H(t,\theta)$ due to Theorem 2.6. Row-vector (3.9) will be defined at the end of this section.

The most simple case in constructing the asymptotic formulae for solutions of Eq. (2.2) occurs when

$$\rho > 2. \tag{3.10}$$

In this situation system on critical manifold (2.27) with account of (2.28) takes the form

$$\dot{u} = \big[D + O(t^{-\rho})\big]u, \tag{3.11}$$

where matrix D is defined by formula (2.18). Since, due to (3.10), the remainder term in (3.11) has the property that

$$O(t^{-\rho})t^{i-j} \in L_1[t_0,\infty), \quad 1 \le i,j \le 2,$$

we can use [2, Corollary 6.2, p. 213]. It follows that the fundamental solutions of system (3.11) have the following asymptotics as $t \to \infty$:

$$u^{(1)}(t) = \begin{pmatrix} 1+o(1) \\ o(t^{-1})\end{pmatrix}, \quad u^{(2)}(t) = \begin{pmatrix} t\big(1+o(1)\big) \\ 1+o(1)\end{pmatrix}. \tag{3.12}$$

We then use (2.21), with account that $H(t,0) = o(1)$, to obtain the following asymptotic representation for all solutions of Eq. (2.2) as $t \to \infty$:

$$y(t) = c_1\big(1+o(1)\big) + c_2 t\big(1+o(1)\big) + O(e^{-\beta t}), \tag{3.13}$$

where c_1, c_2 are arbitrary real constants and $\beta > 0$ is a certain real number.

Thus, the main interest concerns the case

$$\rho \le 2.$$

We use Theorem 2.7 to bring system (2.27) by the change of variable (2.29) to the averaged form (2.30). In (2.30), constant matrices A_1 and A_2 are described by formulae (2.31), (2.32) and the remainder term $R_1(t)$ has form (2.34). We calculate matrix A_1 taking into account (3.7) and also expressions (3.3), (3.5). We have

$$A_1 = \mathrm{M}\big[A_1(t)\big] = \frac{2}{3}\begin{pmatrix} -p_1^{(0)} & q_1^{(0)} + p_1^{(0)} \\ -3p_1^{(0)} & 3\big(q_1^{(0)} + p_1^{(0)}\big)\end{pmatrix}. \tag{3.14}$$

The explicit form for matrix A_2 will be obtained later. The asymptotics for solutions of system (2.30) will differ depending on the mean value of the function $p_1(t)$. We now proceed to analysis of these cases.

I. $$p_1^{(0)} \neq 0$$

The eigenvalues of the matrix

$$A(t) = D + A_1 t^{-\rho} + A_2 t^{-2\rho} + \cdots + A_{k+1} t^{-(k+1)\rho}$$

in the main part of system (2.30) have the following asymptotics as $t \to \infty$:

$$\lambda_{1,2}(t) = \pm t^{-\frac{\rho}{2}} \sqrt{-2p_1^{(0)}} \left(1 + O(t^{-\rho})\right) + \left(q_1^{(0)} + \frac{2}{3} p_1^{(0)}\right) t^{-\rho} + O(t^{-2\rho}). \tag{3.15}$$

Here and in what follows the symbol $\sqrt{a}$, where $a \in \mathbb{R}$, stands for the quantity

$$\sqrt{a} = \begin{cases} \sqrt{a}, & a \geq 0, \\ \mathrm{i}\sqrt{-a}, & a < 0. \end{cases} \tag{3.16}$$

Since the eigenvalues (3.15) are distinct for sufficiently large t, the matrix $A(t)$ can be reduced to the diagonal form by certain non-singular matrix $C(t)$:

$$\Lambda(t) = \mathrm{diag}\left(\lambda_1(t), \lambda_2(t)\right) = C^{-1}(t) A(t) C(t). \tag{3.17}$$

Some easy calculations show that the corresponding matrix $C(t)$ has the following asymptotics as $t \to \infty$:

$$C(t) = \begin{pmatrix} 1 & 1 \\ t^{-\frac{\rho}{2}} \sqrt{-2p_1^{(0)}} + O(t^{-\rho}) & -t^{-\frac{\rho}{2}} \sqrt{-2p_1^{(0)}} + O(t^{-\rho}) \end{pmatrix}. \tag{3.18}$$

For the inverse matrix we get

$$C^{-1}(t) = \frac{1}{2\sqrt{-2p_1^{(0)}}} \begin{pmatrix} \sqrt{-2p_1^{(0)}} + O\left(t^{-\frac{\rho}{2}}\right) & t^{\frac{\rho}{2}} + O(1) \\ \sqrt{-2p_1^{(0)}} + O\left(t^{-\frac{\rho}{2}}\right) & -t^{\frac{\rho}{2}} + O(1) \end{pmatrix}, \quad t \to \infty.$$

We note that matrix $C^{-1}(t)$ is unbounded as $t \to \infty$ and has the asymptotic estimate $O\left(t^{\frac{\rho}{2}}\right)$. Keeping this fact in mind, we make in (2.30) the change of variable

$$u_1(t) = C(t) u_2(t)$$

with matrix $C(t)$ having form (3.18). Since

$$C^{-1}(t)\dot{C}(t) = \frac{\rho}{4} t^{-1} \begin{pmatrix} -1 + O\left(t^{-\frac{\rho}{2}}\right) & 1 + O\left(t^{-\frac{\rho}{2}}\right) \\ 1 + O\left(t^{-\frac{\rho}{2}}\right) & -1 + O\left(t^{-\frac{\rho}{2}}\right) \end{pmatrix}, \tag{3.19}$$

we obtain

$$\dot{u}_2 = \left[\Lambda(t) + B t^{-1} + R_2(t)\right] u_2. \tag{3.20}$$

Here the diagonal matrix $\Lambda(t)$ is defined by formula (3.17) with account of (3.15) and the constant matrix B has the following form:

$$B = \frac{\rho}{4} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{3.21}$$

Moreover, the remainder term in (3.20), due to (1.5), (2.34) and (3.19), admits the asymptotic estimate $R_2(t) = O\left(t^{-\frac{\rho}{2}-1}\right)$ as $t \to \infty$. Further, we need to study several alternatives.

Assume first that

$$\rho = 2. \tag{3.22}$$

In this situation system (3.20) takes the following form:

$$\dot{u}_2 = \left[St^{-1} + O(t^{-2})\right]u_2, \tag{3.23}$$

where

$$S = \sqrt{-2p_1^{(0)}}\,\mathrm{diag}(1,-1) + B, \tag{3.24}$$

and matrix B is defined by formula (3.21). The eigenvalues of this matrix are

$$\mu_{1,2} = \frac{1}{2} \pm \sigma, \qquad \sigma = \frac{1}{2}\sqrt{1 - 8p_1^{(0)}}. \tag{3.25}$$

We recall that the square root here means the quantity (3.16). We should consider two cases.

- $p_1^{(0)} \neq \dfrac{1}{8}$

This is the case when $\mu_{1,2}$ are distinct and system (3.23) by the change of variable $u_2 = Cu_3$, where, for instance,

$$C = \begin{pmatrix} \dfrac{1}{2\sqrt{-2p_1^{(0)}} - \sqrt{1-8p_1^{(0)}}} & \dfrac{1}{2\sqrt{-2p_1^{(0)}} + \sqrt{1-8p_1^{(0)}}} \end{pmatrix},$$

can be reduced to L-diagonal form (2.36). In the corresponding L-diagonal system we have

$$\Lambda(t) = \mathrm{diag}(\mu_1,\mu_2)t^{-1}, \qquad R_3(t) = O(t^{-2}), \quad t \to \infty.$$

The asymptotics for the fundamental matrix of this system can be constructed by applying Theorem 2.9. If we return then to Eq. (2.2), we get the following asymptotics for its solutions as $t \to \infty$:

$$y(t) = c_1 t^{\frac{1}{2}} \exp\{\sigma \ln t\}\big(1 + o(1)\big) + c_2 t^{\frac{1}{2}} \exp\{-\sigma \ln t\}\big(1 + o(1)\big) + O\big(e^{-\beta t}\big),$$

where c_1, c_2 are arbitrary, in general, complex constants, $\beta > 0$ is a certain real number and quantity σ is defined by formula (3.25).

- $p_1^{(0)} = \dfrac{1}{8}$

In this situation the eigenvalues of matrix (3.24) coincide:

$$\mu_{1,2} = \frac{1}{2}.$$

First, by the change of variable $u_2 = Cu_3$, where

$$C = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ -i & 2 \end{pmatrix},$$

we bring system (3.23) to the form

$$\dot{u}_3 = \left[Jt^{-1} + O(t^{-2})\right]u_3, \qquad J = \frac{1}{2}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}. \tag{3.26}$$

Next, we apply in (3.26) the transformation $u_3 = t^{\frac{1}{2}} u_4$ to obtain

$$\dot{u}_4 = \left[Dt^{-1} + O(t^{-2})\right] u_4, \tag{3.27}$$

where matrix D is defined by (2.18). Finally, in (3.27) we introduce the new time-variable $\tau = \ln t$ to get

$$u_4' = \left[D + O(e^{-\tau})\right] u_4, \tag{3.28}$$

where the dash denotes the derivative with respect to τ. The construction of the asymptotics for the fundamental matrix of system (3.28) is carried out in the same manner as for system (3.11). This results in the following asymptotic representation for solutions of Eq. (2.2) as $t \to \infty$:

$$y(t) = c_1 t^{\frac{1}{2}}\big(1 + o(1)\big) + c_2 t^{\frac{1}{2}} \ln t \big(1 + o(1)\big) + O\big(e^{-\beta t}\big),$$

where c_1, c_2 are arbitrary real constants and $\beta > 0$ is a certain real number.

Consider now the case

$$\rho < 2.$$

We can write system (3.20) in form (2.35), where, due to (3.15),

$$\alpha = \frac{\rho}{2}, \quad A_0 = \sqrt{-2p_1^{(0)}}\, \mathrm{diag}(1, -1), \quad V(t) = \Big(q_1^{(0)} + \frac{2}{3} p_1^{(0)}\Big) I t^{-\frac{\rho}{2}} + B t^{\frac{\rho}{2} - 1} + O(t^{-\rho}).$$

and $R_2(t) = O\big(t^{-\frac{\rho}{2} - 1}\big)$ as $t \to \infty$. Here matrix B is described by formula (3.21). The asymptotic integration of systems having form (2.35) was described at the end of the previous section. Therefore, we give only the final result concerning the asymptotic formulae for solutions of Eq. (2.2) as $t \to \infty$.

So, if

$$1 < \rho < 2,$$

we have

$$\begin{aligned} y(t) = c_1 t^{\frac{\rho}{4}} \exp\left\{\frac{2}{2-\rho}\sqrt{-2p_1^{(0)}}\, t^{1-\frac{\rho}{2}}\right\} \big(1 + o(1)\big) \\ + c_2 t^{\frac{\rho}{4}} \exp\left\{-\frac{2}{2-\rho}\sqrt{-2p_1^{(0)}}\, t^{1-\frac{\rho}{2}}\right\} \big(1 + o(1)\big) + O\big(e^{-\beta t}\big). \end{aligned}$$

If

$$\rho = 1,$$

then

$$\begin{aligned} y(t) = t^{\frac{1}{4} + q_1^{(0)} + \frac{2}{3} p_1^{(0)}} \Big[c_1 \exp\left\{2\sqrt{-2p_1^{(0)}}\, t\right\} \big(1 + o(1)\big) \\ + c_2 \exp\left\{-2\sqrt{-2p_1^{(0)}}\, t\right\} \big(1 + o(1)\big)\Big] + O\big(e^{-\beta t}\big). \end{aligned}$$

Finally, if

$$\rho < 1,$$

we obtain

$$\begin{aligned} y(t) = t^{\frac{\rho}{4}} \exp\left\{\frac{t^{1-\rho}}{1-\rho}\Big(q_1^{(0)} + \frac{2}{3} p_1^{(0)}\Big)\right\} \Big[c_1 \exp\left\{\frac{2}{2-\rho}\sqrt{-2p_1^{(0)}}\, t^{1-\frac{\rho}{2}} + O\Big(\int t^{-\frac{3\rho}{2}} dt\Big)\right\} \big(1 + o(1)\big) \\ + c_2 \exp\left\{-\frac{2}{2-\rho}\sqrt{-2p_1^{(0)}}\, t^{1-\frac{\rho}{2}} + O\Big(\int t^{-\frac{3\rho}{2}} dt\Big)\right\} \big(1 + o(1)\big)\Big] + O\big(e^{-\beta t}\big). \end{aligned}$$

Everywhere in these asymptotic formulae c_1, c_2 are arbitrary, in general, complex constants and $\beta > 0$ is a certain real number.

We now proceed to a case more complicated in computational sense.

II.
$$p_1^{(0)} = 0 \tag{3.29}$$

The simplest situation in this case occurs when

$$\rho > 1.$$

The averaged system (2.30) takes the form

$$\dot{u}_1 = \left[D + \hat{R}_1(t)\right]u_1, \tag{3.30}$$

where, with account of (2.34),

$$\hat{R}_1(t) = A_1 t^{-\rho} + \cdots + A_{k+1}t^{-(k+1)\rho} + O\left(t^{-(\rho+1)}\right) + O\left(t^{-(2\rho+1)}\right) + O\left(t^{-(k+2)\rho}\right).$$

We remark that, due to (3.29), matrix A_1, that is described by formula (3.14), has the following form:

$$A_1 = \frac{2}{3}q_1^{(0)}\begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix}. \tag{3.31}$$

It follows that the entries $\hat{r}_{ij}(t)$ of the matrix $\hat{R}_1(t)$ have the property

$$t^{i-j}\hat{r}_{ij}(t) \in L_1[t_0, \infty), \qquad 1 \le i, j \le 2.$$

This yields that like in the case (3.10) we can use [2, Corollary 6.2, p. 213] to construct the asymptotics for the fundamental solutions of system (3.30). Hence, we obtain asymptotic formulae (3.12) for the fundamental solutions of this system. Thus, we get asymptotics (3.13) for solutions of Eq. (2.2) as $t \to \infty$.

Assume further that

$$\rho \le 1.$$

In the averaged system (2.30) we make one more averaging change of variable

$$u_1 = \left[I + Q(t)t^{-(\rho+1)}\right]u_2 \tag{3.32}$$

that allows us, due to Theorem 2.7, to exclude the summand having the asymptotic order $O(t^{-(\rho+1)})$ in the remainder term (2.34). Here matrix $Q(t)$, whose entries are trigonometric polynomials, is the solution of the matrix differential equation

$$\dot{Q} - DQ + QD = \rho Y_1(t)$$

with zero mean value. The main part of the transformed system has the same form as the main part of system (2.30) but the new remainder term has now the following asymptotic estimate as $t \to \infty$:

$$R_2(t) = O\left(t^{-(\rho+2)}\right) + O\left(t^{-(2\rho+1)}\right) + O\left(t^{-(k+2)\rho}\right). \tag{3.33}$$

Then in the obtained system we make the so-called shearing transformation

$$u_2 = \begin{pmatrix} t^{\frac{\rho}{2}} & \\ 0 & t^{-\frac{\rho}{2}} \end{pmatrix} u_3. \tag{3.34}$$

With account of formulae (2.18) and (3.31), that describe matrices D and A_1, we get the following system:

$$\dot{u}_3 = \Big[B_1 t^{-\rho} + B_2 t^{-2\rho} + \cdots + B_k t^{-k\rho} + B_0 t^{-1} + R_3(t)\Big] u_3. \tag{3.35}$$

Here $B_0, \ldots, B_k$ are certain constant matrices and, in particular,

$$B_0 = \frac{\rho}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 1 \\ a_{21}^{(2)} & 2q_1^{(0)} \end{pmatrix}, \quad B_2 = \begin{pmatrix} a_{11}^{(2)} & \frac{2}{3}q_1^{(0)} \\ a_{21}^{(3)} & a_{22}^{(2)} \end{pmatrix}.$$

Symbols $a_{ij}^{(2)}$ in the above expressions denote the entries of the matrix A_2, situated in the corresponding positions, and symbol $a_{21}^{(3)}$ denotes the corresponding entry of the matrix A_3 from the averaged system (2.30). In what follows we only need the explicit formula for the entry $a_{21}^{(2)}$ of the matrix A_2. We devote the conclusive part of this section to computation of this entry. Finally, we note that the remainder term in (3.35), due to (3.33) and (3.34), has the asymptotic estimate

$$R_3(t) = O\big(t^{-2}\big) + O\big(t^{-(\rho+1)}\big) + O\big(t^{-(k+1)\rho}\big), \quad t \to \infty \tag{3.36}$$

and, therefore, belongs to $L_1[t_0, \infty)$ taking into account (1.5).

We start with the case

$$\rho = 1.$$

System (3.35) due to (3.36) gets the form

$$\dot{u}_3 = \Big[W t^{-1} + O\big(t^{-2}\big)\Big] u_3, \tag{3.37}$$

where

$$W = B_0 + B_1 = \begin{pmatrix} -\frac{1}{2} & 1 \\ a_{21}^{(2)} & \frac{1}{2} + 2q_1^{(0)} \end{pmatrix}.$$

The eigenvalues of the matrix W are

$$\nu_{1,2} = q_1^{(0)} \pm \xi, \qquad \xi = \sqrt{\Big(q_1^{(0)} + \frac{1}{2}\Big)^2 + a_{21}^{(2)}}. \tag{3.38}$$

Here the square root is defined according to (3.16). The further asymptotic analysis of system (3.37) is conducted in the same way as for the case (3.22). Thus, we give here only the final result with account of the transformation (3.34).

- $\big(q_1^{(0)} + \frac{1}{2}\big)^2 + a_{21}^{(2)} \neq 0$
  We have the following asymptotic representation for solutions of Eq. (2.2) as $t \to \infty$:

$$y(t) = t^{\frac{1}{2}+q_1^{(0)}}\Big[c_1 \exp\{\xi \ln t\}\big(1 + o(1)\big) + c_2 \exp\{-\xi \ln t\}\big(1 + o(1)\big)\Big] + O\big(e^{-\beta t}\big),$$

where c_1, c_2 are arbitrary, in general, complex constants, $\beta > 0$ is a certain real number and the quantity ξ is defined by formula (3.38).

- $\big(q_1^{(0)} + \frac{1}{2}\big)^2 + a_{21}^{(2)} = 0$
  In this case the behaviour of solutions of Eq. (2.2) as $t \to \infty$ is described by the asymptotic formula

$$y(t) = c_1 t^{\frac{1}{2}+q_1^{(0)}}\big(1 + o(1)\big) + c_2 t^{\frac{1}{2}+q_1^{(0)}} \ln t\big(1 + o(1)\big) + O\big(e^{-\beta t}\big),$$

where c_1, c_2 are arbitrary real constants and $\beta > 0$ is a certain real number.

Let

$$\rho < 1.$$

System (3.35) takes form (2.35), where

$$\alpha = \rho, \quad A_0 = B_1, \quad V(t) = B_2 t^{-\rho} + \cdots + B_k t^{(-k+1)\rho} + B_0 t^{\rho-1}, \quad R_2(t) = R_3(t). \tag{3.39}$$

The eigenvalues of the matrix $A_0 = B_1$ are

$$\nu_{1,2} = q_1^{(0)} \pm \kappa, \qquad \kappa = \sqrt{\left(q_1^{(0)}\right)^2 + a_{21}^{(2)}}, \tag{3.40}$$

where the square root means (3.16). Further in this paper we study only the case

$$\left(q_1^{(0)}\right)^2 + a_{21}^{(2)} \neq 0, \tag{3.41}$$

when these eigenvalues are distinct. Provided condition (3.41) holds, system (3.35), due to Lemma 2.8, can be reduced to L-diagonal form (2.36), where the entries of the diagonal matrix $\Lambda(t) = \operatorname{diag}\big(\lambda_1(t), \lambda_2(t)\big)$ are the eigenvalues of the matrix $\big(A_0 + V(t)\big)t^{-\rho}$. By (3.39), these eigenvalues have the following form:

$$\lambda_{1,2}(t) = q_1^{(0)} t^{-\rho} \pm \kappa t^{-\rho}\Big(1 + O\big(t^{-\rho}\big) + O\big(t^{2\rho-2}\big) + O\big(t^{-1}\big)\Big) + \frac{a_{11}^{(2)} + a_{22}^{(2)}}{2} t^{-2\rho} + O\big(t^{-3\rho}\big).$$

Here all the terms denoted by the order symbol $O(\cdot)$ are real valued. The asymptotics for the fundamental matrix of system (3.35) can be constructed according to Theorem 2.9. If we then return to Eq. (2.2), we get the following asymptotic formulae for its solutions as $t \to \infty$.

If

$$\frac{1}{2} < \rho < 1,$$

we have

$$\begin{aligned} y(t) = t^{\frac{\rho}{2}} \exp\left\{\frac{q_1^{(0)}}{1-\rho} t^{1-\rho}\right\} \bigg[c_1 \exp\left\{\frac{\kappa}{1-\rho} t^{1-\rho}\right\} (1 + o(1)) \\ + c_2 \exp\left\{-\frac{\kappa}{1-\rho} t^{1-\rho}\right\} (1 + o(1)) \bigg] + O\big(e^{-\beta t}\big). \end{aligned} \tag{3.42}$$

If

$$\rho = \frac{1}{2},$$

then

$$\begin{aligned} y(t) = t^{\frac{1}{4} + \frac{a_{11}^{(2)} + a_{22}^{(2)}}{2}} \exp\left\{2 q_1^{(0)} \sqrt{t}\right\} \Big[c_1 \exp\left\{2\kappa \left(\sqrt{t} + O(\ln t)\right)\right\} (1 + o(1)) \\ + c_2 \exp\left\{-2\kappa \left(\sqrt{t} + O(\ln t)\right)\right\} (1 + o(1)) \Big] + O\big(e^{-\beta t}\big). \end{aligned} \tag{3.43}$$

Finally, if

$$\rho < \frac{1}{2},$$

we obtain

$$
\begin{aligned}
y(t) = t^{\frac{\rho}{2}} \exp\left\{ \frac{q_1^{(0)}}{1-\rho} t^{1-\rho} + \frac{a_{11}^{(2)} + a_{22}^{(2)}}{2(1-2\rho)} t^{1-2\rho} + O\left(\int t^{-3\rho} dt \right) \right\} \\
\times \left[c_1 \exp\left\{ \frac{\kappa}{1-\rho} t^{1-\rho} \left(1 + O(t^{-\rho})\right) \right\} (1 + o(1)) \right. \\
\left. + c_2 \exp\left\{ -\frac{\kappa}{1-\rho} t^{1-\rho} \left(1 + O(t^{-\rho})\right) \right\} (1 + o(1)) \right] + O(e^{-\beta t}).
\end{aligned}
\tag{3.44}
$$

Everywhere in these asymptotic formulae c_1, c_2 are arbitrary, in general, complex constants, $\beta > 0$ is a certain real number and the quantity κ is defined in (3.40).

Computation of the quantity $a_{21}^{(2)}$ in case (3.29)

It follows from the asymptotic formulae (3.42)–(3.44) that the key role in the oscillation problem for Eq. (1.1) plays the quantity $a_{21}^{(2)}$. It defines, due to (3.16) and (3.40), whether the number κ is real or pure imaginary. We recall that the quantity $a_{21}^{(2)}$ is the corresponding entry of the matrix A_2. The latter is defined by formula (2.32) with account of (3.8). First, we calculate the matrix $Y_1(t)$ as the solution of the matrix differential equation (2.33) with zero mean value. Recalling the form of the matrix D (see (2.18)) and also formulae (3.7), (3.31), we conclude that the entries $y_{ij}(t)$ of the matrix $Y_1(t)$ satisfy the following linear differential system with constant coefficients:

$$
\begin{aligned}
\dot{y}_{11} &= y_{21} - \frac{2}{3} p_1(t), & \dot{y}_{12} &= y_{22} - y_{11} + \frac{2}{3}\left(q_1^{(0)}(t) + p_1(t)\right), \\
\dot{y}_{21} &= -2p_1(t), & \dot{y}_{22} &= -y_{21} + 2\left(q_1^{(0)}(t) + p_1(t)\right).
\end{aligned}
$$

Here

$$
q_1^{(0)}(t) = q_1(t) - q_1^{(0)}, \tag{3.45}
$$

function $q_1(t)$ is defined in (1.4) (see also (3.3)), and the real number $q_1^{(0)}$ is its mean value according to (3.5). After some easy calculations we obtain

$$
\begin{aligned}
y_{11}(t) &= -2 \iint p_1(t)(dt)^2 - \frac{2}{3} \int p_1(t) dt, \\
y_{12}(t) &= 4 \iiint p_1(t)(dt)^3 + \iint \left(2q_1^{(0)}(t) + \frac{8}{3} p_1(t)\right)(dt)^2 + \frac{2}{3} \int \left(q_1^{(0)}(t) + p_1(t)\right) dt, \\
y_{21}(t) &= -2 \int p_1(t) dt, \\
y_{22}(t) &= 2 \iint p_1(t)(dt)^2 + 2 \int \left(q_1^{(0)}(t) + p_1(t)\right) dt.
\end{aligned}
\tag{3.46}
$$

Symbol $\int$ denotes the antiderivative having zero mean value. Further we will use the following relations that can be proved simply by integration by parts. If $f(t)$ is a trigonometric polynomial (or T-periodic function as well) with zero mean value then the following equalities hold:

$$
\begin{aligned}
&\mathrm{M}\left[f(t) \iint f(t)(dt)^2\right] = -\mathrm{M}\left[\left(\int f(t) dt\right)^2\right], \quad \mathrm{M}\left[f(t) \int f(t) dt\right] = 0, \\
&\mathrm{M}\left[f(t) \iiint f(t)(dt)^3\right] = -\mathrm{M}\left[\int f(t) dt \iint f(t)(dt)^2\right] = 0.
\end{aligned}
\tag{3.47}
$$

We recall now (3.7), (3.31) and take into account (3.46), (3.47) to conclude that

$$\begin{aligned} &\mathrm{M}\big[A_1(t)Y_1(t)\big] \\ &\quad= \mathrm{M}\big[(A_1(t)-A_1)Y_1(t)\big] \\ &\quad= \frac{4}{9}\begin{pmatrix} -3\,\mathrm{M}\Big[\big(\int p_1(t)dt\big)^2\Big]+3\,\mathrm{M}\Big[p_1(t)\int q_1^{(0)}(t)dt\Big] & \mathrm{M}\Big[\big(\int p_1(t)dt\big)^2\Big]-\mathrm{M}\Big[p_1(t)\int q_1^{(0)}(t)dt\Big] \\ -9\,\mathrm{M}\Big[\big(\int p_1(t)dt\big)^2\Big]+9\,\mathrm{M}\Big[p_1(t)\int q_1^{(0)}(t)dt\Big] & 3\,\mathrm{M}\Big[\big(\int p_1(t)dt\big)^2\Big]-3\,\mathrm{M}\Big[p_1(t)\int q_1^{(0)}(t)dt\Big] \end{pmatrix}. \end{aligned} \tag{3.48}$$

To calculate the entries of the matrix A_2 we also need to find row-vector (3.9). This is done as follows. We substitute (2.23) in (2.22) and collect terms corresponding to factor $t^{-\rho}$. With account of (2.5) and (3.6) we obtain the following problem:

$$\begin{aligned} &\Phi(\theta)\Psi(0)\big[q_1(t)\Phi'(-1)-p_1(t)\Phi(-1)\big]+H_1(t,\theta)D+\frac{\partial H_1}{\partial t} \\ &\qquad\qquad= \begin{cases} \dfrac{\partial H_1}{\partial \theta}, & -h\le\theta<0, \\ H_1(t,0)-H_1(t,-1)+q_1(t)\Phi'(-1)-p_1(t)\Phi(-1), & \theta=0. \end{cases} \end{aligned}$$

We apply (2.18) to get the following partial differential system for finding the entries of row-vector (3.9):

$$\begin{aligned} \frac{\partial h_{11}}{\partial\theta} &= \frac{\partial h_{11}}{\partial t}-\left(\frac{2}{3}+2\theta\right)p_1(t), \\ \frac{\partial h_{12}}{\partial\theta} &= \frac{\partial h_{12}}{\partial t}+h_{11}+\left(\frac{2}{3}+2\theta\right)(q_1(t)+p_1(t)), \end{aligned} \tag{3.49}$$

where $-h\le\theta<0$. At the point $\theta=0$ the solution of this system should satisfy the condition

$$\begin{aligned} \frac{\partial h_{11}}{\partial t}(t,0) &= h_{11}(t,0)-h_{11}(t,-1)-\frac{p_1(t)}{3}, \\ \frac{\partial h_{12}}{\partial t}(t,0) &= h_{12}(t,0)-h_{11}(t,0)-h_{12}(t,-1)+\frac{1}{3}(q_1(t)+p_1(t)). \end{aligned} \tag{3.50}$$

Due to (3.3), we seek the solution of (3.49), (3.50) in the form

$$h_{11}(t,\theta)=\sum_{j=-N}^{N} g_1^{(j)}(\theta)e^{\mathrm{i}\omega_j t}, \qquad h_{12}(t,\theta)=\sum_{j=-N}^{N} g_2^{(j)}(\theta)e^{\mathrm{i}\omega_j t}, \tag{3.51}$$

where the infinitely differentiable functions $g_1^{(j)}(\theta)$ and $g_2^{(j)}(\theta)$ belong to subspace Q_Λ. Hence, by (2.17), these functions should satisfy the following additional condition:

$$\big(\Psi(\xi),g_i^{(j)}(\theta)\big)=0, \quad i=1,2, \quad j=-N,\dots,N. \tag{3.52}$$

Here the bilinear form $(\cdot,\cdot)$ is defined according to (2.13) and the column-vector $\Psi(\xi)$ has form (2.18).

It follows from (2.32) and (3.8) that to compute the quantity $a_{21}^{(2)}$ we need to find only the function $h_{11}(t,\theta)$. We substitute (3.3), (3.51) in (3.49), (3.50) and match the coefficients of the corresponding exponentials $e^{\mathrm{i}\omega_j t}$. Thus, we get the following boundary value problems for functions $g_1^{(j)}(\theta)$:

$$\begin{aligned} &\frac{dg_1^{(j)}}{d\theta}=\mathrm{i}\omega_j g_1^{(j)}(\theta)-\Big(\frac{2}{3}+2\theta\Big)p_1^{(j)}, \\ &(1-\mathrm{i}\omega_j)g_1^{(j)}(0)-g_1^{(j)}(-1)=\frac{p_1^{(j)}}{3}, \quad j=-N,\dots,N. \end{aligned} \tag{3.53}$$

It is easy to verify that

$$g_1^{(j)}(\theta) = \left(\frac{e^{\mathrm{i}\omega_j\theta}}{1 - \mathrm{i}\omega_j - e^{-\mathrm{i}\omega_j}} - \frac{2\mathrm{i}\theta}{\omega_j} - \frac{6 + 2\mathrm{i}\omega_j}{3\omega_j^2} \right) p_1^{(j)}, \quad j \neq 0. \tag{3.54}$$

If $j = 0$ then, by (3.4) and (3.29), we have $\omega_0 = 0$ and $p_1^{(0)} = 0$. This yields that the corresponding solution of (3.53) has the form $g_1^{(0)}(\theta) \equiv c$, where c is a certain constant. The quantity c is uniquely defined from equality (3.52). Finally, we deduce that

$$g_1^{(0)}(\theta) \equiv 0. \tag{3.55}$$

Therefore, taking into account (2.32), (3.8) and also expression (3.48), we get the following representation for the quantity $a_{21}^{(2)}$:

$$a_{21}^{(2)} = -4\,\mathrm{M}\left[\left(\int p_1(t)dt \right)^2 \right] + 4\,\mathrm{M}\left[p_1(t) \int q_1^{(0)}(t)dt \right] - 2\,\mathrm{M}\left[p_2(t)\right]$$
$$+ 2\,\mathrm{M}\left[q_1(t) \frac{\partial h_{11}}{\partial \theta}(t,\theta)\Big|_{\theta=-1} \right] - 2\,\mathrm{M}\left[p_1(t)h_{11}(t,-1)\right]. \tag{3.56}$$

Here the function $q_1^{(0)}(t)$ is defined according to (3.45) and the function $h_{11}(t,\theta)$ has form (3.51) with account of (3.54) and (3.55). If we calculate in (3.56) all the mean values and use (3.4) we obtain the more compact form for $a_{21}^{(2)}$. Namely, we conclude that

$$a_{21}^{(2)} = 2 \sum_{\substack{j=-N \\ j\neq 0}}^{N} \frac{(\mathrm{i}\omega_j p_1^{(j)} \bar{q}_1^{(j)} - |p_1^{(j)}|^2) e^{-\mathrm{i}\omega_j}}{1 - \mathrm{i}\omega_j - e^{-\mathrm{i}\omega_j}} - 2\,\mathrm{M}\big[p_2(t)\big]. \tag{3.57}$$

Function $p_2(t)$ in this expression is defined by formula (3.2).

4 Conclusions and examples

We begin this section by analyzing the asymptotic formulae obtained in the previous section as applied to the oscillation problem of Eq. (1.1) with conditions (1.3), (1.4). The results of the analysis are given in Tables 4.1 and 4.2.

	$\rho > 2$	$\rho = 2$	$\rho < 2$
o	–	$p_1^{(0)} > \frac{1}{8}$	$p_1^{(0)} > 0$
p	+	$p_1^{(0)} \leq \frac{1}{8}$	$p_1^{(0)} < 0$

Table 4.1: Case $p_1^{(0)} \neq 0$.

In these tables the line titled «o» contains the conditions for existence of oscillatory solutions and the line titled «p» contains the conditions for existence of nonoscillatory (positive) solutions. Symbol «–» means the situation when the oscillatory solutions are not found by

	$\rho > 1$	$\rho = 1$	$\rho < 1$
o	−	$\left(q_1^{(0)} + \frac{1}{2}\right)^2 + a_{21}^{(2)} < 0$	$\left(q_1^{(0)}\right)^2 + a_{21}^{(2)} < 0$
p	+	$\left(q_1^{(0)} + \frac{1}{2}\right)^2 + a_{21}^{(2)} \geq 0$	$\left(q_1^{(0)}\right)^2 + a_{21}^{(2)} > 0$

Table 4.2: Case $p_1^{(0)} = 0$.

means of the main parts of the asymptotic formulae in the prescribed interval of the parameter ρ (highly likely oscillatory solutions don't exist at all). Symbol «+» stands for the situation when there exist nonoscillatory (positive) solutions for all values of the parameter ρ in the prescribed interval. In all the other positions of these tables the conditions for existence of oscillatory and nonoscillatory (positive) solutions of Eq. (1.1) in the prescribed intervals of the parameter ρ are collected. We also remind that the real numbers $p_1^{(0)}, q_1^{(0)}$ are defined in (3.5) and the real number $a_{21}^{(2)}$ is described by formula (3.57) with account of (3.2) and (3.3).

We now demonstrate the obtained results by a number of illustrating examples.

Example 4.1. In paper [13] the authors illustrate the obtained criteria for existence of the positive solutions by the following equation:

$$\frac{d\hat{x}}{ds} = -\hat{a}(s)\hat{x}\left(s - c - \frac{d}{s}\right), \tag{4.1}$$

where $c, d > 0$. It is claimed that if

$$\hat{a}(s) \leq \frac{1}{ec} - \frac{d}{ec^2} \cdot \frac{1}{s} + \frac{1}{e} \cdot \left(\frac{d^2}{c^3} + \frac{c}{8}\right) \cdot \frac{1}{s^2} + o\left(\frac{1}{s^2}\right) \tag{4.2}$$

or

$$\hat{a}(s) \leq \frac{1}{ec} - \frac{d}{ec^2} \cdot \frac{1}{s} + \frac{1}{e} \cdot \left(\frac{d^2}{c^3} + \frac{d}{2c}\right) \cdot \frac{1}{s^2} + o\left(\frac{1}{s^2}\right) \tag{4.3}$$

as $s \to \infty$ then Eq. (4.1) has positive solution.

We consider the case when function $\hat{a}(s)$ in Eq. (4.1) has the following asymptotic representation as $s \to \infty$:

$$\hat{a}(s) = \frac{1}{ec} + \hat{a}_1(s)s^{-1} + \hat{a}_2(s)s^{-2} + O(s^{-3}), \tag{4.4}$$

where $\hat{a}_1(s), \hat{a}_2(s)$ are real-valued trigonometric polynomials. In particular,

$$\hat{a}_1(s) = \sum_{j=-N}^{N} \hat{a}_1^{(j)} e^{i\omega_j s}, \tag{4.5}$$

and, besides,

$$\hat{a}_1^{(-j)} = \overline{\hat{a}}_1^{(j)}, \quad \omega_{-j} = -\omega_j \ (\omega_l \neq \omega_l,\ l \neq m), \quad j = 1, \ldots, N.$$

In Eq. (4.1) we make the change of independent variable $s = tc$ that transforms it to form (1.1), where

$$x(t) = \hat{x}(ct), \quad a(t) = c\hat{a}(tc), \quad \tau(t) = 1 + \frac{d}{c^2} \cdot \frac{1}{t}. \tag{4.6}$$

Due to (4.4) and (4.6), we conclude that in the considered case the coefficients in the expansions (1.3), (1.4) have the following form:

$$a_1(t) = \hat{a}_1(tc), \quad a_2(t) = \frac{\hat{a}_2(tc)}{c}, \quad q_1(t) \equiv \frac{d}{c^2}, \quad q_m(t) \equiv 0, \quad m \geq 2 \tag{4.7}$$

and $\rho = 1$. It follows from (3.5) with account of (3.2) that

$$p_1^{(0)} = ea_1^{(0)} + q_1^{(0)}, \quad q_1^{(0)} = \frac{d}{c^2},$$

where

$$a_1^{(0)} = \mathrm{M}[a_1(t)] = \hat{a}_1^{(0)}, \quad \hat{a}_1^{(0)} = \mathrm{M}[\hat{a}_1(s)].$$

We deduce from Table 4.1 that equation (4.1), (4.4) has oscillatory solutions if

$$\hat{a}_1^{(0)} > -\frac{d}{ec^2}$$

and positive solutions if

$$\hat{a}_1^{(0)} < -\frac{d}{ec^2}.$$

We also need to study the case when $p_1^{(0)} = 0$, i.e.,

$$\hat{a}_1^{(0)} = -\frac{d}{ec^2}.$$

By (3.2) and (4.7), we have

$$\mathrm{M}[p_2(t)] = ea_2^{(0)} + \frac{d^2}{2c^4} + ea_1^{(0)} \cdot \frac{d}{c^2} = ea_2^{(0)} - \frac{d^2}{2c^4},$$

where

$$a_2^{(0)} = \mathrm{M}\left[a_2(t)\right] = \frac{\hat{a}_2^{(0)}}{c}, \quad \hat{a}_2^{(0)} = \mathrm{M}\left[\hat{a}_2(s)\right]. \tag{4.8}$$

We then compute quantity (3.57) using (3.2), (4.5), (4.7) and (4.8). We obtain

$$a_{21}^{(2)} = -2e^2 \sum_{\substack{j=-N \\ j \neq 0}}^{N} \frac{\left|\hat{a}_1^{(j)}\right|^2 e^{-\mathrm{i}c\omega_j}}{1 - \mathrm{i}c\omega_j - e^{-\mathrm{i}c\omega_j}} - \frac{2e\hat{a}_2^{(0)}}{c} + \frac{d^2}{c^4}.$$

It follows from Table 4.2 that equation (4.1), (4.4) has oscillatory solutions if

$$\hat{a}_2^{(0)} > \frac{1}{e}\left(\frac{d^2}{c^3} + \frac{d}{2c} + \frac{c}{8}\right) - ec \sum_{\substack{j=-N \\ j \neq 0}}^{N} \frac{\left|\hat{a}_1^{(j)}\right|^2 e^{-\mathrm{i}c\omega_j}}{1 - \mathrm{i}c\omega_j - e^{-\mathrm{i}c\omega_j}} \tag{4.9}$$

and positive solutions if

$$\hat{a}_2^{(0)} \leq \frac{1}{e}\left(\frac{d^2}{c^3} + \frac{d}{2c} + \frac{c}{8}\right) - ec \sum_{\substack{j=-N \\ j \neq 0}}^{N} \frac{\left|\hat{a}_1^{(j)}\right|^2 e^{-\mathrm{i}c\omega_j}}{1 - \mathrm{i}c\omega_j - e^{-\mathrm{i}c\omega_j}}. \tag{4.10}$$

We now consider the special case when the following identity holds in (4.4):

$$\hat{a}_1(s) \equiv \hat{a}_1^{(0)} = -\frac{d}{ec^2}. \tag{4.11}$$

In this situation formulae (4.9), (4.10) take the simple form. It is easily seen that in this case equation (4.1), (4.4) with the coefficient $\hat{a}_1(s)$ described by (4.11) has oscillatory solutions if

$$\hat{a}_2^{(0)} > \frac{1}{e}\left(\frac{d^2}{c^3} + \frac{d}{2c} + \frac{c}{8}\right)$$

and positive solutions if

$$\hat{a}_2^{(0)} \le \frac{1}{e}\left(\frac{d^2}{c^3} + \frac{d}{2c} + \frac{c}{8}\right).$$

This fact allows us to propose the **hypothesis** that the condition for existence of positive solutions in Eq. (4.1) is described by the inequality

$$\hat{a}(s) \le \frac{1}{ec} - \frac{d}{ec^2}\cdot\frac{1}{s} + \frac{1}{e}\cdot\left(\frac{d^2}{c^3} + \frac{d}{2c} + \frac{c}{8}\right)\cdot\frac{1}{s^2} + o\left(\frac{1}{s^2}\right), \quad s \to \infty, \tag{4.12}$$

instead of (4.2) and (4.3).

Example 4.2. This example concerns equation (1.1), where

$$a(t) = \frac{1}{e}\left(1 + \frac{K(\sin^2 \pi t - \gamma)}{t^\rho}\right), \quad 0 < \rho \le 2, \quad \tau(t) \equiv 1, \tag{4.13}$$

and $K > 0$, $\gamma \in \mathbb{R}$. Equation (1.1), (4.13) was considered in [15, 17]. In [17], this equation was studied provided that $\gamma = 0$. In this case it was shown that all solutions of this equation oscillate if $K > 0$ and $0 \le \rho < 2$, and also if $K > 1$ and $\rho = 2$. If $K < \frac{1}{8}$ and $\rho = 2$, then equation (1.1), (4.13) has nonoscillatory solution. In paper [15], equation (1.1), (4.13) was studied in the case $\rho = 2$. It was shown that if $\gamma < \frac{1}{2}$ and $K > \frac{1}{4(1-2\gamma)}$ then all solutions of this equation oscillate. In particular, the authors improved the results from [17] for the case $\gamma = 0$.

We write (4.13) in form (1.3), (1.4) and obtain

$$a_1(t) = \frac{K}{e}\left(\sin^2 \pi t - \gamma\right), \quad q_1(t) \equiv 0, \quad a_m(t) = q_m(t) \equiv 0, \quad m \ge 2.$$

We deduce from (3.2) and (3.5) that

$$p_1(t) = K\left(\sin^2 \pi t - \gamma\right) = K\left(\frac{1}{2} - \gamma\right) - \frac{K}{2}\cos 2\pi t, \quad p_2(t) \equiv 0 \tag{4.14}$$

and

$$p_1^{(0)} = \mathrm{M}\left[p_1(t)\right] = K\left(\frac{1}{2} - \gamma\right), \quad q_1^{(0)} = \mathrm{M}\left[q_1(t)\right] = 0. \tag{4.15}$$

It follows from Table 4.1 that if $\rho = 2$ then equation (1.1), (4.13) has oscillatory solutions provided that inequality $4K(1-2\gamma) > 1$ holds and positive solutions if $4K(1-2\gamma) \le 1$. Parameter γ in these inequalities may take all the values except $\gamma = \frac{1}{2}$. If $0 < \rho < 2$ then the considered equation has oscillatory solutions if $\gamma < \frac{1}{2}$ and positive solutions if $\gamma > \frac{1}{2}$. Parameter K may take all the positive values.

The more difficult case occurs when

$$\gamma = \frac{1}{2}, \tag{4.16}$$

since we have $p_1^{(0)} = 0$. We calculate coefficients in (3.3) with account of (4.14), (4.16) and conclude that

$$N = 1, \quad p_1^{(1)} = p_1^{(-1)} = -\frac{K}{4}, \quad \omega_1 = 2\pi, \quad \omega_{-1} = -2\pi. \tag{4.17}$$

We compute quantity (3.57) using (4.17) to get

$$a_{21}^{(2)} = -\frac{K^2}{8}\left(\frac{e^{2\pi i}}{1 + i2\pi - e^{2\pi i}} + \frac{e^{-2\pi i}}{1 - i2\pi - e^{-2\pi i}}\right) = 0. \tag{4.18}$$

We then deduce from Table 4.2 that equation (1.1), (4.13) under condition (4.16) has nonoscillatory solutions for all values of the parameter $K > 0$ if $1 < \rho \leq 2$. If $\rho = 1$ then, by (4.15), we also conclude that this equation has nonoscillatory solutions for all values of the parameter $K > 0$.

Unfortunately, the obtained results don't allow us to analyze the oscillation problem for equation (1.1), (4.13) under condition (4.16) for the case $\rho < 1$. In this situation condition (3.41), under which the asymptotic representations were constructed in this paper, fails. Nevertheless, certain advance in the analysis of the oscillation problem for this case can still be made. Note that in relation to the studied equation system (3.35) in the case $\rho < 1$ takes the following form:

$$\dot{u}_3 = \left[B_1 t^{-\rho} + B_0 t^{-1} + O\left(t^{-2\rho}\right)\right] u_3. \tag{4.19}$$

Here matrices B_0, B_1 with account of (4.15), (4.18) are described by the formulae

$$B_0 = \frac{\rho}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We make the change of variable

$$u_3 = \begin{pmatrix} 1 & 1 \\ 0 & \rho t^{\rho-1} \end{pmatrix} u_4$$

to reduce system (4.19) to form (2.35) for new unknown variable $u_4(t)$, where

$$\alpha = 1, \quad A_0 = \begin{pmatrix} -\frac{\rho}{2} & \rho - 1 \\ 0 & 1 - \frac{\rho}{2} \end{pmatrix}, \quad V(t) \equiv 0, \quad R_2(t) = O\left(t^{1-3\rho}\right).$$

We remark that the eigenvalues of the matrix A_0 are distinct and the remainder term $R_2(t)$ belongs to $L_1[t_0, \infty)$ if $\rho > \frac{2}{3}$. Thus, we can bring the obtained system to L-diagonal form (2.36) by certain transformation with constant coefficients and then apply Levinson's Theorem to get the asymptotics for its fundamental matrix. Some easy calculations show that in this situation we obtain the asymptotic representation (3.13) for solutions of Eq. (2.2) as $t \to \infty$. Hence, equation (1.1), (4.13) under condition (4.16) has nonoscillatory solutions for all the values of the parameter $K > 0$ if $\frac{2}{3} < \rho < 1$. Evidently, to study the case $\rho \leq \frac{2}{3}$ under condition (4.16) we need to compute the entries of the matrix B_2 in system (3.35).

Example 4.3. Our last example deals with equation (1.1), where

$$a(t) = \frac{1}{e} + \frac{a \sin \omega t}{t^\rho}, \quad \tau(t) = 1 + \frac{b \sin \omega t}{t^\rho}, \quad \rho > 0 \tag{4.20}$$

and $a, b \in \mathbb{R}$, $\omega > 0$. Therefore,

$$a_1(t) = a \sin \omega t, \quad q_1(t) = b \sin \omega t, \quad a_m(t) = q_m(t) \equiv 0, \quad m \geq 2.$$

It follows from (3.2) and (3.5) that

$$p_1(t) = (ea+b)\sin\omega t, \quad p_2(t) = \left(\frac{b^2}{2} + eab\right)\sin^2\omega t \tag{4.21}$$

and, moreover,

$$p_1^{(0)} = \mathrm{M}\left[p_1(t)\right] = 0, \quad q_1^{(0)} = \mathrm{M}\left[q_1(t)\right] = 0, \quad p_2^{(0)} = \mathrm{M}\left[p_2(t)\right] = \frac{b^2}{4} + \frac{eab}{2}. \tag{4.22}$$

By calculating coefficients in (3.3) with account of (4.21), we get

$$N = 1, \quad p_1^{(1)} = -p_1^{(-1)} = \frac{ea+b}{2\mathrm{i}}, \quad q_1^{(1)} = -q_1^{(-1)} = \frac{b}{2\mathrm{i}}, \quad \omega_1 = -\omega_{-1} = \omega.$$

We then compute quantity (3.57) and conclude that

$$\begin{aligned} a_{21}^{(2)} = {} & 2\left(\frac{\mathrm{i}\omega e^{-\mathrm{i}\omega}}{1-\mathrm{i}\omega - e^{-\mathrm{i}\omega}} - \frac{\mathrm{i}\omega e^{\mathrm{i}\omega}}{1+\mathrm{i}\omega - e^{\mathrm{i}\omega}}\right)\frac{(ea+b)b}{4} \\ & - 2\left(\frac{e^{-\mathrm{i}\omega}}{1-\mathrm{i}\omega - e^{-\mathrm{i}\omega}} + \frac{e^{\mathrm{i}\omega}}{1+\mathrm{i}\omega - e^{\mathrm{i}\omega}}\right)\frac{(ea+b)^2}{4} - \left(\frac{b^2}{2} + eab\right). \end{aligned}$$

We can write this expression in the real form. In particular, we used the mathematical package Wolfram Mathematica to obtain the following real-valued expression:

$$a_{21}^{(2)} = -\frac{(2eab+b^2)\omega^2 - 2e^2a^2 + 2\left(e^2a^2 + (eab+b^2)\omega^2\right)\cos\omega - 2\omega\left(b^2 - e^2a^2 + eab\right)\sin\omega}{2\left(\omega^2 - 2\omega\sin\omega - 2\cos\omega + 2\right)}. \tag{4.23}$$

If we consider the quantity $a_{21}^{(2)}$ as the function of ω we can write the following limit relations (again we used *Wolfram Mathematica*):

$$a_{21}^{(2)} = -\frac{2(ea+b)^2}{\omega^2} + \frac{1}{18}\left(7e^2a^2 + 20eab + 22b^2\right) + O\left(\omega^2\right), \quad \omega \to 0, \tag{4.24}$$

$$a_{21}^{(2)} = -\frac{1}{2}\left(2eab + b^2\right) - \left(eab + b^2\right)\cos\omega + O\left(\omega^{-1}\right), \quad \omega \to \infty. \tag{4.25}$$

In particular, we conclude from (4.25) that $a_{21}^{(2)}$ as the function of ω is asymptotically 2π-periodic as $\omega \to \infty$. In Fig. 4.1 we give the graph of quantity $a_{21}^{(2)}$ as the function $f(\omega) = a_{21}^{(2)}(\omega)$ for the values of parameters $a = b = 1$.

To obtain the conditions for existence of oscillatory or nonoscillatory solutions of (1.1), (4.20) we can use Table 4.2 with account of (4.22) and (4.23). In particular, if $\rho \leq 1$ then it follows from (4.24) that for all sufficiently small ω equation (1.1), (4.20) has oscillatory solutions for all the values of parameters $a, b \in \mathbb{R}$ not simultaneously equal to zero.

It is highly likely that the obtained results are still valid in the case when $a_j(t)$, $q_j(t)$, $j = 1, \dots, k+1$ in (1.3), (1.4) are sufficiently smooth ω-periodic functions. In this situation the periodic coefficients are described in terms of the infinite Fourier series having form (3.3) with $N = +\infty$. Of course, the problem of convergence of the corresponding series (2.24) and its partial derivatives arises in these case. This question is not discussed here.

In conclusion we note that the oscillation problem in critical case can be also studied for the difference analog of equation (1.1):

$$\Delta y(n) = -g(n)y(n-k), \qquad k \in \mathbb{N},$$

where $g(n) > 0$ for all $n \in \mathbb{N}$. The corresponding results are discussed in paper [28].

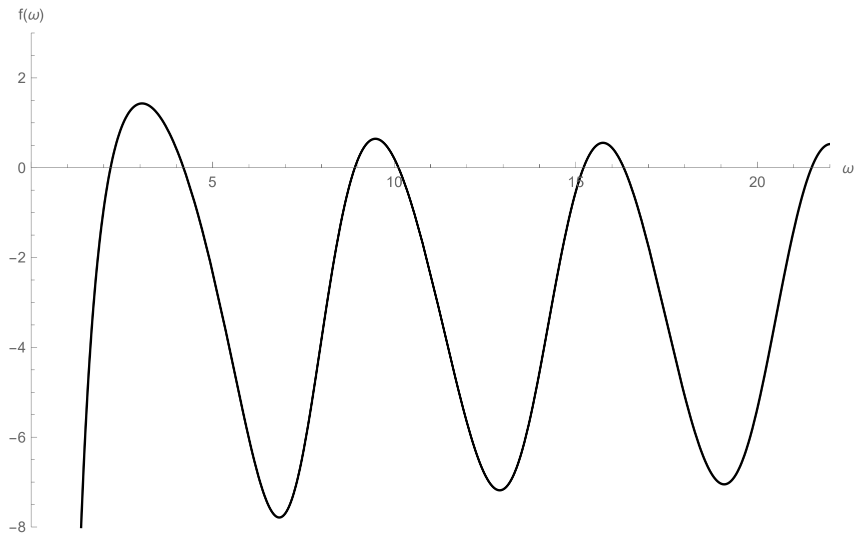

Figure 4.1: The graph of the quantity $a_{21}^{(2)}$, defined by (4.23), as the function $f(\omega) = a_{21}^{(2)}(\omega)$ for the values of parameters $a = b = 1$.

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