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EXISTENCE RESULTS FOR IMPULSIVE FUNCTIONAL AND NEUTRAL FUNCTIONAL DIFFERENTIAL INCLUSIONS WITH LOWER SEMICONTINUOUS RIGHT HAND SIDE

Mouffak Benchohra[1], Johnny Henderson[2*], Sotiris Ntouyas[3] and Abdelghani Ouahabi[1]

[1] Laboratoire de Mathématiques, Université de Sidi Bel Abbès, BP 89, 22000 Sidi Bel Abbès, Algérie,

[2] Department of Mathematics, Baylor University, Waco, TX 76798-7328, USA,

[3] Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece

[*] Corresponding Author. Email : Johnny_Henderson@yahoo.com

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Abstract: In this paper, Schaefer's fixed point theorem combined with a selection theorem due to Bressan and Colombo is used to investigate the existence of solutions for first and second order impulsive functional and neutral differential inclusions with lower semicontinuous and nonconvex-valued right-hand side.

Keywords: Impulsive differential inclusions, selection, existence, fixed point.

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1. Introduction

In this paper, we are concerned with the existence of solutions to some classes of initial value problems for first and second order impulsive functional and neutral functional differential inclusions. Initially, we will consider the first order impulsive functional differential inclusion,

$$y'(t) \in F(t, y_t), \quad a.e.\ t \in J := [0, T],\ t \neq t_k, \quad k = 1, \ldots, m, \tag{1.1}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \ldots, m, \tag{1.2}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \tag{1.3}$$

where $F : J \times C([-r, 0], \mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n)$ is a multivalued map with nonempty compact values, $\phi \in C([-r, 0], \mathbb{R}^n)$, $\mathcal{P}(\mathbb{R}^n)$ is the family of all nonempty subsets of $\mathbb{R}^n$, $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T$, $I_k \in C(\mathbb{R}^n, \mathbb{R}^n)$ $(k = 1, 2, \ldots, m)$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^-)$ and $y(t_k^+)$ represent the left and right limits of $y(t)$ at $t = t_k$, respectively, $\mathbb{R}^n$ the Euclidian Banach space with norm $|\cdot|$.

For any continuous function y defined on $[-r, T] - \{t_1, \ldots, t_m\}$ and any $t \in J$, we denote by y_t the element of $C([-r, 0], \mathbb{R}^n)$ defined by $y_t(\theta) = y(t + \theta)$, $\theta \in [-r, 0]$. Here $y_t(\cdot)$ represents the history of the state from time $t - r$, up to the present time t. Later, we study the second order impulsive functional differential inclusion of the form

$$y'' \in F(t, y_t),\ a.e.\ t \in J := [0, T],\ t \neq t_k, \quad k = 1, \ldots, m, \tag{1.4}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \ldots, m, \tag{1.5}$$

$$\Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k = 1, \ldots, m, \tag{1.6}$$

$$y(t) = \phi(t), \quad t \in [-r, 0],\ y'(0) = \eta, \tag{1.7}$$

where F, I_k, and ϕ are as in problem (1.1)-(1.3), $\overline{I}_k \in C(\mathbb{R}^n, \mathbb{R}^n)$ and $\eta \in \mathbb{R}^n$.

Sections 5 and 6 are devoted to the existence of solutions, for initial value problems for first and second order impulsive neutral functional differential inclusions. More precisely, in these last sections we consider the IVPs

$$\frac{d}{dt}[y(t) - g(t, y_t)] \in F(t, y_t), \quad \text{a.e. } t \in J := [0, T],\ t \neq t_k, \quad k = 1, \ldots, m, \tag{1.8}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \ldots, m, \tag{1.9}$$

$$y(t)=\phi(t), \quad t\in[-r,0], \tag{1.10}$$

where $F,\ I_k, \overline{I}_k$ and ϕ are as in problem (1.1)–(1.3) and (1.4)–(1.7), $g:[0,T]\times C([-r,0],\mathbb{R}^n)\to\mathbb{R}^n$, and

$$\frac{d}{dt}[y'(t)-g(t,y_t)]\in F(t,y_t), \quad \text{a.e. } t\in J:=[0,T],\ t\neq t_k, \quad k=1,\dots,m, \tag{1.11}$$

$$\Delta y|_{t=t_k}=I_k(y(t_k^-)), \quad k=1,\dots,m, \tag{1.12}$$

$$\Delta y'|_{t=t_k}=\bar{I}_k(y(t_k^-)), \quad k=1,\dots,m, \tag{1.13}$$

$$y(t)=\phi(t), \quad t\in[-r,0],\ y'(0)=\eta, \tag{1.14}$$

where $F,g,I_k,\overline{I}_k,\eta$ and ϕ are as in problem (1.1)–(1.3) and (1.4)–(1.7). Impulsive differential equations have become more important in recent years in some mathematical models of real world phenomena, especially in biological or medical domains; see the monographs of Bainov and Simeonov 2, Lakshmikantham, et al 20, Samoilenko and Perestyuk 22, and papers of Agur et al 1, Coldbeter et al 15, where numerous properties of their solutions are studied, and detailed bibliographies are given. Recently, by means of the Leray-Schauder alternative for convex-valued multivalued maps and a fixed point theorem due to Martelli for condensing multivalued maps, existence results of solutions for first and second order impulsive differential inclusions were given by Benchohra et al in 6-7 and Benchohra and Ntouyas in 4. For other results on functional differential equations, we refer the interested reader to the monograph of Erbe et al 12, Hale 17, Henderson 18, and the survey paper of Ntouyas 21. Notice that the fundamental tools such as the topological transversality theorem of Granas in 11, the Leray-Schauder alternative in 14, the lower and upper solutions method in 3 and 5, and fixed point argument in 10, have been used recently for various initial and boundary value problems for impulsive differential inclusions. However, in all the above works, the right-hand side, $F(t,y_t)$, was assumed to be convex-valued. Here we drop this restriction and consider problems with a nonconvex-valued right-hand side. Our approach here is based on fixed point argument combined with a selection theorem due to Bressan and Colombo in 8 for lower semicontinuous multivalued operators.

This paper will be divided into six sections. Particular problems are discussed in last four sections. In Section 2 we will recall briefly some basic definitions and preliminary facts from multivalued analysis which will be used in the following sections. In Section 3 we establish an existence theorem for (1.1)-(1.3). In Section 4 we shall establish an existence theorem for (1.4)-(1.7). In Sections 5 and 6 we study the existence of solutions for first and second impulsive neutral functional differential inclusions, respectively.

2. Preliminaries

In this section, we introduce notation, definitions, and preliminary facts from multivalued analysis which are used throughout this paper.

$C([-r,0],\mathbb{R}^n)$ is the Banach space of all continuous functions from $[-r,0]$ into $\mathbb{R}^n$ with the norm

$$\|\phi\| := \sup\{|\phi(\theta)| : -r \le \theta \le 0\}.$$

By $C(J,\mathbb{R}^n)$ we denote the Banach space of all continuous functions from J into $\mathbb{R}^n$ with the norm

$$\|y\|_J := \sup\{|y(t)| : t \in J\}.$$

$L^1([0,T],\mathbb{R}^n)$ denotes the Banach space of measurable functions $y : J \longrightarrow \mathbb{R}^n$ which are Lebesgue integrable normed by

$$\|y\|_{L^1} := \int_0^T |y(t)|dt \quad \text{for all} \quad y \in L^1(J,\mathbb{R}^n).$$

$AC^i([0,T],\mathbb{R}^n)$ is the space of i-times differentiable functions $y : [0,T] \to \mathbb{R}^n$, whose i^{th} derivative, $y^{(i)}$, is absolutely continuous.

Let A be a subset of $[0,T]\times C([-r,0],\mathbb{R}^n)$. A is $\mathcal{L}\otimes\mathcal{B}$ measurable if A belongs to the σ-algebra generated by all sets of the form $\mathcal{J}\times D$, where $\mathcal{J}$ is Lebesgue measurable in J and D is Borel measurable in $C([-r,0],\mathbb{R}^n)$. A subset A of $L^1([0,T],\mathbb{R}^n)$ is decomposable, if for all $u,v \in A$ and $\mathcal{J} \subset [0,T]$ measurable, the function $u\chi_{\mathcal{J}} + v\chi_{J-\mathcal{J}} \in A$, where χ_J stands for the characteristic function of J. Let E be a Banach space, X a nonempty closed subset of E and $G : X \to \mathcal{P}(E)$ a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{x \in X : G(x)\cap B \neq \emptyset\}$ is open for any open set B in E. G has a fixed point if there is $x \in X$ such that $x \in G(x)$. For more details on multivalued maps we refer to the books of Deimling 9, Gorniewicz 16, Hu and Papageorgiou 19 and Tolstonogov 24.

Definition 2.1 *Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,T], \mathbb{R}^n))$ be a multivalued operator. We say N has property (BC) if*

1) N is lower semi-continuous (l.s.c.);

2) N has nonempty closed and decomposable values.

Let $F : J \times C([-r,0], \mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n)$ be a multivalued map with nonempty compact values. Assign to F the multivalued operator

$$\mathcal{F} : C([-r,T], \mathbb{R}^n) \to \mathcal{P}(L^1([0,T], \mathbb{R}^n))$$

by letting

$$\mathcal{F}(y) = \{w \in L^1([0,T], \mathbb{R}^n) : w(t) \in F(t, y_t) \text{ for a.e. } t \in [0,T]\}.$$

The operator $\mathcal{F}$ is called the Niemytzki operator associated with F.

Definition 2.2 *Let $F : J \times C([-r,0], \mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n)$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Niemytzki operator $\mathcal{F}$ is lower semi-continous and has nonempty closed and decomposable values.*

Next we state a selection theorem due to Bressan and Colombo in 8.

Theorem 2.3. *Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,T], \mathbb{R}^n))$ be a multivalued operator which has property (BC). Then N has a continuous selection; i.e., there exists a continuous function (single-valued) $g : Y \to L^1(J, \mathbb{R}^n)$ such that $g(y) \in N(y)$ for every $y \in Y$.*

3. First Order Impulsive FDI

The main result of this section concerns the IVP (1.1)-(1.3). Before stating and proving the main result, we give the definition of a solution of the IVP (1.1)-(1.3). In order to the define the solution of such a problem, we shall consider the space

$$\Omega := \{y : [-r,T] \to \mathbb{R}^n : y_k \in C(J_k, \mathbb{R}^n), k = 0, \ldots, m \text{ and there exist } y(t_k^-) \text{ and } y(t_k^+), k = 1, \ldots, m \text{ with } y(t_k^-) = y(t_k)\},$$

which is a Banach space with the norm

$$\|y\|_\Omega := \max\{\|y_k\|_J, k = 0, \ldots, m\},$$

where y_k is the restriction of y to $J_k = (t_k, t_{k+1}],\ k = 0, \ldots, m$.

Definition 3.1 *A function $y \in \Omega \cap AC((t_k, t_{k+1}), \mathbb{R}^n)$ for all $k = 0, \ldots, m$, is said to be a solution of (1.1)–(1.3) if y satisfies the differential inclusion $y'(t) \in F(t, y_t)$ a.e. on $J - \{t_1, \ldots, t_m\}$, and the conditions $\Delta y|_{t=t_k} = I_k(y(t_k^-))$, and $y(t) = \phi(t),\ t \in [-r, 0]$.*

Let us introduce the following hypotheses which are assumed hereafter:

(H1) $F : [0, T] \times C([-r, 0], \mathbb{R}^n) \longrightarrow \mathcal{P}(\mathbb{R}^n)$ is a nonempty, compact-valued, multivalued map such that:

a) $(t, u) \mapsto F(t, u)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable;

b) $u \mapsto F(t, u)$ is lower semi-continuous for a.e. $t \in [0, T]$;

(H2) for each $r > 0$, there exists a function $h_r \in L^1([0, T], \mathbb{R}^+)$ such that

$$\|F(t, u)\|_{\mathcal{P}} := \sup\{|v| : v \in F(t, u)\} \leq h_r(t) \text{ for a.e. } t \in [0, T];$$

and for $u \in C([-r, 0], \mathbb{R}^n)$ with $\|u\| \leq r$.

The following lemma from 13 is crucial in the proof of our main theorem:

Lemma 3.2. *Let $F : [0, T] \times C([-r, 0], \mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n)$ be a multivalued map with nonempty, compact values. Assume $(H1)$ and $(H2)$ hold. Then F is of l.s.c. type.*

Theorem 3.3. *Suppose that hypotheses (H1), (H2), and*

(H3) there exist constants d_k, such that $|I_k(x)| \leq d_k,\ k = 1, \ldots, m$ for each $x \in \mathbb{R}^n$,

(H4) there exists a continuous nondecreasing function $\psi : [0, \infty) \longrightarrow (0, \infty)$ and $p \in L^1([0, T], \mathbb{R}_+)$ such that

$$\|F(t, u)\|_{\mathcal{P}} \leq p(t)\psi(\|u\|) \ \text{ for a.e. } t \in J \text{ and each } u \in C([-r, 0], \mathbb{R}^n),$$

with

$$\int_0^T p(s)ds < \int_c^\infty \frac{du}{\psi(u)}, \quad c = \|\phi\| + \sum_{k=1}^m d_k,$$

are satisfied. Then the impulsive initial value problem (1.1)–(1.3) has at least one solution.

Proof. (H1) and (H2) imply, by Lemma 3.2, that F is of lower semi-continuous type. Then from Theorem 2.3 there exists a continuous function $f : \Omega \to L^1([0,T], \mathrm{I\!R}^n)$ such that $f(y) \in \mathcal{F}(y)$ for all $y \in \Omega$.

Consider the problem,

$$y'(t) = f(y_t), \quad t \in [0,T], \ t \neq t_k, \quad k = 1, \dots, m \tag{3.1}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m \tag{3.2}$$

$$y(t) = \phi(t), \quad t \in [-r, 0]. \tag{3.3}$$

It is obvious that if $y \in \Omega$ is a solution of the problem (3.1)–(3.3), then y is a solution to the problem (1.1)-(1.3).

Transform the problem into a fixed point problem. Consider the operator $N : \Omega \to \Omega$ defined by:

$$N(y)(t) := \begin{cases} \phi(t) & \text{if } t \in [-r, 0], \\ \phi(0) + \displaystyle\int_0^t f(y_s)ds + \sum_{0<t_k<t} I_k(y(t_k^-)) & \text{if } t \in [0, T]. \end{cases}$$

We shall show that N is a compact operator.

Step 1: N *is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n \longrightarrow y$ in Ω. Then

$$\begin{aligned} |N(y_n(t)) - N(y(t))| &\leq \int_0^t |f(y_{n,s}) - f(y_s)|ds \\ &+ \sum_{0<t_k<t} |I_k(y_n(t_k)) - I_k(y(t_k))| \\ &\leq \int_0^T |f(y_{n,s}) - f(y_s)|ds \\ &+ \sum_{0<t_k<t} |I_k(y_n(t_k)) - I_k(y(t_k))|. \end{aligned}$$

Since the functions f and $I_k, \ k = 1, \dots, m$, are continuous, then

$$\|N(y_n) - N(y)\|_\Omega \leq \|f(y_n) - f(y)\|_{L^1} + \sum_{k=1}^m |I_k(y_n(t_k)) - I_k(y(t_k))| \to 0 \ \ as \ n \to \infty.$$

Step 2: *N maps bounded sets into bounded sets in* Ω.

Indeed, it is enough to show that for any $q > 0$ there exists a positive constant ℓ such that, for each $y \in B_q := \{y \in \Omega : \|y\|_\Omega \leq q\}$, we have $\|N(y)\|_\Omega \leq \ell$.

From (H2) and (H3) we have

$$\begin{aligned} |N(y)(t)| &\leq \|\phi\| + \int_0^t |f(y_s)|ds + \sum_{0<t_k<t} |I_k(y(t_k))| \\ &\leq \|\phi\| + \|h_q\|_{L^1} + \sum_{k=1}^m d_k = \ell. \end{aligned}$$

Step 3: *N maps bounded sets into equicontinuous sets of* Ω.

Let $\tau_1, \tau_2 \in [0, T]$, $\tau_1 < \tau_2$ and B_q be a bounded set of Ω. Let $y \in B_q$. Then

$$|N(y)(\tau_2) - N(y)(\tau_1)| \leq \int_{\tau_1}^{\tau_2} h_q(s)ds + \sum_{\tau_1<t_k<\tau_2} d_k.$$

As $\tau_2 \longrightarrow \tau_1$ the right-hand side of the above inequality tends to zero.

The equicontinuity for the cases $\tau_1 < \tau_2 \leq 0$ and $\tau_1 \leq 0 \leq \tau_2$ is obvious.

As a consequence of Steps 1 to 3, together with the Arzela-Ascoli theorem, we conclude that $N : \Omega \longrightarrow \Omega$ is completely continuous.

Step 4: *Now it remains to show that the set*

$$\mathcal{E}(N) := \{y \in \Omega : y = \lambda N(y), \quad \textit{for some} \;\; 0 < \lambda < 1\}$$

is bounded.

Let $y \in \mathcal{E}(N)$. Then $y = \lambda N(y)$ for some $0 < \lambda < 1$. Thus for each $t \in J$

$$y(t) = \lambda[\phi(0) + \int_0^t f(y_s)\, ds + \sum_{0<t_k<t} I_k(y(t_k^-))], \quad t \in J.$$

This implies by (H3)-(H4) that for each $t \in [0, T]$ we have

$$|y(t)| \leq \|\phi\| + \int_0^t p(s)\psi(\|y_s\|)ds + \sum_{k=1}^m d_k. \tag{3.4}$$

We consider the function μ defined by

$$\mu(t) := \sup\{|y(s)| : -r \leq s \leq t\}, \quad 0 \leq t \leq T.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in J$, by inequality (3.4) we have for $t \in J$

$$\mu(t) \le \|\phi\| + \int_0^t p(s)\psi(\mu(s))ds + \sum_{k=1}^m d_k. \tag{3.5}$$

If $t^* \in [-r, 0]$ then $\mu(t) = \|\phi\|$ and the inequality (3.5) holds. Let us take the right-hand side of the inequality (3.5) as $v(t)$. Then we have

$$c = v(0) = \|\phi\| + \sum_{k=1}^m d_k, \quad \mu(t) \le v(t), \quad t \in J,$$

and

$$v'(t) = p(t)\psi(\mu(t)), \;\; t \in [0, T].$$

Using the nondecreasing character of ψ we get

$$v'(t) \;\; \le p(t)\psi(v(t)), \;\; t \in [0, T].$$

By using (H4) this implies for each $t \in [0, T]$ that

$$\int_{v(0)}^{v(t)} \frac{d\tau}{\psi(\tau)} \le \int_0^T p(s)ds < \int_{v(0)}^{\infty} \frac{d\tau}{\psi(\tau)}.$$

This inequality implies that there exists a constant K such that $v(t) \le K, \; t \in J$, and hence $\mu(t) \le K, \; t \in [0, T]$. Since for every $t \in [0, T]$, $\|y_t\| \le \mu(t)$, we have

$$\|y\|_\Omega \le K' := \max\{\|\phi\|, K\},$$

where K' depends only T and on the functions p and ψ. This shows that $\mathcal{E}(N)$ is bounded.

Set $X := \Omega$. As a consequence of Schaefer's theorem in 23 (see p. 29), we deduce that N has a fixed point y which is a solution to problem (3.1)–(3.3). Then y is a solution to the problem (1.1)–(1.3). ■

4. Second Order Impulsive FDIs

In this section we give an existence result for the IVP (1.4)–(1.7).

Definition 4.1 *A function $y \in \Omega \cap AC^1((t_k, t_{k+1}), \mathbb{R}^n)$ for all $k = 0, \ldots, m$, is said to be a solution of (1.4)–(1.7) if y satisfies the differential inclusion $y''(t) \in F(t, y_t)$ a.e. on $J - \{t_1, \ldots, t_m\}$, the conditions $\Delta y|_{t=t_k} = I_k(y(t_k^-))$, $\Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-))$, $k = 1, \ldots, m$, $y(t) = \phi(t)$, $t \in [-r, 0]$, and $y'(0) = \eta$.*

Theorem 4.2. *Assume (H1)-(H3) and the conditions,*

(H5) there exist constants $\overline{d}_k$*, such that*

$$|\overline{I}_k(y)| \leq \overline{d}_k \ \text{ for each } y \in \mathbb{R}^n, \ k = 1, \ldots, m,$$

(H6) there exists a continuous nondecreasing function $\psi : [0, \infty) \longrightarrow (0, \infty)$ *and* $p \in L^1([0,T], \mathbb{R}_+)$ *such that*

$$\|F(t,u)\|_{\mathcal{P}} \leq p(t)\psi(\|u\|) \ \textit{ for a.e. } t \in J \textit{ and each } \ u \in C([-r,0], \mathbb{R}^n)$$

with

$$\int_0^T (T-s)p(s)ds < \int_c^\infty \frac{du}{\psi(u)}, \quad c = \|\phi\| + T|\eta| + \sum_{k=1}^m [d_k + (T-t_k)\overline{d}_k,$$

are satisfied. Then the IVP (1.4)-(1.7) has at least one solution.

Proof. (H1) and (H2) imply, by Lemma 3.2, that F is of lower semi-continuous type. Then from Theorem 2.3 there exists a continuous function $f : \Omega \to L^1([0,T], \mathbb{R}^n)$ such that $f(y) \in \mathcal{F}(y)$ for all $y \in \Omega$.

Consider the problem,

$$y''(t) = f(y_t), \quad t \in [0,T], \ t \neq t_k, \quad k = 1, \ldots, m \tag{4.1}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)) \quad k = 1, \ldots, m \tag{4.2}$$

$$\Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k = 1, \ldots, m \tag{4.3}$$

$$y(t) = \phi(t), \quad t \in [-r,0], \ y'(0) = \eta. \tag{4.4}$$

Transform the problem into a fixed point problem. Consider the operator $\overline{N} : \Omega \to \Omega$ defined by

$$\overline{N}(y)(t) := \begin{cases} \phi(t) & \text{if } t \in [-r,0], \\ \phi(0) + t\eta + \displaystyle\int_0^t (t-s)f(y_s)ds & \\ + \displaystyle\sum_{0<t_k<t} [I_k(y(t_k^-)) + (t-t_k)\overline{I}_k(y(t_k))] & \text{if } t \in [0,T]. \end{cases}$$

As in Theorem 3.3 we can show that $\overline{N}$ is completely continuous. Now we prove only that the set

$$\mathcal{E}(\overline{N}) := \{y \in \Omega : y = \lambda \overline{N}(y), \quad \text{for some} \quad 0 < \lambda < 1\}$$

is bounded. Let $y \in \mathcal{E}(\overline{N})$. Then $y = \lambda \overline{N}(y)$ for some $0 < \lambda < 1$. Thus

$$\begin{aligned} y(t) &= \lambda[\phi(0) + t\eta + \int_0^t (t-s) f(y_s) ds \\ &\quad + \sum_{0<t_k<t} [I_k(y(t_k)) + (t - t_k)\overline{I}_k(y(t_k))]]. \end{aligned}$$

This implies by (H5) and (H6) that for each $t \in J$ we have

$$|y(t)| \leq \|\phi\| + T|\eta| + \int_0^t (T-s)p(s)\psi(\|y_s\|)ds + \sum_{k=1}^m [d_k + (T - t_k)\overline{d}_k]. \tag{4.5}$$

We consider the function μ defined by

$$\mu(t) := \sup\{|y(s)| : -r \leq s \leq t\}, 0 \leq t \leq T.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in J$, by the inequality (4.5) we have for $t \in J$

$$|\mu(t)| \leq \|\phi\| + T|\eta| + \sum_{k=1}^m [d_k + (T - t_k)\overline{d}_k] + \int_0^t (T-s)p(s)\psi(\mu(s))ds. \tag{4.6}$$

If $t^* \in [-r, 0]$ then $\mu(t) = \|\phi\|$ and the inequality (4.6) holds. Let us take the right-hand side of inequality (4.6) as $v(t)$. Then we have

$$v(0) = \|\phi\| + T|\eta| + \sum_{k=1}^m \left(d_k + (T - t_k)\overline{d}_k\right)$$

and

$$v'(t) = (T-t)p(t)\psi(\mu(t)), \ t \in [0, T].$$

Using the nondecreasing character of ψ we get

$$v'(t) \ \leq (T-t)p(t)\psi(v(t)), \ \ t \in [0, T].$$

This implies together with (H4) for each $t \in [0, T]$ that

$$\int_{v(0)}^{v(t)} \frac{d\tau}{\psi(\tau)} \leq \int_0^T (T-s)p(s)ds < \int_{v(0)}^{\infty} \frac{d\tau}{\psi(\tau)}.$$

This inequality implies that there exists a constant b such that $v(t) \leq b, \ t \in [0,T]$, and hence $\mu(t) \leq b, \ t \in J$. Since for every $t \in [0,T]$, $\|y_t\| \leq \mu(t)$, we have

$$\|y\|_\Omega \leq \max\{\|\phi\|, b\},$$

where b depends only T and on the functions p and ψ. This shows that $\mathcal{E}(\overline{N})$ is bounded.

Set $X := \Omega$. As a consequence of Schaefer's theorem we deduce that $\overline{N}$ has a fixed point y which is a solution to problem (4.1)–(4.4) and hence a solution to the problem (1.4)–(1.7). ■

5. First Order Impulsive NFDEs

Let us start by defining what we mean by a solution of IVP (1.8)–(1.10).

Definition 5.1 *A function $y \in \Omega \cap AC((t_k, t_{k+1}), \mathbb{R}^n)$ for all $k = 0, \ldots, m$, is said to be a solution of (1.8)–(1.10) if y satisfies the differential inclusion $\frac{d}{dt}[y(t) - g(t, y_t)] \in F(t, y_t)$ a.e. on $[0,T] - \{t_1, \ldots, t_m\}$ and the conditions $\Delta y|_{t=t_k} = I_k(y(t_k^-))$, and $y(t) = \phi(t), \quad t \in [-r, 0]$.*

Let us introduce the following hypotheses which are assumed hereafter:

(A1) there exist constants $0 \leq c_1 < 1$, and $c_2 \geq 0$ such that

$$|g(t,u)| \leq c_1\|u\| + c_2, \ \ t \in [0,T], \ \ u \in C([-r,0], \mathbb{R}^n);$$

(A2) the function g is completely continuous;

(A3) there exists a continuous nondecreasing function $\psi : [0,\infty) \longrightarrow (0,\infty)$ and $p \in L^1([0,T], \mathbb{R}_+)$ such that

$$\|F(t,u)\|_{\mathcal{P}} \leq p(t)\psi(\|u\|) \ \text{ for a.e. } t \in [0,T] \text{ and each } u \in C([-r,0], \mathbb{R}^n)$$

with

$$\int_0^T p(s)ds < \int_c^\infty \frac{du}{\psi(u)},$$

where

$$c = \frac{1}{1-c_1}\left[(1+c_1)\|\phi\| + 2c_2 + \sum_{k=1}^{m} d_k\right].$$

Theorem 5.2. *Assume that hypotheses (H1)-(H3) and (A1)-(A3) hold. Then the problem (1.8)-(1.10) has at least one solution.*

Proof. (H1) and (H2) imply by Lemma 3.2 that F is of lower semi-continuous type. Then from Theorem 2.3 there exists a continuous function $f : \Omega \to L^1([0,T], \mathrm{IR}^n)$ such that $f(y) \in \mathcal{F}(y)$ for all $y \in \Omega$. Consider the following problem

$$\frac{d}{dt}[y(t) - g(t, y_t)] = f(y_t), \quad t \in J, \ t \neq t_k, \quad k = 1, \ldots, m, \tag{5.1}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \ldots, m, \tag{5.2}$$

$$y(t) = \phi(t), \quad t \in [-r, 0]. \tag{5.3}$$

Transform the problem into a fixed point problem. Consider the operator, $\overline{N}_1 : \Omega \to \Omega$ defined by

$$\overline{N}_1(y)(t) := \begin{cases} \phi(t) & \text{if } t \in [-r, 0], \\ \phi(0) - g(0, \phi(0)) + g(t, y_t) + \displaystyle\int_0^t f(y_s)ds & \\ + \displaystyle\sum_{0<t_k<t} I_k(y(t_k^-)) & \text{if } t \in [0, T]. \end{cases}$$

We shall show that $\overline{N}_1$ is a completely continuous multivalued operator. Using (A2) it suffices to show that the operator $\widetilde{N}_1 : \Omega \to \Omega$ defined by:

$$\widetilde{N}_1(y)(t) := \begin{cases} \phi(t) & \text{if } t \in [-r, 0], \\ \phi(0) + \displaystyle\int_0^t f(y_s)ds + \sum_{0<t_k<t} I_k(y(t_k^-)) & \text{if } t \in [0, T], \end{cases}$$

is completely continuous.

Step 1: $\widetilde{N}_1$ is continuous. Let $\{y_n\}$ be a sequence such that $y_n \longrightarrow y$ in Ω. Then

$$\begin{aligned} |\widetilde{N}_1(y_n(t)) - \widetilde{N}_1(y(t))| \ &\leq \ \int_0^t |f(y_{n,s}) - f(y_s)|ds \\ &+ \ \sum_{0<t_k<t} |I_k(y_n(t_k)) - I_k(y(t_k))| \\ &\leq \ \int_0^T |f(y_{n,s}) - f(y_s)|ds \\ &+ \ \sum_{0<t_k<t} |I_k(y_n(t_k)) - I_k(y(t_k))|. \end{aligned}$$

Since the functions f, and I_k, $k = 1, \ldots, m$, are continuous. Then

$$\|\widetilde{N}_1(y_n) - \widetilde{N}_1(y)\|_\Omega \leq \|f(y_n) - f(y)\|_{L^1} + \sum_{k=1}^{m} |I_k(y_n(t_k)) - I_k(y(t_k))| \to 0.$$

Step 2: $\widetilde{N}_1$ maps bounded sets into bounded sets in Ω.

Indeed, it is enough to show that there exists a positive constant ℓ such that for each $y \in B_q = \{y \in \Omega : \|y\|_\Omega \leq q\}$,

$$\begin{aligned} |\widetilde{N}_1(y)(t)| &\leq \|\phi\| + \int_0^t |f(y_s)|ds + \sum_{0<t_k<t} |I_k(y(t_k))| \\ &\leq \|\phi\| + \int_0^T h_q(s)|ds + \sum_{k=1}^{m} d_k \\ &\leq T\|h_q\|_{L^1} + \sum_{k=1}^{m} d_k = \ell. \end{aligned}$$

Step 3: $\widetilde{N}_1$ maps bounded sets into equicontinuous sets of Ω.

Let $r_1, r_2 \in [0, T]$, $r_1 < r_2$ and B_q be a bounded set of Ω. Then

$$|\widetilde{N}_1(y)(r_2) - \widetilde{N}_1(y)(r_1)| \leq \int_{r_1}^{r_2} |h_q(s)|ds + \sum_{r_1<t_k<r_2} d_k.$$

As $r_2 \longrightarrow r_1$ the right-hand side of the above inequality tends to zero. The equicontinuous for the cases $r_1 < r_2 \leq 0$ and $r_1 \leq 0 \leq r_2$ is obvious.

As a consequence of Steps 1 to 3, and (A2), together with the Arzela-Ascoli theorem, we can conclude that $\widetilde{N}_1$ is a completely continuous operator.

Step 4 The set $\mathcal{E}(\overline{N}_1) := \{y \in \Omega : \; y = \lambda N_1(y)$ for some $\lambda \in (0, 1)\}$ is bounded.

Let $y \in \mathcal{E}(\overline{N}_1)$. Then $\lambda \overline{N}_1(y) = y$ for some $0 < \lambda < 1$ and

$$y(t) = \lambda \left[\phi(0) - g(0, \phi(0)) + g(t, y_t) + \int_0^t f(y_s)ds + \sum_{0<t_k<t} I_k(y(t_k)) \right].$$

This implies by (H3), (A1) and (A3) that for each $t \in [0, T]$ we have

$$|y(t)| \leq |\phi(0)| + |g(0, \phi(0))| + |g(t, y_t)| + \int_0^t p(s)\psi(\|y_s\|)ds + \sum_{k=1}^{m} |I_k(y(t_k))|$$

or

$$|y(t)| \le (1+c_1)\|\phi\| + 2c_2 + c_1\|y_t\| + \int_0^t p(s)\psi(\|y_s\|)ds + \sum_{k=1}^{m} d_k. \tag{5.4}$$

We consider the function μ defined by

$$\mu(t) := \sup\{|y(s)| : -r \le s \le t\}, \quad t \in [0,T].$$

Let $t^* \in [-r,t]$ be such that $\mu = |y(t^*)|$. If $t^* \in [0,T]$, by the inequality (5.4), we have for $t \in [0,T]$

$$\mu(t) \le (1+c_1)\|\phi\| + 2c_2 + c_1\mu(t) + \int_0^t p(s)\psi(\mu(s))ds + \sum_{k=1}^{m} d_k.$$

Thus

$$\mu(t) \le \frac{1}{1-c_1}\left[(1+c_1)\|\phi\| + 2c_2 + \int_0^t p(s)\psi(\mu(s))ds + \sum_{k=1}^{m} d_k\right]. \tag{5.5}$$

If $t^* \in [-r,0]$ then $\mu(t) = \|\phi\|$ and the inequality (5.5) holds. Let us take the right-hand side of the inequality (5.5) as $v(t)$, then we have

$$v(0) = \frac{1}{1-c_1}\left[(1+c_1)\|\phi\| + 2c_2 + \sum_{k=1}^{m} d_k\right] \text{ and } v'(t) = p(t)\psi(\mu(t)).$$

Since ψ is nondecreasing we have

$$v'(t) = p(t)\psi(\mu(t)) \le p(t)\psi(v(t)) \text{ for all } t \in [0,T].$$

From this inequality, it follows that

$$\int_0^t \frac{v'(s)}{\psi(v(s))}ds \le \int_0^t p(s)ds.$$

By using (A3) we then have

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \le \int_0^t p(s)ds \le \int_0^T p(s)ds < \int_{v(0)}^{\infty} \frac{du}{\psi(u)}.$$

This inequality implies that there exists a constant b depending only on T and on the function p such that

$$|y(t)| \le b \text{ for each } t \in [0,T].$$

Hence

$$\|y\|_\Omega \le b.$$

This shows that $\mathcal{E}(\overline{N}_1)$ is bounded. As a consequence of Schaefer's theorem we deduce that $\overline{N}_1$ has a fixed point y which is a solution to problem (5.1)–(5.2). Then y is a solution to the problem (1.8)–(1.10). ■

6. Second Order Impulsive NFDEs

In this section we study the initial value problem (1.11)–(1.14). We give first the definition of solution of the IVP (1.11)–(1.14).

Definition 6.1 *A function* $y \in \Omega \cap AC^1((t_k, t_{k+1}), \mathbb{R}^n)$ *for all* $k = 0, \ldots, m$, *is said to be a solution of (1.11)–(1.14) if* y *satisfies the equation* $\frac{d}{dt}[y'(t) - g(t, y_t)] \in F(t, y_t)$ *a.e. on* $[0,T] - \{t_1, \ldots, t_m\}$ *and the conditions* $\Delta y|_{t=t_k} = I_k(y(t_k^-))$, $\Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-))$, $k = 1, \ldots, m$, $y(t) = \phi(t)$ *on* $[-r, 0]$ *and* $y'(0) = \eta$.

Theorem 6.2. *Assume that hypotheses (H1)-(H3), (H5), (A1)-(A2) and*

(A4) there exists a continuous nondecreasing function $\psi : [0, \infty) \longrightarrow (0, \infty)$ *and* $p \in L^1([0,T], \mathbb{R}_+)$ *such that*

$$\|F(t,u)\|_{\mathcal{P}} \leq p(t)\psi(\|u\|) \quad \textit{for a.e. } t \in [0,T] \textit{ and each } u \in C([-r,0], \mathbb{R}^n)$$

with

$$\int_0^T M(s)ds < \int_c^\infty \frac{d\tau}{\tau + \psi(\tau)},$$

where

$$c = \|\phi\| + [|\eta| + c_1\|\phi\| + 2c_2]T + \sum_{k=1}^{m}(d_k - (T - t_k)\overline{d}_k),$$

and

$$M(t) = \max\{1, c_1, (T-t)p(t)\}$$

are satisfied. Then the IVP (1.4)–(1.7) has a least one solution.

Proof. (H1) and (H2) imply by Lemma 3.2 that F is of lower semi-continuous type. Then from Theorem 2.3 there exists a continuous function $f : \Omega \to L^1([0,T], \mathbb{R}^n)$ such that $f(y) \in \mathcal{F}(y)$ for all $y \in \Omega$. Consider the problem,

$$\frac{d}{dt}[y'(t) - g(t, y_t)] = f(y_t), \quad t \in [0,T], \ t \neq t_k, \quad k = 1, \ldots, m, \tag{6.1}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \ldots, m, \tag{6.2}$$

$$\Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k = 1, \ldots, m, \tag{6.3}$$

$$y(t) = \phi(t), \quad t \in [-r, 0] \ y'(0) = \eta. \tag{6.4}$$

Transform the problem into a fixed point problem. Consider the operator, $\overline{N}_2 : \Omega \to \Omega$ defined by:

$$\overline{N}_2(y)(t) := \begin{cases} \phi(t) & \text{if } t \in [-r, 0]; \\ \phi(0) + [\eta - g(0, \phi(0))]t + \displaystyle\int_0^t g(s, y_s)ds & \\ + \displaystyle\int_0^t (t-s) f(y_s) ds & \\ + \displaystyle\sum_{0<t_k<t} [I_k(y(t_k^-)) + (t - t_k)\overline{I}_k] & \text{if } t \in [0, T]. \end{cases}$$

As in Theorem 5.2 we can show that $\overline{N}_2$ is completely continuous.

Now we prove only that the set

$$\mathcal{E}(\overline{N}_2) := \{y \in \Omega : y = \lambda \overline{N}_2(y), \quad \text{for some} \ \ 0 < \lambda < 1\}$$

is bounded.

Let $y \in \mathcal{E}(\overline{N}_2)$. Then $y = \lambda \overline{N}_2(y)$ for some $0 < \lambda < 1$. Thus

$$\begin{aligned} y(t) &= \lambda\phi(0) + \lambda[\eta - g(0, \phi(0))]t \\ &+ \lambda \int_0^t g(s, y_s)ds + \lambda \int_0^t (t-s) f(y_s) ds \\ &+ \lambda \sum_{0<t_k<t} [I_k(y(t_k^-)) + (t - t_k)\overline{I}_k]. \end{aligned}$$

This implies by (H3), (H5) and (A1), (A4) that, for each $t \in [0, T]$, we have

$$\begin{aligned} |y(t)| &\leq \|\phi\| + T(|\eta| + c_1\|\phi\| + 2c_2) + \int_0^t c_1 \|y_s\| ds \\ &+ \int_0^t (T-s) p(s) \psi(\|y_s\|) ds + \sum_{k=1}^m [d_k + (T - t_k)\overline{d}_k]. \end{aligned} \tag{6.5}$$

We consider the function μ definied by

$$\mu(t) := \sup\{|y(s)| : -r \leq s \leq t\}, \ \ 0 \leq t \leq T.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in J$, by the inequality (6.5) we have for $t \in [0, T]$

$$\begin{aligned} \mu(t) &\leq \|\phi\| + T(|\eta| + c_1\|\phi\| + 2c_2) + \int_0^t M(s)\mu(s) ds \\ &+ \int_0^t M(s) \psi(\mu(s)) ds + \sum_{k=1}^m [d_k + (T - t_k)\overline{d}_k]. \end{aligned} \tag{6.6}$$

If $t^* \in [-r, 0]$ then $\mu(t) = \|\phi\|$ and the inequality (6.6) holds. Let us take the right-hand side of inequality (6.6) as $v(t)$, then we have

$$v(0) = \|\phi\| + T(|\eta| + c_1\|\phi\| + 2c_2) + \sum_{k=1}^{m} \left(d_k + (T - s)\overline{d}_k\right),$$

and

$$v'(t) = M(t)\mu(t) + M(t)\psi(\mu(t)), \ t \in [0, T].$$

Using the nondecreasing character of ψ we get

$$v'(t) \leq M(t)[\mu(t) + \psi(v(t))], \ \ t \in [0, T].$$

This together with (A4) implies for each $t \in [0, T]$ that

$$\int_{v(0)}^{v(t)} \frac{d\tau}{\tau + \psi(\tau)} \leq \int_0^T M(s)ds < \int_{v(0)}^{\infty} \frac{d\tau}{\tau + \psi(\tau)}.$$

This inequality implies that there exists a constant b such that $v(t) \leq b, \ t \in [0, T]$, and hence $\mu(t) \leq b, \ t \in [0, T]$. Since for every $t \in [0, T]$, $\|y_t\| \leq \mu(t)$, we have

$$\|y\|_\Omega \leq \max\{\|\phi\|, b\},$$

where b depends only on T and on the functions p and ψ. This shows that $\mathcal{E}(\overline{N}_2)$ is bounded.

Set $X := \Omega$. As a consequence of Schaefer's theorem we deduce that $\overline{N}_2$ has a fixed point y which is a solution to problem (6.1)–(6.4). Then y is a solution to the problem (1.11)–(1.14). ■

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