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ON THE ANALYSIS OF A VISCOPLASTIC CONTACT PROBLEM WITH TIME DEPENDENT TRESCA'S FRICTION LAW

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Abstract: This paper deals with the study of a nonlinear problem of frictional contact between an elastic-viscoplastic body and a rigid obstacle. We model the frictional contact by a version of Tresca's friction law where the friction bound depends on time. Firstly, we obtain an existence and uniqueness result in a weak sense for a model including the bilateral contact. To this end we use a time discretization method and the Banach fixed point theorem. Secondly, we show an existence result for a mechanical problem with the unilateral contact conditions (Signorini's contact) using an iterative method.

Keywords: Quasistatic frictional contact, bilateral contact, unilateral contact, Tresca's friction law, fixed point, discretization.

AMS Mathematical Subject Classification Codes: 74D10, 74A55, 49B40.

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1. Introduction

In this paper we consider a mathematical model for the frictional contact between a deformable body and a rigid obstacle. We consider here materials having an elastic-viscoplastic constitutive law of the form

$$\dot{\sigma} = \mathcal{E}\varepsilon(\dot{u}) + G(\sigma, \varepsilon(u)), \tag{1.1}$$

where $\mathcal{E}$ and G are constitutive functions. In this paper, we consider the case of small deformations, we denote by $\varepsilon = (\varepsilon_{ij})$ the small strain tensor and by $\sigma = (\sigma_{ij})$ the stress tensor. A dot above a variable represents the time derivative. The contact is modeled with a bilateral contact or a Signorini's contact conditions and the associated friction law is chosen as

$$|\sigma_\tau| \leq g(t), \qquad \begin{array}{l} |\sigma_\tau| < g(t) \Rightarrow \dot{u}_\tau = 0, \\ |\sigma_\tau| = g(t) \Rightarrow \text{ there exists } \lambda \geq 0 \text{ such that } \sigma_\tau = -\lambda \dot{u}_\tau, \end{array} \tag{1.2}$$

where $\dot{u}_\tau$ (respectively σ_τ) represents the tangential velocity (respectively tangential force).

The engineering literature concerning this topic is extensive. Existence and uniqueness results for quasistatic problems involving (1.1) and Tresca's friction law, in which the friction bound is given, have been obtained by Amassad and Sofonea in 2 for the bilateral case, by Licht in 7 and Cocou, Pratt and Raous in 5 for linearly elastic materials and by Amassad, Sofonea and Shillor in 3 in the case of perfectly plastic materials. Here we extend these results to the case of the friction yield limit g depends on time and of Signorini's contact conditions.

The paper is organised as follows. In section 2 some functional and preliminary material are recalled. In section 3, the mechanical model including *bilateral contact* and a version of *Tresca's friction law* where the *friction bound depends on time* (1.2) is stated together with a variational formulation coupling of the constitutive law (1.1) and a variational inequality including the equilibrium equation and the boundary conditions. In section 4, we show the existence and uniqueness result for this first problem (Theorem 3.1). Sections 5 and 6 are devoted to an analysis of problem with *Signorini's nonpenetration conditions* and Tresca's friction law (1.2). The existence of a solution to the problem is stated in Theorem 5.2 and proved by using an iterative method. The uniqueness part is an open problem.

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2. Notation and preliminaries

In this section we present the notation we shall use and some preliminary material. For further details we refer the reader to references 1. and 2. We denote by S_M the space of second order symmetric tensors on $\mathbb{R}^M$ $(M = 2, 3)$, "$\cdot$" and $|\cdot|$ represent the inner product and the Euclidean norm on S_M and $\mathbb{R}^M$, respectively. Let $\Omega \subset \mathbb{R}^M$ be a bounded and regular domain with a boundary Γ. We shall use the notation

$$\begin{array}{ll} H = L^2(\Omega)^M, & \mathcal{H} = \{\ (\sigma_{ij})\ |\ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\ \} \\ H_1 = H^1(\Omega)^M, & \mathcal{H}_1 = \{\ \sigma \in \mathcal{H}|\ (\sigma_{ij,j}) \in H\ \}. \end{array}$$

Here and below, $i, j = 1, .., M$, summation over repeated indices is implied, and the index that follows a comma indicates a partial derivative. H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are real Hilbert spaces endowed with the inner products given by

$$\langle u, v\rangle_H = \int_\Omega u_i v_i\ dx, \quad \langle \sigma, \tau\rangle_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij}\ dx,$$

with

$$\langle u, v\rangle_{H_1} = \langle u, v\rangle_H + \langle \varepsilon(u), \varepsilon(v)\rangle_{\mathcal{H}}$$

and

$$\langle \sigma, \tau\rangle_{\mathcal{H}_1} = \langle \sigma, \tau\rangle_{\mathcal{H}} + \langle Div\ \sigma, Div\ \tau\rangle_H$$

respectively. Here $\varepsilon : H_1 \longrightarrow \mathcal{H}$ and $Div : \mathcal{H}_1 \longrightarrow H$ are the deformation and the divergence operators, respectively, defined by $\varepsilon(v) = (\varepsilon_{ij}(v))$, $\varepsilon_{ij}(v) = \frac{1}{2}(v_{i,j} + v_{j,i})$ and $Div\ \sigma = (\sigma_{ij,j})$.

Since the boundary Γ is Lipschitz continuous, the unit outward normal vector ν on the boundary is defined a.e. For every vector field $v \in H_1$ we denote by v_ν and v_τ the *normal* and *the tangential* components of v on the boundary given by

$$v_\nu = v \cdot \nu, \qquad v_\tau = v - v_\nu \nu. \tag{2.1}$$

Similary, for a regular (say $\mathcal{C}^1$) tensor field $\sigma : \Omega \longrightarrow S_M$ we define its *normal* and *tangential* components by

$$\sigma_\nu = (\sigma\nu) \cdot \nu, \qquad \sigma_\tau = \sigma\nu - \sigma_\nu \nu \tag{2.2}$$

and we recall that the following Green formula holds (valid in regular cases):

$$\langle \sigma, \varepsilon(v)\rangle_{\mathcal{H}} + \langle Div\ \sigma, v\rangle_H = \int_\Gamma \sigma\nu \cdot v\ da \quad \forall v \in H_1 \tag{2.3}$$

where da is the surface measure element.

3. Persistent contact and time dependent Tresca friction law

In this section we describe a model for the process, present its variational formulation, list the assumptions imposed on the problem data and state our first result.

The setting is as follows. An elastic-viscoplastic body occupies the domain Ω and is acted upon by given forces and tractions. We assume that the boundary Γ of Ω is partitioned into three disjoint measurable parts Γ_1, Γ_2, and Γ_3, such that $meas\Gamma_1 > 0$. The body is clamped on $\Gamma_1 \times (0,T)$ and surface tractions φ_2 act on $\Gamma_2 \times (0,T)$. The solid is frictional contact with a rigid obstacle on $\Gamma_3 \times (0,T)$ and this is where our main interest lies. Moreover, a volume force of density φ_1 acts on the body in $\Omega \times (0,T)$.

We assume a quasistatic process and use (1.1) as the constitutive law and (1.2) as the boundary contact conditions. With these assumptions, the mechanical problem of frictional contact of the viscoplastic body may be formulated classicaly as follows:
Find a displacement field $u : \Omega\times[0,T] \longrightarrow \mathbb{R}^M$ and a stress field $\sigma : \Omega\times[0,T] \longrightarrow S_M$ such that

$$\dot\sigma = \mathcal{E}\varepsilon(\dot u) + G(\sigma, \varepsilon(u)) \qquad \text{in } \Omega \times (0,T), \tag{3.1}$$

$$Div\ \sigma + \varphi_1 = 0 \qquad \text{in } \Omega \times (0,T), \tag{3.2}$$

$$u = 0 \qquad \text{on } \Gamma_1 \times (0,T), \tag{3.3}$$

$$\sigma\nu = \varphi_2 \qquad \text{on } \Gamma_2 \times (0,T), \tag{3.4}$$

$$\begin{aligned} u_\nu = 0, \quad & |\sigma_\tau| \le g(t) \qquad \text{on } \Gamma_3 \times (0,T), \\ & |\sigma_\tau| < g(t) \Rightarrow \dot u_\tau = 0, \\ & |\sigma_\tau| = g(t) \Rightarrow \text{ there exists } \lambda \ge 0 \text{ such that } \sigma_\tau = -\lambda \dot u_\tau, \end{aligned} \tag{3.5}$$

$$u(0) = u_0, \qquad \sigma(0) = \sigma_0 \qquad \text{in } \Omega. \tag{3.6}$$

To obtain a variational formulation of the contact problem (3.1)-(3.6) we need additional notations. Let V denote the closed subspace of H_1 defined by

$$V = \{\ v \in H_1 \mid v = 0 \ \ on \ \ \Gamma_1\}.$$

We note that the Korn's inequality holds, since $meas(\Gamma_1) > 0$, thus

$$|\varepsilon(u)|_{\mathcal{H}} \geq C|u|_{H_1} \qquad \forall u \in V. \tag{3.7}$$

Here and below, C represents a positive generic constant which may depend on Ω, Γ, G and T, and do not depend on time or on the input data φ_1, φ_2, g, u_0 or σ_0 and whose value may change from line to line.
Let $\langle u, v\rangle_V = \langle \varepsilon(u), \varepsilon(v)\rangle_{\mathcal{H}}$ be the inner product on V, then by (3.7) the norms $|\cdot|_{H_1}$ and $|\cdot|_V$ are equivalent on V, and $(V, |\cdot|_V)$ is a Hilbert space.
Next, we denote by $f(t)$ the element of V' given by (γ is the trace operator)

$$\langle f(t), v\rangle_{V',V} = \langle \varphi_1(t),\ v\rangle_H + \langle \varphi_2(t),\ \gamma v\rangle_{L^2(\Gamma_2)^M} \qquad \forall v \in V, \quad t \in [0, T], \tag{3.8}$$

and let $j : L^2(\Gamma_3) \times V \longrightarrow \mathbb{R}$ be the friction functional

$$j(g(t), v) = \int_{\Gamma_3} |g(t, x)||v_\tau(x)| da \qquad \forall v \in V, \quad t \in [0, T], \tag{3.9}$$

and let us denote by U_{ad} the space of admissible displacements defined by

$$U_{ad} = \{\ v \in V \mid v_\nu = 0 \ \ on \ \ \Gamma_3\}.$$

The space U_{ad} is closed in V and is endowed with this topology.
In the study of the contact problem (3.1)-(3.6) we make the following assumptions on the data :

$$\begin{array}{l}
\mathcal{E} : \Omega \times S_M \to S_M \ \text{ is a symmetric and positively definite tensor, i.e.} \\
(a)\ \mathcal{E}_{ijkh} \in L^\infty(\Omega) \quad \forall i, j, k, h = 1, .., M \\
(b)\ \mathcal{E}\sigma \cdot \tau = \sigma \cdot \mathcal{E}\tau \quad \forall \sigma, \tau \in S_M,\ \text{a.e. in } \Omega \\
(c)\ \text{there exists } \ \alpha > 0 \ \text{ such that } \ \mathcal{E}\sigma \cdot \sigma \geq \alpha|\sigma|^2 \quad \forall \sigma \in S_M,
\end{array} \tag{3.10}$$

$$\begin{array}{l}
G : \Omega \times S_M \times S_M \to S_M \quad \text{and} \\
(a)\ \text{there exists } \ k > 0 \ \text{ such that} \\
\qquad |G(x, \sigma_1, \varepsilon_1) - G(x, \sigma_2, \varepsilon_2)| \leq k(|\sigma_1 - \sigma_2| + |\varepsilon_1 - \varepsilon_2|) \\
\qquad\qquad \forall\ \sigma_1, \sigma_2, \varepsilon_1, \varepsilon_2 \in S_M,\ \text{a.e. in } \Omega \\
(b)\ x \mapsto G(x, \sigma, \varepsilon) \ \text{ is a measurable function with respect to the} \\
\qquad\qquad \text{Lebesgue measure on } \ \Omega,\ \text{for all } \sigma, \varepsilon \in S_M \\
(c)\ x \mapsto G(x, 0, 0) \in \mathcal{H},
\end{array} \tag{3.11}$$

$$\varphi_1 \in H^1(0,T;H), \quad \varphi_2 \in H^1(0,T;L^2(\Gamma_2)^M), \tag{3.12}$$

$$g \in H^1(0,T;L^2(\Gamma_3)), \tag{3.13}$$

$$u_0 \in U_{ad}, \quad \langle \sigma_0, \varepsilon(v)\rangle_{\mathcal{H}} + j(g(0),v) \geq \langle f(0), v\rangle_{V',V} \quad \forall v \in U_{ad}. \tag{3.14}$$

Using (3.1)-(3.6),(2.3) we obtain the following variational formulation of the mechanical problem

Problem FV : Find a displacement field $u : [0,T] \longrightarrow U_{ad}$ and $\sigma : [0,T] \longrightarrow \mathcal{H}$ such that

$$\dot{\sigma}(t) = \mathcal{E}\varepsilon(\dot{u}(t)) + G(\sigma(t), \varepsilon(u(t))) \quad \text{a.e. } t \in (0,T), \tag{3.15}$$

$$\begin{gathered}\langle \sigma(t), \varepsilon(v) - \varepsilon(\dot{u}(t))\rangle_{\mathcal{H}} + j(g(t),v) - j(g(t),\dot{u}(t)) \geq \langle f(t), v - \dot{u}(t)\rangle_{V',V} \\ \forall v \in U_{ad}, \text{ a.e. } t \in (0,T),\end{gathered} \tag{3.16}$$

$$u(0) = u_0, \qquad \sigma(0) = \sigma_0. \tag{3.17}$$

Our main result of this section, which will be established in the next is the following theorem:

Theorem 3.1. *Assume that* (3.10) − (3.14) *hold. Then there exists a unique solution* (u,σ) *of the problem* FV *satisfying*

$$u \in H^1(0,T;U_{ad}), \qquad \sigma \in H^1(0,T;\mathcal{H}_1).$$

4. Proof of Theorem 3.1

The proof of Theorem 3.1 is based on a time discretization method followed by a fixed point arguments, similar to those in 2 and is carried out in several steps.

In the first step we assume that the viscoplastic part of the stress field is a known function $\eta \in L^2(0,T;\mathcal{H})$. Let $z_\eta \in H^1(0,T;\mathcal{H})$ be given by

$$z_\eta(t) = \int_0^t \eta(s)ds + z_0, \qquad \text{where} \quad z_0 = \sigma_0 - \mathcal{E}\varepsilon(u_0). \tag{4.1}$$

We consider the following nonlinear variational problem

Problem FV_η : Find a displacement field $u_\eta : [0,T] \longrightarrow U_{ad}$ and $\sigma_\eta : [0,T] \longrightarrow \mathcal{H}$ such that

$$\sigma_\eta(t) = \mathcal{E}\varepsilon(u_\eta(t)) + z_\eta(t) \quad \text{a.e.} \ \ t \in (0,T), \tag{4.2}$$

$$\langle \sigma_\eta(t), \varepsilon(v) - \varepsilon(\dot{u}_\eta(t))\rangle_{\mathcal{H}} + j(g(t),v) - j(g(t),\dot{u}_\eta(t)) \geq \langle f(t), v - \dot{u}_\eta(t)\rangle_{V',V}$$
$$\forall v \in U_{ad}, \ \text{a.e.} \ \ t \in (0,T), \tag{4.3}$$

$$u_\eta(0) = u_0. \tag{4.4}$$

We have the following result

Proposition 4.1. *There exists a unique solution* (u_η, σ_η) *to problem* FV_η. *Moreover* $u_\eta \in H^1(0,T;U_{ad}), \quad \sigma_\eta \in H^1(0,T;\mathcal{H}_1)$.

Proposition 4.1 may be obtained using similar arguments as in reference 2. However, for the convenience of the reader, we summarize here the main ideas of the proof. For this, let $N \in \mathbb{N}$, $h = \frac{T}{N}$, $t_n = nh$, $g^n = g(t_n)$, $f^n = f(t_n)$, $z_\eta^n = z_\eta(t_n)$, $\forall n = 0,..,N$. We introduce the bilinear form $a : V \times V \longrightarrow \mathbb{R}$ defined by $a(u,v) = \langle \mathcal{E}\varepsilon(u), \varepsilon(v)\rangle_{\mathcal{H}}$ and we consider the sequence of variational inequalities

Problem FV_η^{n+1} : Find $u_\eta^{n+1} \in U_{ad}$ such that

$$a(u_\eta^{n+1}, v - u_\eta^{n+1}) + j(g^{n+1}, v - u_\eta^n) - j(g^{n+1}, u_\eta^{n+1} - u_\eta^n) \geq$$
$$\langle f^{n+1}, v - u_\eta^{n+1}\rangle_{V',V} - \langle z_\eta^{n+1}, \varepsilon(v) - \varepsilon(u_\eta^{n+1})\rangle_{\mathcal{H}} \quad \forall v \in U_{ad} \tag{4.5}$$

$$u_\eta^0 = u_0. \tag{4.6}$$

Lemma 4.2. *For all* $n = 0,..,N-1$, *there exists a unique solution* u_η^{n+1} *to problem* $(4.5)-(4.6)$. *Moreover, there exists* $C > 0$ *such that*

$$|u_\eta^n|_V \leq C(|g^n|_{L^2(\Gamma_3)} + |f^n|_{V'} + |z_\eta^n|_{\mathcal{H}}) \quad \forall n = 0,...,N, \tag{4.7}$$

$$|u_\eta^{n+1}-u_\eta^n|_V \leq C(|g^{n+1}-g^n|_{L^2(\Gamma_3)}+|f^{n+1}-f^n|_{V'}+|z_\eta^{n+1}-z_\eta^n|_{\mathcal{H}}) \quad \forall n=0,...,N-1. \quad (4.8)$$

Proof. The problem (4.5) is equivalent to the following minimization problem

Find $u_\eta^{n+1} \in U_{ad}$ such that $J_\eta^n(u_\eta^{n+1}) = \inf_{v\in U_{ad}} J_\eta^n(v)$ where

$$J_\eta^n(v) = \frac{1}{2}a(v,v) + j(g^{n+1}, v-u_\eta^n) - \langle f^{n+1}, v\rangle_{V',V} + \langle z_\eta^{n+1}, \varepsilon(v)\rangle_{\mathcal{H}}. \quad (4.9)$$

The functional J_η^n is proper, continuous, strictly convex and coercive on U_{ad}. Therefore, the problem (4.9) has a unique solution $u_\eta^{n+1} \in U_{ad}$, a.e. $t \in (0,T)$. In the case $n \in \{1,2,...,N\}$, the inequality (4.7) may be obtained by taking $v=0$ in (4.5) and using (3.10), (3.11), in the case $n=0$, the same inequality may be obtained using (3.14). The inequality (4.8) also follows from (4.5),(3.10) and (3.11). ■

We now consider the function $u_\eta^N : [0,T] \longrightarrow U_{ad}$ defined by

$$u_\eta^N(t) = u_\eta^n + \frac{(t-t_n)}{h}(u_\eta^{n+1}-u_\eta^n) \quad \forall t \in [t_n, t_{n+1}],\ n=0,...,N-1. \quad (4.10)$$

We obtain

Lemma 4.3. *There exists an element $u_\eta \in H^1(0,T;U_{ad})$ such that, passing to a subsequence again denoted $(u_\eta^N)_N$, we have*

$$u_\eta^N \rightharpoonup u_\eta \ \textit{weak}\star \ \textit{in}\ L^\infty(0,T;U_{ad}), \quad (4.11)$$

$$\dot{u}_\eta^N \rightharpoonup \dot{u}_\eta \ \textit{weak in}\ L^2(0,T;U_{ad}). \quad (4.12)$$

Proof. Using (4.7)-(4.8) and having in mind the regularities $g \in H^1(0,T;L^2(\Gamma_3))$, $f \in H^1(0,T;V')$ and $z_\eta \in H^1(0,T;\mathcal{H})$, we obtain that

$$\begin{aligned} |u_\eta^N(t)|_V &\leq |u_\eta^n|_V + |u_\eta^{n+1}|_V \qquad \forall t \in [t_n,t_{n+1}], \\ &\leq C(|g|_{C([0,T];L^2(\Gamma_3))} + |f|_{C([0,T];V')} + |\eta|_{L^2(0,T;\mathcal{H})}), \end{aligned} \quad (4.13)$$

$$|\dot{u}_\eta^N(t)|_{L^2(0,T;V)} \leq C(|\dot{g}|_{L^2(0,T;L^2(\Gamma_3))} + |\dot{f}|_{L^2(0,T;V')} + |\eta|_{L^2(0,T;\mathcal{H})}). \quad (4.14)$$

Lemma 4.3 follows now from (4.13)-(4.14) and using standard compactness arguments. ■

We turn now to prove Proposition 4.1:

Proof of Proposition 4.1. Let $N \in \mathbb{N}$ and let us consider the functions $\widetilde{u}_\eta^N : [0,T] \to U_{ad}$, $\widetilde{g}^N : [0,T] \to L^2(\Gamma_3)$, $\widetilde{f}^N : [0,T] \to V'$ and $\widetilde{z}_\eta^N : [0,T] \to \mathcal{H}$ defined by

$$\begin{array}{ll} \widetilde{u}_\eta^N(t) = u_\eta^{n+1}, \quad \widetilde{g}^N(t) = g^{n+1}, \quad \widetilde{f}^N(t) = f^{n+1}, \\ \widetilde{z}_\eta^N(t) = z_\eta^{n+1} \quad \forall t \in [t_n, t_{n+1}], \quad n = \overline{0, N-1}. \end{array} \tag{4.15}$$

Substituting (4.10) and (4.15) in (4.4), after integration on $[0,T]$, we obtain

$$\begin{gathered} \int_0^T a(\widetilde{u}_\eta^N(t), v(t) - \dot{u}_\eta^N(t))dt + \int_0^T j(\widetilde{g}^N(t), v(t))dt - \int_0^T j(\widetilde{g}^N(t), \dot{u}_\eta^N(t))dt \\ \geq \int_0^T \langle \widetilde{f}^N(t), v(t) - \dot{u}_\eta^N(t)\rangle_{V'\times V} dt - \int_0^T \langle \widetilde{z}_\eta^N(t), \varepsilon(v(t)) - \varepsilon(\dot{u}_\eta^N(t))\rangle_{\mathcal{H}} dt \\ \forall v \in L^2(0,T;U_{ad}). \end{gathered} \tag{4.16}$$

From (4.10),(4.14) and (4.15) it results that

$$\int_0^T |\widetilde{u}_\eta^N(t) - u_\eta^N(t)|_V^2 dt \leq Ch^2(|\dot{g}|^2_{L^2(0,T;L^2(\Gamma_3))} + |\dot{f}|^2_{L^2(0,T;V')} + |\eta|^2_{L^2(0,T;\mathcal{H})}) \tag{4.17}$$

and, therefore

$$|\widetilde{u}_\eta^N - u_\eta^N|_{L^2(0,T;U_{ad})} \longrightarrow 0. \tag{4.18}$$

Let now consider the element $u_\eta \in H^1(0,T;V)$ given by Lemma 4.3, it follows, for all $v \in L^2(0,T;V)$

$$\int_0^T a(\widetilde{u}_\eta^N(t), v(t))dt \longrightarrow \int_0^T a(u_\eta(t), v(t))dt, \tag{4.19}$$

$$\int_0^T j(\widetilde{g}^N(t), v(t))dt \longrightarrow \int_0^T j(g(t), v(t))dt, \tag{4.20}$$

$$\int_0^T \langle \widetilde{z}_\eta^N(t), \varepsilon(v(t)) - \varepsilon(\dot{u}_\eta^N(t))\rangle_{\mathcal{H}} dt \longrightarrow \int_0^T \langle z_\eta(t), \varepsilon(v(t)) - \varepsilon(\dot{u}_\eta(t))\rangle_{\mathcal{H}} dt, \tag{4.21}$$

$$\int_0^T \langle \widetilde{f}^N(t), v(t) - \dot{u}_\eta^N(t)\rangle_{V',V} dt \longrightarrow \int_0^T \langle f(t), v(t) - \dot{u}_\eta(t)\rangle_{V',V} dt. \tag{4.22}$$

Moreover, we can write

$$\int_0^T a(\widetilde{u}_\eta^N(t), \dot{u}_\eta^N(t))dt = \int_0^T a(\widetilde{u}_\eta^N(t) - u_\eta^N(t), \dot{u}_\eta^N(t))dt + \int_0^T a(u_\eta^N(t), \dot{u}_\eta^N(t))dt, \tag{4.23}$$

using again (4.11)-(4.12), (4.18) and standard lower semicontinuity arguments, we obtain

$$\lim_N \int_0^T a(\widetilde{u}_\eta^N(t) - u_\eta^N(t), \dot{u}_\eta^N(t))dt = 0, \tag{4.24}$$

$$\begin{aligned}\liminf_N \int_0^T a(u_\eta^N(t), \dot{u}_\eta^N(t))dt &= \tfrac{1}{2}[\liminf_N a(u_\eta^N(T), u_\eta^N(T)) - a(u_0, u_0)] \\ &\geq \int_0^T a(u_\eta(t), \dot{u}_\eta(t))dt,\end{aligned} \tag{4.25}$$

$$\liminf_N \int_0^T j(\widetilde{g}^N(t), \dot{u}_\eta^N(t))dt \geq \int_0^T j(g(t), \dot{u}_\eta(t))dt. \tag{4.26}$$

Using now (4.19)-(4.26) and Lebesgue points for an L^1 function we obtain

$$\begin{gathered} a(u_\eta(t), v - \dot{u}_\eta(t)) + \langle z_\eta(t), \varepsilon(v) - \varepsilon(\dot{u}_\eta(t))\rangle_{\mathcal{H}} + j(g(t), v) - j(g(t), \dot{u}_\eta(t)) \\ \geq \langle f(t), v - \dot{u}_\eta(t)\rangle_{V',V}, \quad \forall v \in V, \quad a.e.\ t \in (0,T). \end{gathered} \tag{4.27}$$

Let now $\sigma_\eta \in H^1(0,T;\mathcal{H})$ be given by (4.2). Using (4.27) and (4.1) it follows that (u_η, σ_η) is a solution for (4.2),(4.3). Moreover, since $u_\eta^N(0) = u_0 \quad \forall N \in \mathbb{N}$, using (4.11) and (4.12) we deduce (4.4). Using (4.27) and (4.2) we obtain (4.3) and by choosing $v = u_\eta(t) \pm \psi$ with $\psi \in \mathcal{D}(\Omega)^M$, as test functions in (4.3) we get

$$Div\sigma_\eta(t) + \varphi_1(t) = 0 \quad \text{ in } \Omega, \quad \forall t \in [0,T].$$

Therefore, by (3.12) we obtain that

$$\sigma_\eta \in H^1(0,T;\mathcal{H}_1).$$

This concludes the existence part of Proposition 4.1. The uniqueness part is an easy consequence of (4.3),(4.4). ■

Proposition 4.1 and (3.11) allow us to consider the operator $\Lambda : L^2(0,T;\mathcal{H}) \longrightarrow L^2(0,T;\mathcal{H})$ defined by

$$\Lambda\eta(t) = G(\sigma_\eta(t), \varepsilon(u_\eta(t))) \qquad \forall \eta \in L^2(0,T;\mathcal{H}), \tag{4.28}$$

for $t \in [0,T]$, where, for every $\eta \in L^2(0,T;\mathcal{H})$, (u_η, σ_η) denotes the solution of the variational problem FV_η. We have

Lemma 4.4. *The operator Λ has a unique fixed point $\eta^\star \in L^2(0,T;\mathcal{H})$.*

Proof. Let $\eta_1,\ \eta_2, \in L^2(0,T;\mathcal{H})$ and $t \in [0,T]$. For the sake of simplicity we denote $z_i = z_{\eta_i}$, $u_i = u_{\eta_i}$, $\sigma_i = \sigma_{\eta_i}$, for $i = 1,2$. Using (4.2),(4.3) and after some manipulations, we obtain

$$a(u_1 - u_2, \dot{u}_1 - \dot{u}_2) \leq -\frac{d}{dt}\langle z_1 - z_2, \varepsilon(u_1) - \varepsilon(u_2)\rangle_{\mathcal{H}} + \langle \eta_1 - \eta_2, \varepsilon(u_1) - \varepsilon(u_2)\rangle_{\mathcal{H}}. \tag{4.29}$$

Using (3.9) we deduce

$$C|u_1(t)-u_2(t)|_V^2 \le |z_1(t)-z_2(t)|_{\mathcal{H}} + \int_0^t |\eta_1(s)-\eta_2(s)|_{\mathcal{H}}|u_1(s)-u_2(s)|_V ds, \quad (4.30)$$

for all $t \in [0,T]$. Using (4.1) we obtain

$$C|u_1(t)-u_2(t)|_V^2 \le \int_0^t |\eta_1(s)-\eta_2(s)|_{\mathcal{H}}^2 ds + \int_0^t |u_1(s)-u_2(s)|_V^2 ds, \quad (4.31)$$

and, by Gronwall-type inequality, we find

$$|u_1(t)-u_2(t)|_V^2 \le C\int_0^t |\eta_1(s)-\eta_2(s)|_{\mathcal{H}}^2 ds. \quad (4.32)$$

Using now (4.2), (3.10), (4.1) and (4.32) we obtain

$$|\sigma_1(t)-\sigma_2(t)|_{\mathcal{H}}^2 \le C\int_0^t |\eta_1(s)-\eta_2(s)|_{\mathcal{H}}^2 ds. \quad (4.33)$$

Therefore, form (4.28), (3.11), (4.32) and (4.33) we get

$$|\Lambda\eta_1(t)-\Lambda\eta_2(t)|_{\mathcal{H}}^2 \le C\int_0^t |\eta_1(s)-\eta_2(s)|_{\mathcal{H}}^2 ds, \quad (4.34)$$

for all $t \in [0,T]$. Iterating this inequality n times we obtain

$$|\Lambda^n\eta_1-\Lambda^n\eta_2|_{L^2(0,T;\mathcal{H})}^2 \le \frac{C^nT^n}{n!}|\eta_1-\eta_2|_{L^2(0,T;\mathcal{H})}^2, \quad (4.35)$$

which implies that for n large enough a power Λ^n of Λ is a contraction in $L^2(0,T;\mathcal{H})$. Thus, there exists a unique element $\eta^\star \in L^2(0,T;\mathcal{H})$ such that $\Lambda^n\eta^\star = \eta^\star$. Moreover, $\eta^\star$ is the unique fixed point of Λ. ■

We now have all the ingredients needed to prove Theorem 3.1.

Proof of Theorem 3.1. Using Proposition 4.1 and Lemma 4.4 it is easy to see that the couple of functions $u = u_{\eta^\star}$, $\sigma = \sigma_{\eta^\star}$, given by (4.2),(4.4) for $\eta = \eta^\star$ represents a solution of the problem (3.15)-(3.17). So, we proved the existence part in Theorem 3.1. The uniqueness part in this Theorem follows from the uniqueness of the fixed point of the operator Λ defined by (4.28). ■

5. Unilateral contact and time dependent Tresca friction law

In this section we consider a version of the problem which involves the unilateral contact with Tresca's friction law. The physical setting is the same as in section 3. In the model we replace the bilateral contact ($u_\nu = 0$) in (3.5) by the Signorini's contact conditions given by

$$u_\nu \leq 0, \quad \sigma_\nu \leq 0, \quad u_\nu \sigma_\nu = 0 \quad \text{on } \Gamma_3 \times [0,T]. \tag{5.1}$$

The associated friction law is a version of Tresca's law considered in the first problem i.e:

$$\begin{array}{ll} |\sigma_\tau| \leq g(t), \;\; |\sigma_\tau| < g(t) \Rightarrow \dot{u}_\tau = 0, & \text{on } \Gamma_3 \times [0,T] \\ \qquad\qquad\quad\; |\sigma_\tau| = g(t) \Rightarrow \text{ there exists } \lambda \geq 0 & \text{such that } \sigma_\tau = -\lambda \dot{u}_\tau. \end{array} \tag{5.2}$$

The classical formulation of the mechanical problem is to find a displacement field $u : \Omega \times [0,T] \longrightarrow \mathbb{R}^M$ and a stress field $\sigma : \Omega \times [0,T] \longrightarrow S_M$ such that (3.1)-(3.4), (3.6), (5.1),(5.2) hold.

In order to obtain a variational formulation for the problem, we need additional notations and assumptions. We denote by K the set of admissible displacement functions

$$K = \{ \; v \in V \mid v_\nu \leq 0 \;\; on \;\; \Gamma_3 \; \}. \tag{5.3}$$

K is a closed and convex subset of V and it is endowed with the $V-$ topology.

For every $\sigma \in \mathcal{H}_1$, let $\langle \cdot , \cdot \rangle$ denote the duality pairing between $H(\Gamma_3)$ and its dual with

$$\langle \sigma_\nu, v_\nu \rangle = \int_{\Gamma_3} \sigma_\nu v_\nu da \quad \forall v \in V$$

where

$$H(\Gamma_3) = \{ w|_{\Gamma_3} \mid w \in H^{\frac{1}{2}}(\Gamma), \; w = 0 \; on \; \Gamma_1 \; \}$$

and we assume that

$$u_0 \in K, \quad \langle \sigma(0), \varepsilon(v) - \varepsilon(u_0) \rangle_{\mathcal{H}} + j(g(0), v - u_0) \geq \langle f(0), v - u_0 \rangle_{V',V} \quad \forall v \in K. \tag{5.4}$$

Using the notation and arguments as those in section 3 and (5.3), we are in a position to give this lemma

Lemma 5.1. *If* (u, σ) *are sufficiently regular functions satisfying* (3.1) − (3.4),(5.1), (5.2) *and* (3.6) *then*

$$u(t) \in K \quad \forall t \in [0, T], \tag{5.5}$$

$$\begin{aligned} &\langle \sigma(t), \varepsilon(v) - \varepsilon(\dot{u}(t))\rangle_{\mathcal{H}} + j(g(t), v) - j(g(t), \dot{u}(t)) \geq \langle f(t), v - \dot{u}(t)\rangle_{V',V} + \\ &+\langle \sigma_\nu(t), v_\nu - \dot{u}_\nu(t)\rangle \qquad \forall v \in V, \ \textit{a.e.} \ \ t \in (0, T), \end{aligned} \tag{5.6}$$

$$\langle \sigma_\nu(t), w_\nu - u_\nu(t)\rangle \geq 0 \quad \forall w \in K, \quad \forall t \in [0, T]. \tag{5.7}$$

Lemma 5.1, (3.1) and (3.6) lead us to consider the following variational formulation of the problem with Signorini's contact conditions and a version of Tresca's law:

Problem FV_s: Find a displacement field $u : [0, T] \longrightarrow K$ and a stress field $\sigma : [0, T] \longrightarrow \mathcal{H}$ such that

$$\dot{\sigma}(t) = \mathcal{E}\varepsilon(\dot{u}(t)) + G(\sigma(t), \varepsilon(u(t))) \quad \text{a.e. } t \in (0, T), \tag{5.8}$$

$$\begin{aligned} &\langle \sigma(t), \varepsilon(v) - \varepsilon(\dot{u}(t))\rangle_{\mathcal{H}} + j(g(t), v) - j(g(t), \dot{u}(t)) \geq \langle f(t), v - \dot{u}(t)\rangle_{V',V} + \\ &+\langle \sigma_\nu(t), v_\nu - \dot{u}_\nu(t)\rangle \qquad \forall v \in V, \ \text{a.e.} \ \ t \in (0, T), \end{aligned} \tag{5.9}$$

$$\langle \sigma_\nu(t), w_\nu - u_\nu(t)\rangle \geq 0 \quad \forall w \in K, \quad \forall t \in [0, T], \tag{5.10}$$

$$u(0) = u_0, \quad \sigma(0) = \sigma_0. \tag{5.11}$$

One has the following theorem

Theorem 5.2. *Let* $T > 0$ *and assume that* (3.10) − (3.14) *and* (5.4) *hold. Then there exists a solution* (u, σ) *of problem* FV_s. *Moreover, the solution satisfies*

$$u \in H^1(0, T; V) \cap C([0, T]; K), \quad \sigma \in H^1(0, T; \mathcal{H}_1).$$

Remark 5.3. The question of uniqueness of a solution is still an open problem.

6. Proof of Theorem 5.2

Let us first notice that it is sufficient to prove Theorem 5.2 for a time T_0 small enough independent of the data (initial data, right hand side). Indeed, suppose that we have proved existence of a solution (u,σ) on the interval $[0,T_0]$. In order to construct a solution (u,σ) which will be in $H^1(0,T;V)\times H^1(0,T;\mathcal{H})$ on $[0,2T_0]$, we just have to obtain the compatibility condition (5.4) at time T_0. Taking $v=\dot{u}(t)$ and $v=0$ in (5.9) for $0\le t\le T_0$, we obtain

$$\langle\sigma(t),\varepsilon(v)\rangle_{\mathcal{H}}+j(g(t),v)\ge\langle f(t),v\rangle_{V',V}+\langle\sigma_\nu(t),v_\nu\rangle\quad\forall v\in V,\ \text{a.e. }\ t\in(0,T),\tag{6.1}$$

on the other hand, (5.10) yields easily to

$$\langle\sigma_\nu(t),u_\nu(t)\rangle=0,\qquad\langle\sigma_\nu(t),w_\nu\rangle\ge 0\qquad\forall w\in K,\forall t\in[0,T].$$

Then for $v=w-u(T_0)$ in (6.1) for $t=T_0$ we get

$$\langle\sigma(T_0),\varepsilon(w)-\varepsilon(u(T_0))\rangle_{\mathcal{H}}+j(g(T_0),w-u(T_0))\ge\langle f(T_0),w-u(T_0)\rangle_{V',V}\quad\forall w\in K,$$

which is the compatibility condition written at time T_0. If $(u_1,\sigma_1)\in H^1(0,T;V)\times H^1(0,T;\mathcal{H})$ solution of FV_s taken on $(0,T_0)$ and $(u_2,\sigma_2)\in H^1(0,T;V)\times H^1(0,T;\mathcal{H})$ solution of FV_s taken on $(T_0,2T_0)$ with initial data $(u_1(T_0),\sigma_1(T_0))$. Then, $(u_1 1_{(0,T_0)}+u_2 1_{(T_0,2T_0)},\sigma_1 1_{(0,T_0)}+\sigma_2 1_{(T_0,2T_0)})$ is in $H^1(0,T;V)\times H^1(0,T;\mathcal{H})$ and solves FV_s on $(0,2T_0)$. Theorem 5.2 will then be proved by splitting the interval $[0,T]$ on interval of length T_0.

The proof of Theorem 5.2 will be accomplished out in two steps, we suppose in the sequel that the assumptions of Theorem 5.2 are fulfilled.

Step 1: The first step consists of studying an equivalent incremental formulation that we derive from discretization like in section 4. For this, let $N\in\mathbb{N}$, $h=\frac{T}{N}$, $t_n=nh$, $g^n=g(t_n)$, $f^n=f(t_n)$, we consider the sequence of variational inequalities:

Problem FV_s^{n+1} : Find a displacement field $u^{n+1}\in K$, and a stress field $\sigma^{n+1}\in\mathcal{H}$ such that

$$\sigma^{n+1}=\mathcal{E}\varepsilon(u^{n+1})+\sum_{i=1}^{n+1}hG(\sigma^i,\varepsilon(u^i))+\sigma^0-\mathcal{E}\varepsilon(u^0),\tag{6.2}$$

$$\langle \sigma^{n+1}, \varepsilon(v) - \varepsilon(u^{n+1})\rangle_{\mathcal{H}} + j(g^{n+1}, v - u^n) - j(g^{n+1}, u^{n+1} - u^n)$$
$$\geq \langle f^{n+1}, v - u^{n+1}\rangle_{V',V} + \langle \sigma_\nu^{n+1}, v_\nu - u_\nu^{n+1}\rangle \quad \forall v \in V, \tag{6.3}$$

$$\langle \sigma_\nu^{n+1}, w_\nu - u_\nu^{n+1}\rangle \geq 0 \quad \forall w \in K, \tag{6.4}$$

$$u^0 = u_0, \quad \sigma^0 = \sigma_0. \tag{6.5}$$

Proposition 6.1. *The problem FV_s^{n+1} has a unique solution $(u^{n+1}, \sigma^{n+1}) \in K \times \mathcal{H}$ for all $n = 0, .., N-1$.*

In order to prove Proposition 6.1 we need some preliminary results.

Fixed point technique: We assume that the viscoplastic part of the stress field $\eta^n = G(\sigma^n, \varepsilon(u^n)) \in \mathcal{H}$ is given, and we denote by $z_\eta^n = h\sum_{i=1}^{n} \eta^i + z^0$ where $z^0 = \sigma^0 - \mathcal{E}\varepsilon(u^0)$. We consider the following auxiliary problem

Problem $FV_{s\eta}^{n+1}$: Find a displacement field $u_\eta^{n+1} \in K$ such that

$$a(u_\eta^{n+1}, v - u_\eta^{n+1}) + j(g^{n+1}, v - u_\eta^n) - j(g^{n+1}, u_\eta^{n+1} - u_\eta^n) \geq \langle f^{n+1}, v - u_\eta^{n+1}\rangle_{V',V}$$
$$-\langle z_\eta^{n+1}, \varepsilon(v) - \varepsilon(u_\eta^{n+1})\rangle_{\mathcal{H}} \quad \forall v \in K,$$
$$u_\eta^0 = u_0. \tag{6.6}$$

We have the following result

Lemma 6.2. *There exists a unique solution $u_\eta^{n+1} \in K$ to problem $FV_{s\eta}^{n+1}$.*

Proof. Problem (6.6) is equivalent to the following minimization problem

$$\text{Find } u_\eta^{n+1} \in K, \quad J_\eta^{n+1}(u_\eta^{n+1}) = \inf_{v\in K} J_\eta^{n+1}(v) \tag{6.7}$$

where $J_\eta^{n+1}(v) = \frac{1}{2}a(v,v) + j(g^{n+1}, v - u_\eta^n) - \langle f^{n+1}, v\rangle_{V',V} + \langle z_\eta^{n+1}, \varepsilon(v)\rangle_{\mathcal{H}}$. The functional J_η^{n+1} is proper, continuous, strictly convex and coercive on K. Therefore, problem (6.7) has a unique solution $u_\eta^{n+1} \in K$. ■

Analysis of nonlinear static inequality: The purpose in this paragraph is to investigate the abstract static systems of the from

$$\sigma = \mathcal{E}\varepsilon(u) + Z \quad \text{where} \quad Z = h\eta + z \tag{6.8}$$

$$a(u, v-u) + j(g, v-w) - j(g, u-w) \geq \langle f, v-u\rangle_{V',V} - \langle Z, \varepsilon(v) - \varepsilon(u)\rangle_{\mathcal{H}} \quad \forall v \in K, \tag{6.9}$$

in which the unknowns are the functions $u : \Omega \longrightarrow K$, and $\sigma : \Omega \longrightarrow \mathcal{H}$. We obtain abstract results which will be applied in the study of (6.6). In the study of (6.8)-(6.9) we consider the following assumptions :

$$w \in K, \quad g \in L^2(\Gamma_3), \quad f \in V', \quad Z \in \mathcal{H}. \tag{6.10}$$

It is straightforward to show that (6.8)-(6.9) has a unique solution $u \in K$, $\sigma \in \mathcal{H}$. The previous result and (3.11) allow us to consider the operator $\Lambda : \mathcal{H} \longrightarrow \mathcal{H}$ defined by

$$\Lambda\eta = G(\sigma, \varepsilon(u)), \tag{6.11}$$

where $\Lambda = \Lambda(w, g, f, z, \cdot)$.

Lemma 6.3. *There exists a constant $C > 0$ and N_0 such that*

$$\forall (w, g, f, z) \in K \times L^2(\Gamma_3) \times V' \times \mathcal{H}, \quad \forall N \geq N_0 \qquad |\Lambda(\eta_1) - \Lambda(\eta_2)|_{\mathcal{H}} \leq \frac{C}{N}|\eta_1 - \eta_2|_{\mathcal{H}}.$$

The maps $\Lambda(w, g, f, z, \cdot)$ are then uniform contractions with respect to the variable $(w, g, f, z, \cdot)$ in $\mathcal{H}$ for a large N. In particular, for all (w, g, f, z) there exists a unique $\eta_\star = \eta_\star(w, g, f, z)$ such that

$$\Lambda(w, g, f, z, \eta_\star) = \eta_\star.$$

Proof. Let $\eta_1,\ \eta_2 \in \mathcal{H}$, and take the difference between the two inequalities written for η_i, $(i = 1, 2)$, we obtain

$$a(u_1 - u_2, u_1 - u_2) \leq \langle Z_2 - Z_1, \varepsilon(u_1) - \varepsilon(u_2)\rangle_{\mathcal{H}}, \tag{6.12}$$

after some algebraic manipulations, and (6.8) we find

$$|u_1 - u_2|_V \leq Ch|\eta_1 - \eta_2|_{\mathcal{H}} = \frac{CT}{N}|\eta_1 - \eta_2|_{\mathcal{H}}. \tag{6.13}$$

Here and below C represents a positive generic constant whose value may change from line to line. Using (6.8) and (6.13) we get

$$|\sigma_1 - \sigma_2|_{\mathcal{H}} \quad \leq C|u_1 - u_2|_V + |Z_1 - Z_2|_{\mathcal{H}} \leq \tfrac{CT}{N}|\eta_1 - \eta_2|_{\mathcal{H}}. \tag{6.14}$$

So, from (6.11),(6.13) and (6.14) it results

$$|\Lambda\eta_1 - \Lambda\eta_2|_{\mathcal{H}} \leq C(|\sigma_1 - \sigma_2|_{\mathcal{H}} + |u_1 - u_2|_V) \leq \frac{CT}{N}|\eta_1 - \eta_2|_{\mathcal{H}}. \ \blacksquare \tag{6.15}$$

Lemma 6.4. *For all* $n = 0, .., N - 1$, *there exists a unique* $\eta_\star^{n+1} \in \mathcal{H}$ *such that* $\Lambda(u^n, g^{n+1}, f^{n+1}, z_\star^n, \eta_\star^{n+1}) = \eta_\star^{n+1}$ *where* $u^n = u^n(\eta_\star^n)$ *and* $z_\star^n = z^0 + h\eta_\star^1 + ...h\eta_\star^n$.

Proof. In order to prove this lemma we shall use Lemma 6.3 with the following notations:

$$u = u^{n+1}, \quad \sigma = \sigma^{n+1}, \quad w = u^n, \quad g = g^{n+1}, \quad f = f^{n+1}, \quad z = z_\star^n.$$

1) <u>Initialization.</u> Let $w = u^0$, $g = g^1$, $f = f^1$, $z = z^0 = \sigma^0 - \mathcal{E}\varepsilon(u^0)$. It follows from Lemma 6.3 that there exists a unique fixed point $\eta_\star^1$ such that

$$\Lambda(u^0, g^1, f^1, z^0, \eta_\star^1) = \eta_\star^1.$$

2) <u>Step 2.</u> From the initializing step, $u^1 = u^1(\eta_\star^1)$ is carried out.
Let $w = u^1 = u^1(\eta_\star^1)$, $g = g^2$, $f = f^2$, $z = z_\star^1 = z^0 + h\eta_\star^1$. Using Lemma 6.3, we can prove that there exists a unique fixed point $\eta_\star^2$ such that

$$\Lambda(u^1, g^2, f^2, z_\star^1, \eta_\star^2) = \eta_\star^2.$$

3) <u>Step n+1.</u> In this step, let

$w = u^n = u^n(\eta_\star^n)$, $g = g^{n+1}$, $f = f^{n+1}$, $z = z_\star^n = z_0 + h\eta_\star^1 + ... + h\eta_\star^n$. Since in this case the assumptions (3.12) and (3.13) are satisfied, we may apply Lemma 6.3 and conclude that there exists a unique fixed point $\eta_\star^{n+1}$ such that

$$\Lambda(u^n, g^{n+1}, f^{n+1}, z_\star^n, \eta_\star^{n+1}) = \eta_\star^{n+1},$$

$$z_\star^{n+1} = z_\star^n + h\eta_\star^{n+1}. \ \blacksquare$$

Proof of Proposition 6.1. Let $\eta_\star^{n+1}$ be the unique fixed point of the map $\Lambda(u^n, g^{n+1}, f^{n+1}, z_\star^n, \cdot)$ and let u^{n+1} be the solution of (6.6) for $\eta^{n+1} = \eta_\star^{n+1}$. Then u^{n+1} is a solution of (6.1)-(6.5). The uniqueness part of the solution is obtained from the uniqueness of the fixed point of the operator $\Lambda(u^n, g^{n+1}, f^{n+1}, z_\star^n, \cdot)$. ■

Step 2 : Asymptotic Analysis

By Proposition 6.1 we get that for all $n = 0, .., N-1$ there exists a unique pair of functions $(u^{n+1}, \sigma^{n+1}) \in K \times \mathcal{H}$ satisfying problem (FV_s^{n+1}).

In order to study the behaviour of (u^{n+1}, σ^{n+1}) for all $n = 0, .., N-1$ when $N \to \infty$, we introduce the following notations

$$\widetilde{u}^N(t) = u^{n+1}, \quad \widetilde{\sigma}^N(t) = \sigma^{n+1},$$

$$u^N(t) = u^n + \tfrac{t-t_n}{h}(u^{n+1} - u^n), \tag{6.16}$$

$$\widetilde{z}^N(t) = z^{n+1} = \int_0^{t_{n+1}} G(\widetilde{\sigma}^N(s), \varepsilon(\widetilde{u}^N(s))ds + \sigma_0 - \mathcal{E}\varepsilon(u_0) \ \ \forall t \in [t_n, t_{n+1}].$$

Proposition 6.5. *There exists a couple of functions* $(u, \sigma) \in (H^1(0,T;V) \cap C([0,T],K)) \times L^\infty(0,T;\mathcal{H})$ *such that passing to a subsequence still denoted* (u^N, σ^N), *we have*

$$u^N \rightharpoonup u \ \textit{weak}\star \ \textit{in} \ L^\infty(0,T;K), \tag{6.17}$$

$$\dot{u}^N \rightharpoonup \dot{u} \ \textit{weak in} \ L^2(0,T;V), \tag{6.18}$$

$$\widetilde{\sigma}^N \rightharpoonup \sigma \ \textit{weak}\star \ \textit{in} \ L^\infty(0,T;\mathcal{H}). \tag{6.19}$$

Proof. For $1 \le i \le N$, we write

$$w_i = |\sigma^i|_{\mathcal{H}} + |u^i|_V, \tag{6.20}$$

and i_0 is an index with $w_{i_0} = \sup_{1\leq i\leq N} w_i$. We recall that

$$\sigma^i = \mathcal{E}\varepsilon(u^i) + \sum_{j=1}^{i} hG(\sigma^j, \varepsilon(u^j)) + \sigma^0 - \mathcal{E}\varepsilon(u^0). \tag{6.21}$$

Now we derive a priori estimates for $(u^n, \sigma^n, u^{n+1} - u^n)$:

A priori estimate I. Using (6.20),(6.21) and (3.11), we obtain

$$\begin{aligned} w_i \ &\leq C_0|u^i|_V + hk\sum_{j=1}^{i} w_j + hi|G(0,0)|_{\mathcal{H}} + w_0, \\ &\leq C_0|u^i|_V + hkiw_{i_0} + hi|G(0,0)|_{\mathcal{H}} + w_0, \end{aligned} \tag{6.22}$$

which imply with $i = i_0$, that for $T < \frac{1}{k}$, we have

$$w_{i_0} \leq \frac{C_0|u^{i_0}|_V + T|G(0,0)|_{\mathcal{H}} + w_0}{1 - Tk}. \tag{6.23}$$

Taking $v = 0$ as the test function in (6.6) we obtain

$$\begin{aligned} &a(u^{n+1}, u^{n+1}) \leq j(g^{n+1}, -u^n) - j(g^{n+1}, u^{n+1} - u^n) + \langle f^{n+1}, u^{n+1}\rangle_{V',V} \\ &-\sum_{j=1}^{n+1} h\langle G(\sigma^j, \varepsilon(u^j)), \varepsilon(u^{n+1})\rangle_{\mathcal{H}} - \langle \sigma^0 - \mathcal{E}\varepsilon(u^0), \varepsilon(u^{n+1})\rangle_{\mathcal{H}}, \end{aligned} \tag{6.24}$$

and using the V-ellipticity of a, we obtain for $0 \leq n \leq N-1$

$$|u^{n+1}|_V \leq C(|g^{n+1}|_{L^2(\Gamma_3)} + |f^{n+1}|_{V'} + kh\sum_{j=1}^{n+1} w_j + h(n+1)|G(0,0)|_{\mathcal{H}} + w_0). \tag{6.25}$$

If we take $n + 1 = i_0$ in the previous inequality, we get

$$|u^{i_0}|_V \leq C_1(|g^{i_0}|_{L^2(\Gamma_3)} + |f^{i_0}|_{V'} + khi_0w_{i_0} + hi_0|G(0,0)|_{\mathcal{H}} + w_0), \tag{6.26}$$

therefore, using the estimate (6.26) in (6.23), for $T = T_0 < \frac{1}{(C_0C_1+1)k}$, we get

$$|u^{i_0}|_V \leq C(|g^{i_0}|_{L^2(\Gamma_3)} + |f^{i_0}|_{V'} + \frac{T}{1-Tk}|G(0,0)|_{\mathcal{H}} + \frac{1}{1-Tk}w_0). \tag{6.27}$$

From (6.23) and (6.27), we have the following bound

$$w_{i_0} \leq C(|g|_{H^1(0,T;L^2(\Gamma_3))} + |f|_{H^1(0,T;V')} + |G(0,0)|_{\mathcal{H}} + |\sigma_0|_{\mathcal{H}} + |u_0|_V). \tag{6.28}$$

Hence, from (6.20) and (6.28) we deduce that, for T small enough ($= T_0 << \frac{1}{k}$), the sequences (u^n) and (σ^n) are bounded in K and $\mathcal{H}$ respectively for $n = 1, ..., N$ and we conclude from (6.16) that

$$(u^N) \text{ is a bounded sequence in } L^\infty(0, T; K), \tag{6.29}$$

$$(\widetilde{\sigma}^N) \text{ is a bounded sequence in } L^\infty(0, T; \mathcal{H}). \tag{6.30}$$

A priori estimate II. In the sequel, we derive a priori estimate for the time derivative $\dot{u}^N$. We take the difference between the two inequalities (6.6) written at time t_n and t_{n+1} and take respectively u^n and u^{n+1} as test functions, we obtain

$$\begin{gathered} a(u^{n+1} - u^n, u^{n+1} - u^n) + h\langle G(\sigma^{n+1}, \varepsilon(u^{n+1})), \varepsilon(u^{n+1}) - \varepsilon(u^n)\rangle_{\mathcal{H}} \leq \\ j(g^n, u^{n+1} - u^{n-1}) - j(g^n, u^n - u^{n-1}) - j(g^{n+1}, u^{n+1} - u^n) \\ + \langle f^{n+1} - f^n, u^{n+1} - u^n\rangle_{V',V}, \end{gathered} \tag{6.31}$$

use the V-ellipticity of $a(\cdot, \cdot)$ and (3.9), we get

$$|u^{n+1} - u^n|_V \leq C(|g^{n+1} - g^n|_{L^2(\Gamma_3)} + |f^{n+1} - f^n|_{V'} + h|G(\sigma^{n+1}, \varepsilon(u^{n+1}))|_{\mathcal{H}}). \tag{6.32}$$

With hypothesis on G and (3.11), we obtain

$$\begin{gathered} |u^{n+1} - u^n|_V \leq C(|g^{n+1} - g^n|_{L^2(\Gamma_3)} + |f^{n+1} - f^n|_{V'} + hk(|\sigma^{n+1}|_{\mathcal{H}} + |u^{n+1}|_V) \\ + h|G(0,0)|_{\mathcal{H}}) \leq \\ C(|g^{n+1} - g^n|_{L^2(\Gamma_3)} + |f^{n+1} - f^n|_{V'} + hkw_{i_0} + h|G(0,0)|_{\mathcal{H}}), \end{gathered} \tag{6.33}$$

and after division of (6.33) by h, integration in $[0, T]$ and using (6.28), we get

$$\begin{gathered} \int_0^T |\dot{u}^N(t)|_V^2 dt = \int_0^T \frac{|u^{n+1} - u^n|_V^2}{h^2} dt \\ \leq C(|\dot{g}|^2_{L^2(0,T;L^2(\Gamma_3))} + |\dot{f}|^2_{L^2(0,T;V')} + |\sigma^0|^2_{\mathcal{H}} + |u^0|^2_V + |G(0,0)|^2_{\mathcal{H}}). \end{gathered} \tag{6.34}$$

Inequality (6.34) leads to

$$(\dot{u}^N) \text{ is a bounded sequence in } L^2(0, T; V). \tag{6.35}$$

■

We need the following result

Lemma 6.6. *Any weak limit of the sequence* $(u^N, \tilde{\sigma}^N)$ *in* $H^1(0,T;V) \times L^\infty(0,T;\mathcal{H})$ *is a strong limit point in* $L^2(0,T;V) \times L^2(0,T;\mathcal{H})$.

Proof. Using (6.1)-(6.4) and (6.16) we obtain

$$a(\widetilde{u}^N(t), v - \dot{u}^N(t)) + \langle \widetilde{z}^N(t), \varepsilon(v) - \varepsilon(\dot{u}^N(t))\rangle_{\mathcal{H}} + j(\widetilde{g}^N(t), v)$$

$$-j(\widetilde{g}^N(t), \dot{u}^N(t)) \geq \langle \widetilde{f}^N(t), v - \dot{u}^N(t)\rangle_{V',V} + \langle \widetilde{\sigma}^N_\nu(t), v_\nu - \dot{u}^N_\nu(t)\rangle \tag{6.36}$$

$$\forall v \in V, a.e.\ t,$$

$$\langle \widetilde{\sigma}^N_\nu(t), w_\nu - \widetilde{u}^N_\nu(t)\rangle \geq 0 \quad \forall w \in K, \quad \forall t \in [0,T]. \tag{6.37}$$

Taking $v = 0$ and $v = 2\dot{u}^N(t)$ as test functions in (6.36), we get

$$a(\widetilde{u}^N(t), v) + \langle \widetilde{z}^N(t), \varepsilon(v)\rangle_{\mathcal{H}} + j(\widetilde{g}^N(t), v) \geq \langle \widetilde{f}^N(t), v\rangle_{V',V} + \langle \widetilde{\sigma}^N_\nu(t), v_\nu\rangle \ \forall v \in V. \tag{6.38}$$

To show the strong convergence, we take $v = \widetilde{u}^{N+p}(t) - \widetilde{u}^N(t)$ in (6.38) and $v = \widetilde{u}^N(t) - \widetilde{u}^{N+p}(t)$ in the same inequality satisfied by $\widetilde{u}^{N+p}(t)$, which give

$$a(\widetilde{u}^{N+p}(t), \widetilde{u}^N(t) - \widetilde{u}^{N+p}(t)) + \langle \widetilde{z}^{N+p}(t), \varepsilon(\widetilde{u}^N(t)) - \varepsilon(\widetilde{u}^{N+p}(t))\rangle_{\mathcal{H}}$$

$$+j(\widetilde{g}^{N+p}(t), \widetilde{u}^N(t) - \widetilde{u}^{N+p}(t)) \tag{6.39}$$

$$\geq \langle \widetilde{f}^{N+p}(t), \widetilde{u}^N(t) - \widetilde{u}^{N+p}(t)\rangle_{V',V} + \langle \widetilde{\sigma}^{N+p}_\nu(t), \widetilde{u}^N_\nu(t) - \widetilde{u}^{N+p}_\nu(t)\rangle,$$

$$a(\widetilde{u}^N(t), \widetilde{u}^{N+p}(t) - \widetilde{u}^N(t)) + \langle \widetilde{z}^N(t), \varepsilon(\widetilde{u}^{N+p}(t)) - \varepsilon(\widetilde{u}^N(t))\rangle_{\mathcal{H}}$$

$$+j(\widetilde{g}^N(t), \widetilde{u}^{N+p}(t) - \widetilde{u}^N(t)) \tag{6.40}$$

$$\geq \langle \widetilde{f}^N(t), \widetilde{u}^{N+p}(t) - \widetilde{u}^N(t)\rangle_{V',V} + \langle \widetilde{\sigma}^N_\nu(t), \widetilde{u}^{N+p}_\nu(t) - \widetilde{u}^N_\nu(t)\rangle.$$

We add the two inequalities (6.39),(6.40) to obtain

$$a(\widetilde{u}^{N+p}(t) - \widetilde{u}^N(t), \widetilde{u}^{N+p}(t) - \widetilde{u}^N(t)) \leq j(\widetilde{g}^{N+p}(t), \widetilde{u}^N(t) - \widetilde{u}^{N+p}(t))$$

$$+j(\widetilde{g}^N(t), \widetilde{u}^{N+p}(t) - \widetilde{u}^N(t)) + \langle \widetilde{f}^{N+p}(t) - \widetilde{f}^N(t), \widetilde{u}^{N+p}(t) - \widetilde{u}^N(t)\rangle_{V',V} -$$

$$\langle \widetilde{z}^{N+p}(t) - \widetilde{z}^N(t), \varepsilon(\widetilde{u}^{N+p}(t)) - \varepsilon(\widetilde{u}^N(t))\rangle_{\mathcal{H}} + \langle \widetilde{\sigma}^{N+p}_\nu(t) - \widetilde{\sigma}^N_\nu(t), \widetilde{u}^{N+p}_\nu(t) - \widetilde{u}^N_\nu(t)\rangle. \tag{6.41}$$

By the inequality (6.37) we have

$$\langle \widetilde{\sigma}^{N+p}_\nu(t) - \widetilde{\sigma}^N_\nu(t), \widetilde{u}^{N+p}_\nu(t) - \widetilde{u}^N_\nu(t)\rangle \leq 0 \quad \forall t \in [0,T], \tag{6.42}$$

and from (6.41)-(6.42) it follows that

$$\begin{aligned} |\widetilde{u}^{N+p}(t)-\widetilde{u}^{N}(t)|_V^2 \le C(\sup_N |\widetilde{g}^N(t)|_{L^2(\Gamma_3)} |\widetilde{u}_\tau^{N+p}(t)-\widetilde{u}_\tau^{N}(t)|_{L^2(\Gamma_3)^M} \\ +|\widetilde{f}^{N+p}(t)-\widetilde{f}^{N}(t)|_{V'}^2 + |\widetilde{z}^{N+p}(t)-\widetilde{z}^{N}(t)|_{\mathcal{H}}^2). \end{aligned} \tag{6.43}$$

Having in mind that

$$\begin{aligned} |\widetilde{u}_\tau^{N+p}(t)-\widetilde{u}_\tau^{N}(t)|_{L^2(\Gamma_3)^M} \le |\widetilde{u}_\tau^{N+p}(t)-u_\tau^{N+p}(t)|_{L^2(\Gamma_3)^M} \\ +|u_\tau^{N+p}(t)-u_\tau^{N}(t)|_{L^2(\Gamma_3)^M} + |u_\tau^{N}(t)-\widetilde{u}_\tau^{N}(t)|_{L^2(\Gamma_3)^M}. \end{aligned} \tag{6.44}$$

Since (u_N) is bounded in $H^1(0,T;V)$, the sequence $(u^N_{|\Gamma})$ is relatively compact in $C([0,T],L^2(\Gamma)^M)$ and therefore there exists a subsequence, still denoted by $(u^N)_N$ such that

$$\forall \varepsilon>0,\quad \exists N_\varepsilon,\quad \forall N\ge N_\varepsilon,\quad \forall t\in[0,T]\quad |u_\tau^{N+p}(t)-u_\tau^{N}(t)|_{L^2(\Gamma_3)^M} \le \varepsilon. \tag{6.45}$$

On another hand by (6.16) it follows that

$$|u_\tau^{N}(t)-\widetilde{u}_\tau^{N}(t)|_{L^2(\Gamma_3)^M} \le C|u^{n+1}-u^n|_V \le C\tfrac{T}{N}|\dot{u}_N(t)|_V, \tag{6.46}$$

where $(\dot{u}^N)_N$ is bounded in $L^2(0,T;V)$. Combining (6.44)-(6.46), we obtain that there exists a positive constant $L_2 = L_2(g,f,u_0,\sigma_0,G(0,0))$ depending on all these arguments such that

$$\begin{aligned} \int_0^T |\widetilde{u}_\tau^{N+p}(t)-\widetilde{u}_\tau^{N}(t)|^2_{L^2(\Gamma_3)^M}dt &\le CL_2(\tfrac{1}{N^2}+\tfrac{1}{(N+p)^2}+\varepsilon^2) \\ &\le CL_2(\tfrac{1}{N^2}+\varepsilon^2). \end{aligned} \tag{6.47}$$

Now, we focus on the last term of (6.43). First, recall that

$$z^N(t) = \int_0^t G(\widetilde{\sigma}^N(s),\varepsilon(\widetilde{u}^N(s)))ds + \sigma_0 - \mathcal{E}\varepsilon(u_0). \tag{6.48}$$

Furthermore, we have

$$\begin{aligned} |\widetilde{z}^{N+p}(t)-\widetilde{z}^{N}(t)|_{\mathcal{H}} &\le |\widetilde{z}^{N+p}(t)-z^{N+p}(t)|_{\mathcal{H}} + |z^{N+p}(t)-z^{N}(t)|_{\mathcal{H}} \\ &+|z^{N}(t)-\widetilde{z}^{N}(t)|_{\mathcal{H}}. \end{aligned} \tag{6.49}$$

From (6.16), (6.48) we can rewrite $\widetilde{z}^N$ as

$$\widetilde{z}^N(t) = z^N(t) + \int_t^{t_{n+1}} G(\widetilde{\sigma}^N(s),\varepsilon(\widetilde{u}^N(s)))ds \quad \forall t\in[t_n,t_{n+1}], \tag{6.50}$$

so, we have

$$
\begin{aligned}
|\widetilde{z}^N(t) - z^N(t)|_{\mathcal{H}} &= |\int_t^{t_{n+1}} G(\widetilde{\sigma}^N(s), \varepsilon(\widetilde{u}^N(s)))ds|_{\mathcal{H}} \\
&\le k \int_t^{t_{n+1}} (|\widetilde{\sigma}^N(s)|_{\mathcal{H}} + |\widetilde{u}^N(s)|_V)ds + h|G(0,0)|_{\mathcal{H}} \\
&\le Ch(|g|_{H^1(0,T;L^2(\Gamma_3))} + |f|_{H^1(0,T;V')} + |G(0,0)|_{\mathcal{H}} + |\sigma_0|_{\mathcal{H}} + |u_0|_V),
\end{aligned}
\tag{6.51}
$$

on another hand we have

$$
\begin{aligned}
|z^{N+p}(t) - z^N(t)|_{\mathcal{H}} &= |\int_0^t (G(\widetilde{\sigma}^{N+p}(s), \varepsilon(\widetilde{u}^{N+p}(s))) - G(\widetilde{\sigma}^N(s), \varepsilon(\widetilde{u}^N(s))))ds|_{\mathcal{H}} \\
&\le kC \int_0^t (|\widetilde{\sigma}^{N+p}(s) - \widetilde{\sigma}^N(s)|_{\mathcal{H}} + |\widetilde{u}^{N+p}(s) - \widetilde{u}^N(s)|_V)ds.
\end{aligned}
\tag{6.52}
$$

Using (6.49),(6.51) and (6.52), we get the estimate

$$
\begin{aligned}
|\widetilde{z}^{N+p}(t) - \widetilde{z}^N(t)|^2_{\mathcal{H}} \le\ & \tfrac{C}{N^2}(|g|^2_{H^1(0,T;L^2(\Gamma_3))} + |f|^2_{H^1(0,T;V')} + |G(0,0)|^2_{\mathcal{H}} + |\sigma_0|^2_{\mathcal{H}} + |u_0|^2_V) \\
&+ k^2 C \int_0^t (|\widetilde{\sigma}^{N+p}(s) - \widetilde{\sigma}^N(s)|^2_{\mathcal{H}} + |\widetilde{u}^{N+p}(s) - \widetilde{u}^N(s)|^2_V)ds.
\end{aligned}
\tag{6.53}
$$

We integrate the inequality (6.43) with respect to t over the interval $[0,T]$ and use (6.47) and (6.53) together to yield

$$
\begin{aligned}
\int_0^T |\widetilde{u}^{N+p}(t) - \widetilde{u}^N(t)|^2_V dt &\le CL(\varepsilon + \tfrac{1}{N} + \tfrac{1}{N^2}) + \\
& C \int_0^T \int_0^t (|\widetilde{\sigma}^{N+p}(s) - \widetilde{\sigma}^N(s)|^2_{\mathcal{H}} + |\widetilde{u}^{N+p}(s) - \widetilde{u}^N(s)|^2_V)dsdt.
\end{aligned}
\tag{6.54}
$$

Moreover, we obtain from (6.16),(6.21) and (6.54) that

$$
\begin{aligned}
\int_0^T |\widetilde{\sigma}^{N+p}(t) - \widetilde{\sigma}^N(t)|^2_{\mathcal{H}} dt &\le C(\int_0^T |\widetilde{u}^{N+p}(t) - \widetilde{u}^N(t)|^2_V dt + \int_0^T |\widetilde{z}^{N+p}(t) - \widetilde{z}^N(t)|^2_{\mathcal{H}} dt) \\
&\le LC(\varepsilon + \frac{1}{N} + \frac{1}{N^2}) + C\int_0^T \int_0^t (|\widetilde{\sigma}^{N+p}(s) - \widetilde{\sigma}^N(s)|^2_{\mathcal{H}} + |\widetilde{u}^{N+p}(s) - \widetilde{u}^N(s)|^2_V)dsdt.
\end{aligned}
\tag{6.55}
$$

Summing up the two last inequalities it follows

$$
\begin{aligned}
&\int_0^T (|\widetilde{u}^{N+p}(t) - \widetilde{u}^N(t)|^2_V + |\widetilde{\sigma}^{N+p}(t) - \widetilde{\sigma}^N(t)|^2_{\mathcal{H}})dt \le \\
&CL(\varepsilon + \tfrac{1}{N} + \tfrac{1}{N^2}) + C\int_0^T\int_0^t (|\widetilde{\sigma}^{N+p}(s) - \widetilde{\sigma}^N(s)|^2_{\mathcal{H}} + |\widetilde{u}^{N+p}(s) - \widetilde{u}^N(s)|^2_V)dsdt,
\end{aligned}
\tag{6.56}
$$

and, by a Gronwall-type inequality, we find

$$\int_0^T (|\widetilde{u}^{N+p}(t) - \widetilde{u}^N(t)|_V^2 + |\widetilde{\sigma}^{N+p}(t) - \widetilde{\sigma}^N(t)|_{\mathcal{H}}^2)dt \leq CL(\varepsilon + \frac{1}{N} + \frac{1}{N^2}). \qquad (6.57)$$

where L is a constant wich may depend T, k, g, f, $G(0,0)$, σ_0 and u_0. Therefore, Lemma 6.6 is proved. ■

Proof of Theorem 5.2. Using Lemma 6.6, we have

$$u^N, \widetilde{u}^N \longrightarrow u \text{ strongly in } L^2(0,T;V), \qquad (6.58)$$

$$\widetilde{\sigma}^N \longrightarrow \sigma \text{ strongly in } L^2(0,T;\mathcal{H}). \qquad (6.59)$$

From the convergences (6.58)-(6.59) and (3.11), it results

$$G(\widetilde{\sigma}^N, \varepsilon(\widetilde{u}^N)) \longrightarrow G(\sigma, \varepsilon(u)) \text{ strongly in } L^2(0,T;\mathcal{H}), \qquad (6.60)$$

(6.2) and (6.60) yield

$$\dot{\sigma}(t) = \mathcal{E}\varepsilon(\dot{u}(t)) + G(\sigma(t), \varepsilon(u(t))). \qquad (6.61)$$

Notice that (6.61) proves that $\sigma \in H^1(0,T;\mathcal{H})$.

We now prove that inequalities (6.36) and (6.37) have a limit when N tends to infinity and that these limit are inequalities of the original formulation FV_s. Using the strong convergence of $\widetilde{\sigma}^N$ in $L^2(0,T;\mathcal{H})$ and (6.35), we find that

$$\int_0^T \langle \widetilde{\sigma}_\nu^N(t), \dot{u}_\nu^N(t)\rangle dt \longrightarrow \int_0^T \langle \sigma_\nu(t), \dot{u}_\nu(t)\rangle dt. \qquad (6.62)$$

Furthermore, from (6.58)-(6.59), we have

$$\int_0^T \langle \widetilde{\sigma}_\nu^N(t), u_\nu^N(t)\rangle dt \longrightarrow \int_0^T \langle \sigma_\nu(t), u_\nu(t)\rangle dt. \qquad (6.63)$$

Combining (6.61)-(6.63) and the other straightforward convergences, one can easily prove that all the terms appearing in (6.36) and (6.37) have a limit and that it gives the desired result FV_s. Every solution (u,σ) in $H^1(0,T;V) \times H^1(0,T;\mathcal{H})$ satisfies in $\Omega \times (0,T)$ the equation $\text{Div}(\sigma) + varphi_1 = 0$. This proves that we have the regularity $\sigma \in H^1(0,T;\mathcal{H}_1)$. So far, we have proved Theorem 5.2. ■

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