Special Issue in honor of John W. Neuberger
Electronic Journal of Differential Equations, Special Issue 02 (2023), pp. 239–253.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu or https://ejde.math.unt.edu

NONLINEAR DIFFUSION WITH THE p-LAPLACIAN IN A BLACK-SCHOLES-TYPE MODEL

PETER TAKÁČ

Dedicated to the memory of Professor John W. Neuberger with admiration

Abstract. We present a new nonlinear version of the well-known Black-Scholes model for option pricing in financial mathematics. The nonlinear Black-Scholes partial differential equation is based on the quasilinear diffusion term with the p-Laplace operator Δ_p for $1 < p < \infty$. The existence and uniqueness of a weak solution in a weighted Sobolev space is proved, first, by methods for nonlinear parabolic problems using the Gel'fand triplet and, alternatively, by a method based on nonlinear semigroups. Finally, possible choices of other weighted Sobolev spaces are discussed to produce a function space setting more realistic in financial mathematics.

1. Introduction

When I had the opportunity and agreed to write this contribution to honor Professor John William W. Neuberger, it was clear to me that I have to write something new, something that has hardly ever been touched in a standard manner. When I met John W. Neuberger for the first time in person, I had a strong impression that he lays stress on new mathematical methods for treating old and new problems and open questions. At that time, in the past millenium, sending preprints of papers was a standard way to communicate new results that may have been submitted to a journal, but have not been published yet. I was deeply impressed by John's original approach to nonlinear semigroups [14, 15, 16] and to the minimization of the Ginzburg-Landau energy functional [17, 18], to mention only the topics I myself have worked on. Besides his research achievements, he was mentor to several excellent Ph.D. students. Among them Professor Glenn F. Webb with whom I have very much enjoyed doing joint research on three articles related to mathematical biology. Last but not least, I benefit from collaboration with his son, John Michael Neuberger, on Mathematics and related (mostly open) "organizational" problems.

As a result of these relations to John W. Neuberger, I have decided to "generalize" the nowadays already classical Black-Scholes model for option pricing in Financial Mathematics to a model with degenerate or singular nonlinear diffusion governed by the p-Laplace operator Δ_p for $1 < p < \infty$, where $\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u)$,

2020 *Mathematics Subject Classification.* 35K92, 91B02, 35J92, 91G20.
Key words and phrases. Nonlinear parabolic equation; p-Laplacian; nonlinear semigroup; Black-Scholes equation; option pricing.

Published March 27, 2023.

∇u being the gradient of a function $u\colon \Omega \to \mathbb{R}$ defined on a domain $\Omega \subset \mathbb{R}^N$. I will worry more about questions and problems connected with (hopefully) interesting Mathematics rather than about their practical applicability to mathematical finance. For hints to and greater details on this subject, an interested reader is referred to the books by Björk [6], Fouque, Papanicolaou, and Sircar [8], and Hull [10]. We let the reader do a comparison between the analytical results and (mostly) statistical data obtained on financial markets to numerical analysis and computational simulations.

2. Black-Scholes model with p-Laplacian

To (briefly) justify our Black-Scholes model with the p-Laplacian Δ_p for $1 < p < \infty$, we begin with the original Black-Scholes model (B-S model, for short) with the regular Laplacian $\Delta \equiv \Delta_2$ where $p = 2$. The independent variable $S \in (0,\infty)$ stands for the *stock price* and $t \in (-\infty, T]$ for the time variable. The option matures at time $T \in \mathbb{R} := (-\infty, +\infty)$. Moreover, $\tau = T - t \in \mathbb{R}_+ := [0,\infty)$ denotes the *time to maturity*. This linear partial differential equation of parabolic type for the *option price* $V = V(S,t)$ reads

$$\frac{\partial V}{\partial t} + \mathscr{A}_2 V - rV(S,t) = 0 \quad \text{for } (S,t) \in (0,\infty)\times(0,T)\,; \tag{2.1}$$

with the *terminal condition*

$$V(S,T) = h(S) \quad \text{for } S \in (0,\infty)\,, \tag{2.2}$$

where $\mathscr{A}_2$ stands for the linear Black-Scholes operator (B-S operator, for short) defined by

$$\begin{gathered} (\mathscr{A}_2 V)(S,t) := \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (q_S - \gamma_S) S \frac{\partial V}{\partial S} \\ \text{for } V\colon (0,\infty)\times(0,T) \to \mathbb{R}\colon (S,t) \mapsto V(S,t)\,. \end{gathered} \tag{2.3}$$

As usual, we take the volatility, $\sigma \in (0,\infty)$, to be a positive constant. The value of γ_S reflects the *rate of dividend income* and the value of q_S is the *net share position financing cost* which depends on the *risk-free rate* r and the *repo rate* (repurchase agreement) of $S(t)$. An important technical role will be played by the *logarithmic stock price* $x = \log\ S \in \mathbb{R}^1$.

Taking into account the quadratic homogeneity (i.e., 2-homogeneity) of the linear second-order diffusion operator

$$V \mapsto (\mathscr{L}_2 V)(S) := \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \quad \text{for } V\colon (0,\infty) \to \mathbb{R}\colon S \mapsto V(S)\,, \tag{2.4}$$

which clearly involves the linear differential operator

$$V \mapsto S\frac{\partial V}{\partial S} = \frac{\partial V}{\partial x} \tag{2.5}$$

together with its square

$$\frac{\partial^2 V}{\partial x^2} \equiv \frac{\partial}{\partial x}\Big(\frac{\partial V}{\partial x}\Big) = S\frac{\partial}{\partial S}\Big(S\frac{\partial V}{\partial S}\Big) = S^2\frac{\partial^2 V}{\partial S^2} + S\frac{\partial V}{\partial S}\,, \tag{2.6}$$

in formula (2.3) for the linear B-S operator $\mathscr{A}_2$, we multiply the operator in formula (2.4) by the factor

$$\frac{2(p-1)}{p}\cdot(\sigma S)^{p-2}\Big|\frac{\partial V}{\partial S}\Big|^{p-2} = \frac{2(p-1)}{p}\cdot\sigma^{p-2}S^{p-2}\Big|\frac{\partial V}{\partial x}\Big|^{p-2}, \tag{2.7}$$

for $p \in (1, \infty)$, thus arriving at the nonlinear diffusion operator

$$\begin{aligned} V \mapsto\ & \frac{p-1}{p} (\sigma S)^p \left| \frac{\partial V}{\partial S} \right|^{p-2} \frac{\partial^2 V}{\partial S^2} \\ &= (\mathscr{L}_p V)(S) := \frac{1}{p} (\sigma S)^p \frac{\partial}{\partial S} \Big(\Big| \frac{\partial V}{\partial S} \Big|^{p-2} \frac{\partial V}{\partial S} \Big) \\ &= \frac{1}{p} \sigma^p S^{p-1} \frac{\partial}{\partial x} \Big(S^{-(p-1)} \Big| \frac{\partial V}{\partial x} \Big|^{p-2} \frac{\partial V}{\partial x} \Big) \\ &= \frac{1}{p} \sigma^p \cdot \frac{\partial}{\partial x} \Big(\Big| \frac{\partial V}{\partial x} \Big|^{p-2} \frac{\partial V}{\partial x} \Big) - (1 - \frac{1}{p}) \sigma^p \cdot \Big| \frac{\partial V}{\partial x} \Big|^{p-2} \frac{\partial V}{\partial x} \end{aligned} \tag{2.8}$$

with the p-Laplace operator $(1 < p < \infty)$

$$V \mapsto \Delta_p V := \frac{\partial}{\partial x} \Big(\Big| \frac{\partial V}{\partial x} \Big|^{p-2} \frac{\partial V}{\partial x} \Big) \quad \text{for } V \colon (0, \infty) \to \mathbb{R} \colon S \mapsto V(S) \,. \tag{2.9}$$

We construct the nonlinear analogue of the linear terminal value problem (2.1), (2.2) by replacing the linear diffusion operator $\mathscr{L}_2$ in formula (2.4) by the nonlinear diffusion operator $\mathscr{L}_p$ in formula (2.8) as follows:

$$\frac{\partial V}{\partial t} + \mathscr{A}_p V - rV(S,t) = 0 \quad \text{for } (S,t) \in (0,\infty) \times (0,T) \,; \tag{2.10}$$

with the terminal condition (2.2), i.e., $V(S,T) = h(S)$ for $S \in (0, \infty)$, where $\mathscr{A}_p$ stands for the nonlinear B-S operator defined by

$$(\mathscr{A}_p V)(S,t) := \frac{1}{p} (\sigma S)^p \frac{\partial}{\partial S} \Big(\Big| \frac{\partial V}{\partial S} \Big|^{p-2} \frac{\partial V}{\partial S} \Big) + (q_S - \gamma_S) S \frac{\partial V}{\partial S} \tag{2.11}$$

for $V \colon (0, \infty) \times (0, T) \to \mathbb{R} \colon (S,t) \mapsto V(S,t)$.

Let us recall that the volatility, σ, is taken to be a positive constant, $\sigma \in (0, \infty)$, and both, γ_S and q_S, are some real constants, $\gamma_S, q_S \in \mathbb{R}$; however, notice that we use only their difference $q_S - \gamma_S$. Setting $q_S - \gamma_S = r$ is a very "rough" approximation (often used in practice), cf. [6, 10].

Instead of looking for a solution $V \colon (0,\infty) \times (0,T) \to \mathbb{R}$ to the nonlinear terminal value problem (2.10), (2.2), we look for a solution

$$u(x, \tau) = V(S,t) \equiv V(\mathrm{e}^x, T - \tau) \quad \text{for } (x, \tau) \in \mathbb{R}^1 \times (0, \infty) \,, \tag{2.12}$$

where $x = \log S \in \mathbb{R}^1$ and $\tau = T - t \in \mathbb{R}_+$, to the following initial value problem which is equivalent with the terminal value problem (2.10), (2.2):

$$\begin{aligned} &\frac{\partial u}{\partial \tau} - \frac{1}{p} \sigma^p \cdot \Delta_p u + \big(1 - \frac{1}{p}\big) \sigma^p \cdot \Big| \frac{\partial u}{\partial x} \Big|^{p-2} \frac{\partial u}{\partial x} - (q_S - \gamma_S) \frac{\partial u}{\partial x} \\ &+ r u(x, \tau) = 0 \quad \text{for } (x, \tau) \in \mathbb{R}^1 \times (0, \infty) \,; \end{aligned} \tag{2.13}$$

with the initial condition

$$u(x, 0) = h(\mathrm{e}^x) \equiv h(S) \quad \text{for } x \in \mathbb{R}^1 \,. \tag{2.14}$$

Here, besides the quasilinear diffusion operator

$$\widetilde{\Delta}_p \colon u \ \mapsto\ \frac{1}{p} \sigma^p \cdot \Delta_p u \equiv \frac{1}{p} \sigma^p \cdot \frac{\partial}{\partial x} \Big(\Big| \frac{\partial u}{\partial x} \Big|^{p-2} \frac{\partial u}{\partial x} \Big) \tag{2.15}$$

for $u \colon \mathbb{R}^1 \to \mathbb{R} \colon x \mapsto u(x)$, obtained from (2.8) and (2.9), we have the nonlinear convection operator

$$u \mapsto \widetilde{\mathscr{B}}_p u \equiv -\big(1 - \frac{1}{p}\big) \sigma^p \cdot \Big| \frac{\partial u}{\partial x} \Big|^{p-2} \frac{\partial u}{\partial x} + (q_S - \gamma_S) \frac{\partial u}{\partial x} \tag{2.16}$$

for $u\colon \mathbb{R}^1 \to \mathbb{R}\colon x \mapsto u(x)$.

Finally, let us introduce the nonlinear B-S operator $\widetilde{\mathcal{A}}_p$ corresponding to $\mathscr{A}_p$ defined in formula (2.11) by

$$\begin{aligned}(\widetilde{\mathcal{A}}_p u)(x,\tau) &:= \widetilde{\Delta}_p u + (\widetilde{\mathcal{B}}_p u)(x,\tau)\\ &\equiv \frac{1}{p}\sigma^p \cdot \Delta_p u - \big(1-\frac{1}{p}\big)\sigma^p \cdot \Big|\frac{\partial u}{\partial x}\Big|^{p-2}\frac{\partial u}{\partial x} + (q_S-\gamma_S)\frac{\partial u}{\partial x}\\ &\quad\text{for } u\colon \mathbb{R}^1 \to \mathbb{R}\colon x \mapsto u(x)\,.\end{aligned} \tag{2.17}$$

With this notation, the nonlinear initial value problem (2.13), (2.14) takes the following equivalent abstract form,

$$\frac{\partial u}{\partial \tau} - \widetilde{\mathcal{A}}_p u + r\, u(x,\tau) = 0 \quad \text{for } (x,\tau) \in \mathbb{R}^1 \times (0,\infty)\,; \tag{2.18}$$

with the initial condition (2.14), i.e., $u(x,0) = h(\mathrm{e}^x) \equiv h(S)$ for $x \in \mathbb{R}^1$.

We wish to transform this initial value problem into a simpler, more standard form. This form will turn out to be suitable for treatment by well-known tools from nonlinear functional analysis as presented in [5, Chapt. III, §§2.1] and [12, Chapt. 2, Sect. 1]. To this end, motivated by our definition of the B-S operator $\widetilde{\mathcal{A}}_p$ in (2.17) above, we substitute $\hat{t} = p^{-1}\sigma^p\tau$ for the time variable $\tau \in \mathbb{R}_+$ (not to be confused with $\tau = T - t$ in and before formula (2.12)) together with $\widehat{q}_S = p\sigma^{-p}(q_S - \gamma_S)$ and $\widehat{r} = p\sigma^{-p} r$, where both $\widehat{q}_S, \widehat{r} \in \mathbb{R}$, and the moving coordinate $\hat{x} = x + \widehat{q}_S\,\hat{t}$ for the space variable $x \in \mathbb{R}^1$. Upon this substitution, the initial value problem (2.18), (2.14) for the unknown function $u(x,\tau)$ above becomes the following initial value problem for the unknown function

$$\hat{u}(\hat{x},\hat{t}) = u(x,\tau) \equiv u\big(\hat{x} - \widehat{q}_S\,\hat{t},\, p\sigma^{-p}\hat{t}\big) \quad \text{of } (\hat{x},\hat{t}) \in \mathbb{R}^1 \times \mathbb{R}_+\,. \tag{2.19}$$

Hence, problem (2.18), (2.14) becomes

$$\frac{\partial \hat{u}}{\partial \hat{t}} - \mathcal{A}_p \hat{u} + \widehat{r}\,\hat{u}(\hat{x},\hat{t}) = 0 \quad \text{for } (\hat{x},\hat{t}) \in \mathbb{R}^1 \times (0,\infty)\,; \tag{2.20}$$

with the initial condition (2.14), i.e., $\hat{u}(\hat{x},0) = h(\mathrm{e}^{\hat{x}}) \equiv h(S)$ for $\hat{x} \in \mathbb{R}^1$, where

$$\begin{aligned}(\mathcal{A}_p \hat{u})(\hat{x},\hat{t}) &:= p\sigma^{-p} \cdot \Big((\widetilde{\mathcal{A}}_p \hat{u})(x,\tau) - (q_S - \gamma_S)\frac{\partial \hat{u}}{\partial x}(x,\tau)\Big)\\ &= \Delta_p \hat{u} - (p-1)\cdot \Big|\frac{\partial \hat{u}}{\partial x}\Big|^{p-2}\frac{\partial \hat{u}}{\partial x}\end{aligned} \tag{2.21}$$

for $\hat{u}\colon \mathbb{R}^1 \to \mathbb{R}\colon x \mapsto \hat{u}(x)$.

To keep our notation standard, in the sequel we write $(x,t) \in \mathbb{R}^1 \times \mathbb{R}_+$ again in place of the pair $(\hat{x},\hat{t})$, and $u(x,t)$ in place of the function $\hat{u}(\hat{x},\hat{t})$, so that problem (2.20), (2.14) above becomes

$$\frac{\partial u}{\partial t} - \mathcal{A}_p u + \widehat{r}\,u(x,t) = 0 \quad \text{for } (x,t) \in \mathbb{R}^1 \times (0,\infty)\,; \tag{2.22}$$

with the initial condition (2.14), i.e., $u(x,0) = h(\mathrm{e}^x) \equiv h(S)$ for $x \in \mathbb{R}^1$. Here, the B-S operator $\widetilde{\mathcal{A}}_p$ in (2.17) has been replaced by the new operator $\mathcal{A}_p$ of a similar form acting on a C^1-function $u\colon \mathbb{R}^1 \to \mathbb{R}\colon x \mapsto u(x)$:

$$\begin{aligned}(\mathcal{A}_p u)(x,t) &:= \Delta_p u + (\mathcal{B}_p u)(x,t) \equiv \Delta_p u - (p-1)\cdot \Big|\frac{\partial u}{\partial x}\Big|^{p-2}\frac{\partial u}{\partial x}\\ &= \mathrm{e}^{(p-1)x}\cdot \frac{\partial}{\partial x}\Big(\mathrm{e}^{-(p-1)x}\cdot \Big|\frac{\partial u}{\partial x}\Big|^{p-2}\frac{\partial u}{\partial x}\Big)\,,\end{aligned} \tag{2.23}$$

where $\mathcal{B}_p$ corresponds to the nonlinear convection operator $\widetilde{\mathcal{B}}_p$ defined in (2.16),

$$u \mapsto \mathcal{B}_p u \equiv -(p-1) \cdot \Big|\frac{\partial u}{\partial x}\Big|^{p-2} \frac{\partial u}{\partial x} \quad \text{for } \ u \colon \mathbb{R}^1 \to \mathbb{R} \colon x \mapsto u(x) \,. \tag{2.24}$$

We remark that the linear convection operator (i.e., the transport term)

$$u \mapsto \widehat{q}_S \cdot \frac{\partial u}{\partial x} \quad \text{for } u \colon \mathbb{R}^1 \to \mathbb{R} \colon x \mapsto u(x) \,, \tag{2.25}$$

has been absorbed in the introduction of the moving coordinate.

3. Analytic approach to the B-S model with Δ_p $(1 < p < \infty)$

The widely used European call and put options are calculated from the linear B-S model (where $p = 2$) with the initial values $h(S) = (S-K)^+$ and $h(S) = (S-K)^- = (K-S)^+$, respectively, for $S \in (0,\infty)$, where the constant $K \in (0,\infty)$ stands for the *strike price at maturity*. These kinds of (realistic) initial values would, however, force us to consider our (in general) nonlinear initial value problem (2.22), (2.14), that is to say,

$$\frac{\partial u}{\partial t} - \mathrm{e}^{(p-1)x} \cdot \frac{\partial}{\partial x}\Big(\mathrm{e}^{-(p-1)x} \cdot \Big|\frac{\partial u}{\partial x}\Big|^{p-2} \frac{\partial u}{\partial x}\Big) + \widehat{r} u(x,t) = 0 \tag{3.1}$$

for $(x,t) \in \mathbb{R}^1 \times (0,\infty)$, with the initial condition (2.14), i.e., $u(x,0) = h(\mathrm{e}^x) \equiv h(S)$ for $x \in \mathbb{R}^1$, in a Banach function space X of (exponentially) unbounded functions $u(\cdot,t) \colon \mathbb{R}^1 \to \mathbb{R} \colon x \mapsto u(x,t)$ with respect to the space variable $x \in \mathbb{R}^1$. In particular, this would mean $h \circ \exp \in X \colon x \mapsto h(\mathrm{e}^x) \colon \mathbb{R}^1 \to \mathbb{R}_+ = [0,\infty)$ with $h(S) = (S-K)^+$ for the call option and $h(S) = (K-S)^+$ for the put option, respectively. We plan to treat a suitable function space setting that allows for these more realistic initial values in the near future. A closely related problem of choosing wisely a suitable weighted Sobolev space for our space setting will be discussed later in Remark 5.1 (Section 5).

The initial value problem (3.1), (2.14) to be solved is a typical example of an evolutionary problem with a nonlinear infinitesimal generator $\mathcal{A}_p \colon D = \mathcal{D}(\mathcal{A}_p) \subset H \to H$ defined on a dense subset $D \subset H$ of a (real) Hilbert space $H \subset L^2_{\mathrm{loc}}(\mathbb{R}^1)$, by formula (2.23). In the sequel we use two methods to solve this evolutionary problem: First, by applying a method from nonlinear functional analysis to the partial differential operators that appear in equation (3.1) as presented in the monograph by Lions [12, Chapt. 2, Sect. 1, pp. 155–171] and, second, by applying another, quite similar method, to the abstract evolutionary problem in (3.1) in the Hilbert space H as described in Barbu [5, Chapt. III, §§2.1, pp. 123–138]. This method yields a so-called integral solution to the initial value problem (3.1), (2.14) which is unique in the class of all integral solutions. As interesting alternatives to this monograph ([5]) we recommend Brézis [7] and Miyadera [13]. The methods and tools studied and applied in Neuberger's monograph [17] are closer to those used in [5, 12], but his earlier works [14, 15, 16] prefer to use those that appear in [7, 13].

Our motivation for now is to present a relatively simple, acceptable presentation of applying the standard theory of nonlinear semigroups (in a Hilbert or Banach space; see e.g. [5, 7, 12, 13]) to the evolutionary problem in (3.1). We will set this problem in the real Hilbert space $H \equiv L^2(\mathbb{R}^1; \mathfrak{w})$ of all real-valued Lebesgue-measurable functions $f \colon \mathbb{R}^1 \to \mathbb{R}$ with the finite norm

$$\|f\|_H \equiv \|f\|_{L^2(\mathbb{R}^1;\mathfrak{w})} := \Big(\int_{\mathbb{R}^1} |f(x)|^2 \mathfrak{w}(x)\,\mathrm{d}x\Big)^{1/2} < \infty \,, \tag{3.2}$$

with the weight function $\mathfrak{w}(x) := \mathrm{e}^{-(p-1)x}$ of $x \in \mathbb{R}^1$; recall that $1 < p < \infty$ is a given number. This norm is induced by the (real) inner product

$$(f,g)_H \equiv (f,g)_{L^2(\mathbb{R}^1;\mathfrak{w})} := \int_{-\infty}^{+\infty} f\, g \cdot \mathfrak{w}(x)\,\mathrm{d}x \quad \text{for } f,g \in H\,. \tag{3.3}$$

In addition, we denote by $W^{1,p} \equiv W^{1,p}(\mathbb{R}^1;\mathfrak{w})$ the (real) vector space of all locally absolutely continuous functions $f\colon \mathbb{R}^1 \to \mathbb{R}$ with the finite seminorm

$$\|f\|_{D^{1,p}} := \Big(\int_{\mathbb{R}^1} \Big| \frac{\partial f}{\partial x} \Big|^p \mathfrak{w}(x)\,\mathrm{d}x \Big)^{1/p} < \infty\,, \tag{3.4}$$

Denoting by $C_{\mathrm{c}}^1(\mathbb{R}^1)$ the vector space of all compactly supported, continuously differentiable functions $f\colon \mathbb{R}^1 \to \mathbb{R}$, we define $D^{1,p} \equiv D^{1,p}(\mathbb{R}^1;\mathfrak{w})$ to be the closure of the vector space $C_{\mathrm{c}}^1(\mathbb{R}^1)$ in $W^{1,p}$ with respect to the seminorm $\|\cdot\|_{D^{1,p}}$ on $W^{1,p}$. It is obvious that $\|\cdot\|_{D^{1,p}}$ is a norm on $C_{\mathrm{c}}^1(\mathbb{R}^1)$ and, consequently, on $D^{1,p} \subset W^{1,p}$, as well, the latter being a Banach space. This norm on $D^{1,p}$ is uniformly convex, by s Clarkson's inequalities; see e.g. Adams and Fournier [1], Theorem 2.39 on p. 45 and Theorem 3.6 on p. 61. Hence, $D^{1,p}$ is a reflexive Banach space. We denote by $D^{-1,p'} \equiv D^{-1,p'}(\mathbb{R}^1;\mathfrak{w}) := (D^{1,p})'$ the (strong) dual space of the Banach space $D^{1,p} \equiv D^{1,p}(\mathbb{R}^1;\mathfrak{w})$, where $p' := p/(p-1) \in (1,\infty)$ is the conjugate exponent to $p \in (1,\infty)$; hence, $\frac{1}{p} + \frac{1}{p'} = 1$.

Next, we give a rigorous definition of the operator $\mathcal{A}_p$ formally defined in (2.23). Let us introduce the "kinetic energy" functional $\mathcal{E}_p\colon D^{1,p} \to \mathbb{R}$ by

$$\mathcal{E}_p(u) := \frac{1}{p} \cdot \|u\|_{D^{1,p}}^p = \frac{1}{p} \int_{-\infty}^{+\infty} \Big| \frac{\partial u}{\partial x} \Big|^p \mathrm{e}^{-(p-1)x}\,\mathrm{d}x \quad \text{for } u \in D^{1,p}\,. \tag{3.5}$$

This (nonlinear) functional is Gâteaux-differentiable with the Gâteaux differential $\mathcal{E}_p'(u) \in (D^{1,p})' = D^{-1,p'}$ (often called also Gâteaux derivative) at a point $u \in D^{1,p}$ given by the formula

$$\begin{aligned} [\mathcal{E}_p'(u)]v &= \int_{-\infty}^{+\infty} \Big| \frac{\partial u}{\partial x} \Big|^{p-2} \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} \mathrm{e}^{-(p-1)x}\,\mathrm{d}x \\ &= -\int_{-\infty}^{+\infty} \mathrm{e}^{(p-1)x} \cdot \frac{\partial}{\partial x} \Big(\mathrm{e}^{-(p-1)x} \cdot \Big| \frac{\partial u}{\partial x} \Big|^{p-2} \frac{\partial u}{\partial x} \Big) \cdot v(x)\mathfrak{w}(x)\,\mathrm{d}x \\ &= -\int_{-\infty}^{+\infty} (\mathcal{A}_p u)(x) \cdot v(x)\mathfrak{w}(x)\,\mathrm{d}x \quad \text{for every } v \in D^{1,p}\,. \end{aligned} \tag{3.6}$$

Notice that the integration by parts in the last formula is justified by our choice of the Sobolev space $D^{1,p}$ with the vector space $C_{\mathrm{c}}^1(\mathbb{R}^1)$ being dense in $D^{1,p}$ ($\subset W^{1,p}$). By applying Hölder's inequality to the first formula in (3.6) we infer that $\mathcal{E}_p'(u)\colon D^{1,p} \to \mathbb{R}$ is a bounded linear form with values in the dual space $D^{-1,p'} = (D^{1,p})'$. A direct calculation involving the continuity of the standard L^p-norm reveals that the mapping

$$u \mapsto \mathcal{E}_p'(u)\colon D^{1,p} \to D^{-1,p'}$$

is continuous. Consequently, the Gâteaux differential $\mathcal{E}_p'(u)$ is, in turn, the Fréchet differential (again, called also Fréchet derivative) at the point $u \in D^{1,p}$. The injectivity of the nonlinear mapping $\mathcal{E}_p'\colon D^{1,p} \to D^{-1,p'} = (D^{1,p})'$ follows from its strict monotonicity which is a direct consequence of the "kinetic energy" functional $\mathcal{E}_p\colon D^{1,p} \to \mathbb{R}$ being strictly convex; cf. Lions [12, Chapt. 2, §1.2, pp. 157–158].

Numerous helpful facts about (nonlinear) monotone operators $A\colon \mathcal{D}(A) \subset H \to H$ in Hilbert spaces (even from a reflexive Banach space X into its dual space X') can be found in Barbu's monograph [5, Chapt. II, §1, Sect. 1.1 on pp. 33–41]. In particular, if A is a Fréchet differential or, more generally, the subdifferential mapping of a convex function $\varphi\colon \mathcal{D}(\varphi) \subset X \to \mathbb{R}$ on a reflexive Banach space X, the interested reader is referred to [5, Chapt. II, §2, Sect. 2.2 on pp. 52–60].

The claim that $\mathcal{E}_p'$ is also surjective follows by the minimization of the linearly perturbed "energy" functional $\mathcal{E}_{p,\varphi}\colon D^{1,p} \to \mathbb{R}$ defined by

$$\begin{aligned} \mathcal{E}_{p,\varphi}(u) &:= \mathcal{E}_p(u) - \varphi(u) = \frac{1}{p} \cdot \|u\|_{D^{1,p}}^p - \varphi(u) \\ &= \frac{1}{p} \int_{-\infty}^{+\infty} \Big|\frac{\partial u}{\partial x}\Big|^p \mathrm{e}^{-(p-1)x}\,\mathrm{d}x - \varphi(u) \quad \text{for every } \ u \in D^{1,p}\,, \end{aligned} \tag{3.7}$$

where $\varphi \in D^{-1,p'}$ is a bounded linear form on $D^{1,p}$, given arbitrarily. The (unique) minimizer, denoted by $u_\varphi \in D^{1,p}$, satisfies the equation $\mathcal{E}_{p,\varphi}'(u) = \mathcal{E}_p'(u_\varphi) - \varphi = 0$ in $D^{-1,p'} = (D^{1,p})'$. Furthermore, the mapping $\varphi \mapsto u_\varphi\colon D^{-1,p'} \to D^{1,p}$ is continuous, owing to the norm $\|\cdot\|_{D^{1,p}}$ on the Banach space $D^{1,p}$ being uniformly convex. Further details about this "minimization method" are explained in the lecture notes by Takáč [21, Sect. 2–4, pp. 72–76]. We conclude that the mapping $\mathcal{E}_p'\colon D^{1,p} \to D^{-1,p'} = (D^{1,p})'$, is a homeomorphism.

In our second step we define the operator $\mathcal{A}_p$ in accordance with (2.23), $\mathcal{A}_p\colon D = \mathcal{D}(\mathcal{A}_p) \subset H \to D^{-1,p'}$, as follows: We set $D = \mathcal{D}(\mathcal{A}_p) := D^{1,p} \cap H$ to be the domain of $\mathcal{A}_p$ and

$$\mathcal{A}_p(u) := -\mathcal{E}_p'(u) \in D^{-1,p'} = (D^{1,p})' \tag{3.8}$$

for $u \in D = \mathcal{D}(\mathcal{A}_p) := D^{1,p} \cap H$.

Clearly, the vector space $C_{\mathrm{c}}^1(\mathbb{R}^1)$ is dense in both normed spaces, $D^{1,p}$ and H, and consequently also in their intersection $D = D^{1,p} \cap H$ endowed with the norm

$$\|u\|_D := \|u\|_{D^{1,p}} + \|u\|_H \quad \text{for } u \in D = \mathcal{D}(\mathcal{A}_p) := D^{1,p} \cap H\,.$$

Since both Banach spaces $D^{1,p}$ and H are reflexive, so is D with the dual space $D' = D^{-1,p'} + H'$ as justified below.

In our third step we identify the (real) Hilbert space $H \equiv L^2(\mathbb{R}^1; \mathfrak{w})$ (endowed with the norm in (3.2) and the inner product in (3.3)) with its (strong) dual space H', by the Riesz representation theorem. As a result, since the (continuous) imbedding $D \hookrightarrow H = H'$ is dense, thanks to $C_{\mathrm{c}}^1(\mathbb{R}^1) \subset D \subset H$, we may identify also H' with a dense subspace of the dual space $D' = D^{-1,p'} + H = (D^{1,p})' + H'$ of the reflexive Banach space D. The locally convex topological vector space $\mathscr{D}(\mathbb{R}^1) \equiv C_{\mathrm{c}}^\infty(\mathbb{R}^1)$ of all compactly supported C^∞ test functions $\phi\colon \mathbb{R}^1 \to \mathbb{R}$ being dense in $C_{\mathrm{c}}^1(\mathbb{R}^1)$, we may identify the dual space D' with a vector subspace of $[C_{\mathrm{c}}^1(\mathbb{R}^1)]' \subset \mathscr{D}'(\mathbb{R}^1) \equiv [C_{\mathrm{c}}^\infty(\mathbb{R}^1)]'$, by standard arguments from the Theory of Distributions, see Schwartz [20]. Thus, $D \hookrightarrow H = H' \hookrightarrow D'$ is a Gel'fand triplet (cf. [12, Chapt. 2, §1.1], Remark 1.2 on p. 156). The (real) inner product $(\cdot,\cdot)_H\colon H \times H \to \mathbb{R}$ on $H = H'$ extends uniquely to a duality mapping $(\cdot,\cdot)_{D\times D'}\colon D \times D' \to \mathbb{R}$ on $D \times D'$ which we denote by $(\cdot,\cdot)_{D\times D'} \equiv (\cdot,\cdot)_H\colon D \times D' \to \mathbb{R}$ again, due to the uniqueness of the extension. We will show that, for every number $\alpha \in (0,\infty)$, the (nonlinear) mapping

$$I - \alpha \cdot \mathcal{A}_p\colon D \to D' = D^{-1,p'} + H\colon u \mapsto u - \alpha \cdot \mathcal{A}_p(u)$$

is injective and it maps the domain D onto $D' = D^{-1,p'} + H$. Indeed, the injectivity follows from the strict monotonicity of $\mathcal{E}_p\colon D^{1,p} \to D^{-1,p'}$ combined with (3.8). To verify the surjectivity, let us take any pair of bounded linear functionals $\varphi \in (D^{1,p})' = D^{-1,p'}$ and $\phi \in H' = H$; hence, $\phi(u) = (u,f)_H$ for all $u \in H$, where $f \equiv f_\phi \in H$ is the identification of the functional $\phi \in H'$ with a unique element from H. Now, given any number $\alpha \in (0,\infty)$, let us consider the new "energy" functional $\Phi_{p,\alpha,\varphi,\phi}\colon D \to \mathbb{R}$ defined by

$$\begin{aligned}
\Phi_{p,\alpha,\varphi,\phi}(u) &:= \alpha \cdot \mathcal{E}_p(u) + \frac{1}{2}(u,u)_H - \varphi(u) - \phi(u) \\
&= \frac{\alpha}{p} \cdot \|u\|_{D^{1,p}}^p + \frac{1}{2}\|u\|_H^2 - \varphi(u) - (u,f)_H \\
&= \frac{1}{p}\int_{-\infty}^{+\infty} \Big|\frac{\partial u}{\partial x}\Big|^p \mathrm{e}^{-(p-1)x}\,\mathrm{d}x + \frac{1}{2}\int_{-\infty}^{+\infty} |u(x)|^2 \mathrm{e}^{-(p-1)x}\,\mathrm{d}x \\
&\quad - \varphi(u) - \int_{-\infty}^{+\infty} u(x)\, f(x) \mathrm{e}^{-(p-1)x}\,\mathrm{d}x \quad \text{for all } u \in D\,.
\end{aligned} \tag{3.9}$$

This functional is clearly coercive on its domain $D = D^{1,p} \cap H$, thanks to the powers $\min\{p,\,2\} > 1$, and strictly convex, as well, with the norms $\|\cdot\|_{D^{1,p}}$ and $\|\cdot\|_H$ being uniformly convex on $D^{1,p}$ and H, respectively. Applying the "minimization method" explained in Takáč [21, Sect. 2–4, pp. 72–76], we obtain a (unique) minimizer for $\Phi_{p,\alpha,\varphi,\phi}\colon D \to \mathbb{R}$, denoted by $u_{\varphi,\phi} \in D$; it satisfies the critical point equation

$$\Phi'_{p,\alpha,\varphi,\phi}(u_{\varphi,\phi}) = \alpha \cdot \mathcal{E}'_p(u_{\varphi,\phi}) + u_{\varphi,\phi} - \varphi - f = 0$$

in $D' = D^{-1,p'} + H = (D^{1,p})' + H'$. Recalling (3.8), we conclude that the mapping

$$I - \alpha \cdot \mathcal{A}_p\colon D \to D' = D^{-1,p'} + H\colon u \mapsto u - \alpha \cdot \mathcal{A}_p(u)$$

is also surjective, i.e., it maps D onto its (strong) dual space $D' = D^{-1,p'} + H$. We have proved that, in fact, $I - \alpha \cdot \mathcal{A}_p\colon D \to D'$ is a homeomorphism for each $\alpha > 0$. The strict monotonicity of the mapping $-\mathcal{A}_p = \mathcal{E}'_p|_D\colon D \to D'$, defined in formula (3.8) with respect to the duality $(\cdot,\cdot)_{D\times D'} \equiv (\cdot,\cdot)_H\colon D \times D' \to \mathbb{R}$ (induced by the inner product), is easily derived from the strict convexity of the functional $\mathcal{E}_p\colon D^{1,p} \to \mathbb{R}$; namely, we have

$$(\mathcal{A}_p(u) - \mathcal{A}_p(v),\, u - v)_H \le 0 \quad \text{for every pair } u,v \in D = D^{1,p} \cap H\,. \tag{3.10}$$

Consequently, the inverse mapping $(I - \alpha \cdot \mathcal{A}_p)^{-1}: D' \to D$ restricts to a nonexpansive mapping on H, sometimes called (nonstrict) contraction on H, that is,

$$\|\,(I - \alpha \cdot \mathcal{A}_p)^{-1}(u) - (I - \alpha \cdot \mathcal{A}_p)^{-1}(v)\|_H \le \|u - v\|_H \tag{3.11}$$

holds for every pair $u,v \in H \subset D'$.

We refer to the monograph by Miyadera [13, Chapt. 2, §2], Theorem 2.9 on p. 20, for details.

Finally, using these results on the Gel'fand triplet $D \hookrightarrow H = H' \hookrightarrow D'$ and the nonlinear mapping $\mathcal{A}_p\colon D \to D'$, we are able to apply the general theorem from Lions [12, Chapt. 2, §1.4], Théorème 1.2 on pp. 162–163 and Théorème 1.2 bis on p. 163, to obtain our main result:

Theorem 3.1 (Existence and uniqueness)**.** *Let $T \in (0,\infty)$ and $\widehat{r} \in \mathbb{R}$. For each initial value $u_0 \in H$, $u_0(x) = h(\mathrm{e}^x) \equiv h(S)$ with $x \in \mathbb{R}^1$, there exists a unique* *weak solution* $u\colon [0,T] \to H$ *to our initial value problem* (3.1), (2.14) *that has the following properties:*

(i) $u\colon [0,T] \to H\colon t \mapsto u(\cdot,t)$ *is continuous, i.e.,* $u \in C([0,T] \to H)$.
(ii) *We have* $\lim_{t\to 0+} u(t) = u(0) = u_0$ *in* H.
(iii) $u\colon (0,T) \to D \hookrightarrow D^{1,p}\colon t \mapsto u(\cdot,t)$ *is (strongly) Lebesgue-measurable with the finite integral*

$$\int_0^T \|u(\cdot,t)\|_{D^{1,p}}^p \,\mathrm{d}t < \infty,$$

i.e., $u \in L^p((0,T) \to D^{1,p})$ *or, equivalently,* $u \in L^p((0,T) \to D)$*, thanks to* $u \in$
$C([0,T] \to H)$.
(iv) *The (weak distributional) derivative* $\frac{\partial u}{\partial t}\colon (0,T) \to D' = H + D^{-1,p'}$ *is (strongly) Lebesgue-measurable with the finite integral*

$$\int_0^T \big\|\frac{\partial u}{\partial t}(\cdot,t)\big\|_{D'}^{p'} \,\mathrm{d}t < \infty\,,$$

i.e., $\frac{\partial u}{\partial t} \in L^{p'}((0,T) \to D')$ *or, equivalently,* $u \in W^{1,p'}((0,T) \to D')$*, thanks to* $u \in C([0,T] \to H)$*, again.*
(v) *The partial differential equation* (3.1) *is satisfied in the weak sense with all terms valued in the dual space* $D' = H + D^{-1,p'}$*, that is to say, the following abstract differential equation holds for almost every* $t \in (0,T)$*:*

$$\frac{\partial u}{\partial t} - \mathcal{A}_p(u) + \widehat{r}\,u(x,t) = 0 \quad \text{for a.e. } t \in (0,T)\,, \tag{3.12}$$

where all terms on the left-hand side are in D'.

Notice that we use the Gel'fand triplet $D \hookrightarrow H = H' \hookrightarrow D'$ and the nonlinear mapping $-\mathcal{A}_p = \mathcal{E}_p'|_D\colon D \to D'$ in place of $V \hookrightarrow H = H' \hookrightarrow V'$ and $A\colon V \to V'$, respectively, used in [12, Chapt. 2, §1.4].

Proof of Theorem 3.1. Because $\|\cdot\|_{D^{1,p}}$ is only a seminorm on the Sobolev space $W^{1,p} \equiv W^{1,p}(\mathbb{R}^1;\mathfrak{w})$, inequality (1.36) on p. 162 in Lions [12, Chapt. 2, §1.4, Théorème 1.2] is not satisfied. This means that [12, Théorème 1.2 (pp. 162–163)] cannot be applied to our initial value problem (3.1), (2.14) directly. Instead of this inequality, we have to use a weaker one, inequality (1.42) on p. 163, in the proof of [12, Théorème 1.2 bis (p. 163)]. Thus, [12, Théorème 1.2 (p. 163)] is applicable to our problem (3.1), (2.14) directly, with a small correction discussed in [12, §§1.5.2 on p. 166]. $\square$

4. Nonlinear semigroups - an alternative approach to the B-S model

To recall interesting contributions by John W. Neuberger to the theory of nonlinear (contraction) semigroups, including the works in [14, 15, 16], I decided to provide an alternative proof of an analogue of Theorem 3.1 which uses explicitly only the Hilbert space $H \equiv L^2(\mathbb{R}^1;\mathfrak{w})$ from Theorem 3.1. The monotonicity of the nonlinear operator $-\mathcal{A}_p = \mathcal{E}_p'\colon D \to D'$ has been verified in (3.10). In the previous section (Section 3) we have taken advantage of the theory of monotone operators for $A\colon \mathcal{D}(A) \subset X \to X'$ from a reflexive Banach space X into its dual space X'. Such a (nonlinear) monotone operator, A, is closely related to its restriction (by means of the corresponding graphs) to a monotone operator $\hat{A} = A|_{\mathcal{D}(\hat{A})} \colon \mathcal{D}(\hat{A}) \subset H \to H$ in the Hilbert space H from the Gel'fand triplet $X \hookrightarrow H = H' \hookrightarrow X'$. To be more precise, one prefers to study its negative, $-\hat{A} \colon \mathcal{D}(\hat{A}) \subset H \to H$, within the class of so-called dissipative operators on a Hilbert space H; see e.g. Barbu [5, Chapt. II, §3, Sect. 3.1 and 3.2 on pp. 71–89]. Such (nonlinear) operators are employed in [5, Chapt. III, §1 and §2 on pp. 98–151] to construct time-continuous solutions $u \colon [0,T] \to H \colon t \mapsto u(t)$, i.e., $u \in C([0,T] \to H)$, of autonomous evolutionary equations with the (nonlinear dissipative) generator $-\hat{A}$.

What we need to use in the approach by nonlinear semigroups is the restriction of the graph $\mathscr{G}(\mathcal{A}_p)$ of $\mathcal{A}_p \colon D \to D'$ in the Cartesian product $D \times D'$, i.e., $\mathscr{G}(\mathcal{A}_p) \subset D \times D'$, to the set $\mathscr{G}_H := \mathscr{G}(\mathcal{A}_p) \cap (H \times H)$ in the Cartesian product $H \times H$. We prefer to apply this "restriction" procedure to the inverse mapping $(I - \alpha_0 \cdot \mathcal{A}_p)^{-1} \colon D' \to D$ that has been shown to restrict to a nonexpansive mapping on H, by (3.11), where $\alpha_0 > 0$ is a fixed number taken arbitrarily. We denote by

$$\mathscr{D}_H = (I - \alpha_0 \cdot \mathcal{A}_p)^{-1}(H)$$

the inverse image of the Hilbert space H under the nonlinear operator $I - \alpha_0 \cdot \mathcal{A}_p \colon D \to D'$. Now, given any number $\alpha > 0$, the corresponding differential equation $u - \alpha \cdot \mathcal{A}_p(u) = f(x)$, $x \in \mathbb{R}^1$, for the unknown function $u \in D$, with a given function $f \in H$ on the right-hand side, is a quasilinear ordinary differential equation that is equivalent with

$$\begin{gathered} u - \alpha \cdot \mathcal{A}_p(u) = f(x) \quad \text{for } x \in \mathbb{R}^1,, \quad \text{and further with} \\ u - \alpha_0 \cdot \mathcal{A}_p(u) = \frac{\alpha_0}{\alpha} f(x) + \big(1 - \frac{\alpha_0}{\alpha}\big) u(x) \quad \text{for } x \in \mathbb{R}^1 . \end{gathered}$$

Thus, for any number $\alpha > 0$ and any function $u \in D = D^{1,p} \cap H$ it holds; cf. Barbu [5, Chapt. II, Proposition 3.3 on p. 73]:

$$u \in (I - \alpha \cdot \mathcal{A}_p)^{-1}(H) \iff u \in \mathscr{D}_H = (I - \alpha_0 \cdot \mathcal{A}_p)^{-1}(H) .$$

In other words, the range $(I - \alpha \cdot \mathcal{A}_p)^{-1}(H)$ of the inverse mapping $(I - \alpha \cdot \mathcal{A}_p)^{-1} \colon D' \to D$ restricted to the Hilbert space H, denoted above by $\mathscr{D}_H = (I - \alpha_0 \cdot \mathcal{A}_p)^{-1}(H)$, is independent from a particular choice of $\alpha > 0$. Consequently, we may define the desired restriction of $\mathcal{A}_p \colon D \to D'$ to a densely defined nonlinear operator $A_p \colon \mathscr{D}_H \to H$ on the domain $\mathscr{D}_H$ ($\subset H$) by

$$A_p(u) := \mathcal{A}_p(u) = -\mathcal{E}_p'(u) \in H \quad \text{for all } u \in \mathscr{D}_H . \tag{4.1}$$

By our results on the operator $\mathcal{A}_p \colon D \to D'$ in the previous section (Section 3), we conclude that $-A_p \colon \mathscr{D}_H \to H$ is a monotone operator satisfying $(I - \alpha \cdot \mathcal{A}_p)(\mathscr{D}_H) = H$ for every $\alpha > 0$, by (3.10). Moreover, the operator $I - \alpha \cdot A_p$ is invertible with a non-expansive inverse on H, by (3.11). Such a nonlinear operator $A_p \colon \mathscr{D}_H \subset H \to H$ is called *m-dissipative*, cf. Miyadera [13, Chapt. 2, §2], Definition 2.5 and Lemma 2.13 on p. 22.

Finally, we are ready to apply the result from Barbu [5, Chapt. III, Theorem 2.3 on p. 135] (with $\omega = \widehat{r} \in \mathbb{R}$) on existence and uniqueness of a strong solution $u \in C([0,T] \to H)$ to our initial value problem (3.1), (2.14), which is a generalization of an earlier result in [5, Chapt. III, Theorem 2.2 on p. 131] (with $\omega = 0$). Indeed, this result, [5, Chapt. III, Theorem 2.3 on p. 135], yields the following analogue of Theorem 3.1.

Theorem 4.1 (Existence and uniqueness). *Let $T \in (0,\infty)$ and $\widehat{r} \in \mathbb{R}$. Given any initial value $u_0 \in H$, $u_0(x) = h(\mathrm{e}^x) \equiv h(S)$ for $x \in \mathbb{R}^1$, there exists a unique strong solution $u\colon [0,T] \to H$ to our initial value problem* (3.1), (2.14) *that has the following properties:*

- (i) *$u\colon [0,T] \to H\colon t \mapsto u(\cdot,t)$ is (uniformly) Lipschitz-continuous on $[0,T]$, i.e., $u \in W^{1,\infty}([0,T] \to H)$ meaning that $u \in C([0,T] \to H)$ and u is differentiable almost everywhere with the strong derivative $\frac{\mathrm{d}u}{\mathrm{d}t} \in L^\infty((0,T) \to H)$.*
- (ii) *We have $\lim_{t\to 0+} u(t) = u(0) = u_0$ in H.*
- (iii) *$u(t) \equiv u(\cdot,t) \in \mathscr{D}_H (\subset H)$ holds for almost every $t \in (0,T)$.*
- (iv) *The partial differential equation* (3.1) *is satisfied in the strong sense with all terms valued in the Hilbert space $H' = H$, that is to say, the abstract differential equation* (3.12) *holds for almost every $t \in (0,T)$, where all terms on the left-hand side are in $H' = H$ and $\mathcal{A}_p(u(t)) = A_p(u(t))$ $(\in H)$.*

Remark 4.2. In this article we used three types of solutions $u \in C([0,T] \to H)$ to our initial value problem (3.1), (2.14):

- (a) Weak solutions defined in Lions [12, Chapt. 2, §1.4], Théorème 1.2 on pp. 162–163, and in Barbu [5, Chapt. III, §2, Sect. 2.1], Definition 2.2 on p. 134.
- (b) Strong solutions defined in [5, Chapt. III, §1, Sect. 3.1], Definition 1.2 on p. 110; see also [5, Chapt. III, §2, Sect. 2.1], p. 123. Strong solutions are often called simply “solutions”.
- (c) Integral solutions defined in [5, Chapt. III, §2, Sect. 2.1], Definition 2.1 on pp. 123–124.

A discussion and results on their mutual relations can be found in Barbu [5, Chapt. III, §2, Sect. 2.1].

5. Discussion, comments, and suggestions

Before we discuss the “volatility” character of our nonlinear B-S operator $\mathscr{A}_p$ defined in (2.11), which is evidently written in the so-called divergence form, we would like to mention another, now rather “classical” nonlinear B-S operator written in the general, non-divergence form:

$$(\mathscr{B}V)(S,t) := \frac{1}{2}(\sigma S)^2 \left[1 + \mathcal{S}\left(\mathrm{e}^{r(T-t)}(aS)^2\, \tfrac{\partial^2 V}{\partial S^2}\right)\right] \frac{\partial^2 V}{\partial S^2} + (q_S - \gamma_S) S \frac{\partial V}{\partial S} \quad (5.1)$$
$$\text{for } V\colon (0,\infty)\times(-\infty,T) \to \mathbb{R}\colon (S,t) \mapsto V(S,t)\,.$$

This nonlinear version of the B-S operator has been derived in the work by Barles and Soner [4, Eq. (1.2), p. 372] and studied later by numerical simulations in, e.g., Koleva and Vulkov [11].

Here, $\sigma \in (0,\infty)$ is the “linear” volatility factor (obtained from the linear B-S operator with $\mathcal{S} \equiv 0$), $a \in (0,\infty)$ is a parameter typical for this model (explained in [4, pp. 371–372]), and $\mathcal{S}\colon \mathbb{R} \to [-1,+\infty)$ is a C^1-function that is a classical solution of the ordinary differential equation in [4, [Eq. (3.2), p. 377] with the initial value $\mathcal{S}(0) = 0$. Perhaps the most important property of this function, besides the limits

$$\lim_{A\to-\infty} \mathcal{S}(A) = -1 \quad \text{and} \quad \lim_{A\to+\infty} \frac{\mathcal{S}(A)}{A} = +1\,,$$

is the fact that the function $A \mapsto A(1+\mathcal{S}(A))\colon \mathbb{R} \to \mathbb{R}$ is monotone increasing (i.e., non-decreasing), even strictly monotone increasing in a vicinity of the point

$A = 0$, proved in [4, Appendix A, pp. 385–388]. Consequently, this function is also negative on the negative half-line $(-\infty, 0)$ and nonnegative on $\mathbb{R}_+ = [0, \infty)$. The dummy variable $A \in \mathbb{R}$ in the argument of the function $\mathcal{S}(A)$ stands for the expression $A = \mathrm{e}^{r(T-t)}(aS)^2 \frac{\partial^2 V}{\partial S^2}$ where $\tau = T - t \in \mathbb{R}_+$ stands for the time to maturity with $-\infty < t \le T$.

We conclude that the constant "linear" volatility factor σ (from the linear B-S operator with $\mathcal{S} \equiv 0$) becomes a nonconstant implied volatility of the form following (called also adjusted volatility), cf. [4, p. 372]:

$$\begin{gathered} \hat{\sigma} \equiv \hat{\sigma}\Big(S, t, \tfrac{\partial^2 V}{\partial S^2}\Big) := \sigma \Big[1 + \mathcal{S}\Big(\mathrm{e}^{r(T-t)}(aS)^2 \tfrac{\partial^2 V}{\partial S^2}\Big)\Big]^{1/2} \\ \text{for } V\colon (0,\infty) \times (-\infty, T) \to \mathbb{R}\colon (S,t) \mapsto V(S,t)\,. \end{gathered} \tag{5.2}$$

Hence, the nonlinearity is introduced into this model through the bracket factor $[\dots]^{1/2}$ with a help from the second partial derivative $\frac{\partial^2 V}{\partial S^2}$.

Last but not least, a linear generalization of the classical (linear) Black-Scholes model (2.1), (2.2) with stochastic volatility, the so-called Heston model [9] has been treated mathematically in the works by Alziary and Takáč [2, 3].

In contrast to the adjusted volatility $\hat{\sigma}$ in (5.2) above, which depends on $\partial^2 V/\partial S^2$, our volatility adjustment in (2.11) is of a different nature; it depends on the first partial derivative $\partial V/\partial S$ rather than on $\partial^2 V/\partial S^2$. This derivative, $\partial V/\partial S$, called simply the Greek delta and denoted correspondingly by $\boldsymbol{\Delta} = \frac{\partial V}{\partial S}$ (to distinguish it from the common linear Laplace operator $\Delta \equiv \Delta_2$), is connected with the so-called delta hedging in Mathematical Finance, in order to create a riskless portfolio; see Hull [10, Chapt. 18, pp. 377–399]. To relate our volatility adjustment in (2.11) to the linear B-S operator $\mathscr{A}_2$ in (2.3), we have replaced the factor $\frac{1}{2}(\sigma S)^2$ in the expression $\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$ in (2.4) by the factor $\frac{p-1}{p}(\sigma S)^p \left|\frac{\partial V}{\partial S}\right|^{p-2}$ in (2.8) whenever $1 < p < \infty$. We remark that in mathematical finance the Greek delta, $\boldsymbol{\Delta} = \frac{\partial V}{\partial S}$, measures the sensitivity of the option price $V = V(S,t)$ with respect to the changes in the stock price $S \in [0, \infty)$; see, in particular, [10, §18.4, pp. 380–386] for delta hedging. A position taken by a trader with $\boldsymbol{\Delta} = 0$ is called delta neutral and plays an important role in trading strategy. Thus, our volatility adjustment in (2.11) is based on this sensitivity (the Greek delta) rather than on the value of the linear diffusion term $\frac{\partial^2 V}{\partial S^2} = \frac{\partial \boldsymbol{\Delta}}{\partial S}$ which appears in the expression for $\hat{\sigma}$ in (5.2) above. The sensitivity of the Greek delta, that is to say, the partial derivative $\frac{\partial \boldsymbol{\Delta}}{\partial S}$, does not seem to play as important role in option trading as does the Greek delta itself or its Greek "cousin" gamma denoted by Γ; cf. [10, §18.6, pp. 389–392].

Remark 5.1. The choice of the weight function $\mathfrak{w}(x) := \mathrm{e}^{-(p-1)x}$ of $x \in \mathbb{R}^1$ in the weighted Lebesgue and Sobolev spaces $H = L^2(\mathbb{R}^1; \mathfrak{w})$ and $W^{1,p} = W^{1,p}(\mathbb{R}^1; \mathfrak{w})$ is determined by the dependence of the option price $u(x,t) = u(\log\ S, t)$ on the stock price $S \in (0, \infty)$ for large negative / positive values of the logarithmic stock price $x = \log\ S \in \mathbb{R}^1$, i.e., for $S \to 0+$ and $S \to +\infty$, respectively. Here, the time $t \in (0, \infty)$ is assumed to be fixed. We will see below, that our choice of $\mathfrak{w}(x)$ corresponds to the asymptotic behavior

$$\mathrm{e}^{-(p-1)x} \cdot |u(x,t)| = S^{-(p-1)}\, |u(\log\ S, t)| \to 0 \quad \text{as } x = \log S \to \pm\infty\,. \tag{5.3}$$

Some remarks on this asymptotic behavior for $p = 2$ can be found, e.g., in Alziary and Takáč [2, 3], Björk [6], Heston [9], and Hull [10]. In principle, this asymptotic behavior (meaning "boundary conditions" near infinity $\pm\infty$) is determined by financial markets. The initial condition $u(x, 0) = h(\mathrm{e}^x) \equiv h(S)$ for $x \in \mathbb{R}^1$ in (2.14) plays an important role in (5.3). Typically, the function $h(\mathrm{e}^x) = h(S)$ should satisfy (5.3), which means that the asymptotic behavior in (5.3) is influenced by the initial values in (2.14). These requirements and restrictions also influence the choice of the weight function $\mathfrak{w}(x)$.

To provide a feeling of what our asymptotic behavior in (5.3) actually means in rigorous "mathematical terms", let us recall that we have used only the seminorm $\|\cdot\|_{D^{1,p}}$ in the definition of the weighted Sobolev space $W^{1,p} = W^{1,p}(\mathbb{R}^1; \mathfrak{w})$ introduced in (3.4) in Section 3. There, we have not used the ("finiteness" of the related) norm

$$\|f\|_{L^p} := \Big(\int_{\mathbb{R}^1} |f(x)|^p \mathfrak{w}(x) \,\mathrm{d}x \Big)^{1/p} \tag{5.4}$$

in the weighted Lebesgue space $L^p = L^p(\mathbb{R}^1; \mathfrak{w})$. However, we can easily derive, with a help from Hölder's inequality, that for any $f \in W^{1,p}$, $f(x) = f(\log\, S)$ for $x \in \mathbb{R}^1$, obeying the asymptotic behavior (5.3), we have $\|f\|_{D^{1,p}} < \infty \Rightarrow \|f\|_{L^p} < \infty$. Indeed, given any number $R \in (0, \infty)$ large enough, we calculate

$$\begin{aligned}
&\int_{-R}^{R} |f(x)|^p \mathfrak{w}(x) \,\mathrm{d}x \\
&= -\frac{1}{p-1} \int_{-R}^{R} |f(x)|^p \cdot \frac{\mathrm{d}}{\mathrm{d}x} \mathrm{e}^{-(p-1)x} \,\mathrm{d}x \\
&= -\frac{1}{p-1} \big[|f(x)|^p \cdot \mathrm{e}^{-(p-1)x} \big] \Big|_{x=-R}^{x=R} \\
&\quad + \frac{p}{p-1} \int_{-R}^{R} |f(x)|^{p-2} f(x) \cdot f'(x) \mathrm{e}^{-(p-1)x} \,\mathrm{d}x \\
&= -\frac{1}{p-1} \big[|f(x)|^p \cdot \mathrm{e}^{-(p-1)x} \big] \Big|_{x=-R}^{x=R} \\
&\quad + \frac{p}{p-1} \int_{-R}^{R} \Big(|f(x)| \mathrm{e}^{-x/p'} \Big)^{p-2} f(x) \mathrm{e}^{-x/p'} \cdot f'(x) \mathrm{e}^{-(p-1)x/p} \,\mathrm{d}x \\
&= -\frac{1}{p-1} \big[|f(x)|^p \cdot \mathrm{e}^{-(p-1)x} \big] \Big|_{x=-R}^{x=R} \\
&\quad + \frac{p}{p-1} \int_{-R}^{R} \Big(|f(x)| \mathfrak{w}(x)^{1/p} \Big)^{p-2} f(x) \mathfrak{w}(x)^{1/p} \cdot f'(x) \mathfrak{w}(x)^{1/p} \,\mathrm{d}x \,,
\end{aligned}$$

then apply (5.3) with $R \to +\infty$ followed by Hölder's inequality, thus arriving at the inequality $\|f\|_{L^p} \le p' \|f\|_{D^{1,p}}$, which guarantees also our claim above, namely, $\|f\|_{D^{1,p}} < \infty \Rightarrow \|f\|_{L^p} < \infty$. Indeed, first, let $\varepsilon > 0$ be arbitrary (but small) and choose $R_\varepsilon > 0$ large enough, such that

$$|f(x)|^p \cdot \mathrm{e}^{-(p-1)x} \le \frac{p-1}{2} \varepsilon \quad \text{holds for all } x \in \mathbb{R}^1 \text{ satisfying } |x| \ge R_\varepsilon \,. \tag{5.5}$$

Let us recall that, by eq. (5.3), we have $|f(x)|^p \cdot \mathrm{e}^{-(p-1)x} = |f(\log\ S)|^p \cdot S^{-(p-1)} \to 0$ as $x = \log\ S \to \pm\infty$. Combining this inequality with Hölder's inequality, we obtain

$$\begin{aligned}&\int_{-R}^{+R} |f(x)|^p \mathfrak{w}(x)\,\mathrm{d}x \\ &\leq \varepsilon + \frac{p}{p-1}\Big(\int_{-R}^{+R} |f(x)|^p \mathfrak{w}(x)\,\mathrm{d}x\Big)^{1/p'}\Big(\int_{-R}^{+R} |f'(x)|^p \mathfrak{w}(x)\,\mathrm{d}x\Big)^{1/p} \\ &= \varepsilon + \frac{p}{p-1}\Big(\int_{-R}^{+R} |f(x)|^p \mathfrak{w}(x)\,\mathrm{d}x\Big)^{(p-1)/p}\Big(\int_{-R}^{+R} |f'(x)|^p \mathfrak{w}(x)\,\mathrm{d}x\Big)^{1/p}\end{aligned}$$

for every $R \geq R_\varepsilon$ (> 0). The implication above, $\|f\|_{D^{1,p}} < \infty \Rightarrow \|f\|_{L^p} < \infty$, follows by letting $R \to +\infty$. More precisely, by letting $\varepsilon \to 0+$ which entails also $R_\varepsilon \to +\infty$, we obtain

$$\begin{aligned}\|f\|_{L^p}^p = \|f\|_{L^p(\mathbb{R}^1;\mathfrak{w})}^p &= \int_{-\infty}^{+\infty} |f(x)|^p \mathfrak{w}(x)\,\mathrm{d}x \\ &\leq \frac{p}{p-1}\Big(\int_{-\infty}^{+\infty} |f(x)|^p \mathfrak{w}(x)\,\mathrm{d}x\Big)^{1/p'}\Big(\int_{-\infty}^{+\infty} |f'(x)|^p \mathfrak{w}(x)\,\mathrm{d}x\Big)^{1/p} \\ &= p'\|f\|_{L^p}^{p/p'} \cdot \|f\|_{D^{1,p}} \\ &= p'\|f\|_{L^p}^{p-1} \cdot \|f\|_{D^{1,p}} < \infty\,.\end{aligned}$$

Consequently, $\|f\|_{L^p} \leq p'\, \|f\|_{D^{1,p}}$ as desired.

Analytical techniques of this kind play a crucial role in particular in the Heston model (see Heston [9]) treated in Alziary andTakáč [2, 3]. There, the reader is referred especially to [2, Appendix: Sect. 10, pp. 43–48] and to [3, Appendix B, pp. 35–43].

Acknowledgments. The author would like to express his thanks to an anonymous referee for very careful reading of the manuscript and a few helpful suggestions and corrections.

References

[1] R. A. Adams, J. J. F. Fournier; *Sobolev Spaces*, 2nd ed., Academic Press, New York-Oxford, 2003.

[2] B. Alziary, P. Takáč; *Analytic Solutions and Complete Markets for the Heston Model with Stochastic Volatility*, Electronic J. Diff. Equations, **2018**(168) (2018), 1–54. ISSN: 1072-6691. Online: URL: http://ejde.math.txstate.edu Preprint, arXiv: 1711.04536v1 [math.AR], 13th November 2017 - arxiv.org. http://arxiv.org/abs/1711.04536

[3] B. Alziary, P. Takáč; *The Heston stochastic volatility model has a boundary trace at zero volatility*, Preprint, arXiv: 2004.00444v1 [math.AP], 1st April 2020 - arxiv.org. http://arxiv.org/abs/2004.00444

[4] G. Barles, H. M. Soner; *Option pricing with transaction costs and a nonlinear Black-Scholes equation*, Finance Stochast., **2**(4) (1998), 369–397.

[5] V. Barbu; *Nonlinear Semigroups and differential equations in Banach spaces.* Noordhoff International Publising, Leyden, The Netherlands, 1976.

[6] T. Björk, *Arbitrage Theory in Continuous Time*, 3nd Ed., Oxford Univ. Press, Oxford, 2011.

[7] H. Brézis; *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert.* North-Holland Publ. Co., Amsterdam, The Netherlands, 1973.

[8] J.-P. Fouque, G. Papanicolaou, K. R. Sircar; *Derivatives in Financial Markets with Stochastic Volatility*, Cambridge University Press, Cambridge, U.K., 2000.

[9] S. L. Heston; *A closed-form solution for options with stochastic volatility with applications to bond and currency options*, Review of Financial Studies, **6**(2) (1993), 327–343. Online: http://www.jstor.org/stable/2962057.

[10] J. C. Hull; *Options, Futures, and Other Derivatives*, 7-th ed., Prentice-Hall, Englewood Cliffs, NJ, 2008.

[11] M. N. Koleva, L. G. Vulkov; *A kernel-based algorithm for numerical solution of nonlinear pDEs in finance*, *Numerical Analysis and Its Applications*, Ivan Lirkov, Svetozar Margenov, and Jerzy Waśniewski, Eds., in *Lecture Notes in Computer Science*, Vol. **7116**, pp. 566–573. Springer International Publishing AG, Cham, Switzerland, 2012. DOI: 10.1007/978-3-642-29843-1.

[12] J.-L. Lions; *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod; Gauthier-Villars, Paris, 1969.

[13] I. Miyadera; *Nonlinear Semigroups*, in Translations of Mathematical Monographs, Vol. **109**. American Math. Society, Providence, RI, 1992.

[14] John W. Neuberger; *An exponential formula for one-parameter semi-groups of nonlinear transformations*, J. Math. Soc. Japan, **18** (1966), 154–157.

[15] John W. Neuberger; *Analyticity and quasi-analyticity for one-parameter semigroups*, Proc. Amer. Math. Soc., **25** (1970), 488–494.

[16] John W. Neuberger; *Quasi-analyticity and semigroups*, Bull. Amer. Math. Soc., **78** (1972), 909–922.

[17] John W. Neuberger, *Sobolev Gradients and Differential Equations*, in Lect. Notes in Math., Vol. **1670**. Springer-Verlag, Berlin-Heidelberg-New York, 1997 and 2006.

[18] J. W. Neuberger, R. J. Renka; *Sobolev gradients and the Ginzburg-Landau functional*, SIAM J. Sci. Comput., **20**(2) (1998), 582–590.

[19] D. H. Sattinger; *Monotone Methods in Nonlinear Elliptic and Parabolic Boundary Value Problems*, Indiana Univ. Math. J., **21**(11) (1972), 979–1000.

[20] L. Schwartz; *Théorie des distributions, I, II*, Hermann, Paris, 1950–1951 (nouvelle édition 1966, nouveau tirage 1998).

[21] P. Takáč; *Variational methods and linearization tools towards the spectral analysis of the p-Laplacian, especially for the Fredholm alternative*. Electr. J. Differential Equations, Conf. **18** (2010), 67–105. ISSN: 1072-6691. In "Proceedings of the 2007 Conference on Variational and Topological Methods: Theory, Applications, Numerical Simulations, and Open Problems, II, May 23–27, 2007, Flagstaff, Arizona, USA.

[22] H. Tanabe; *Equations of Evolution'*, in Monographs and Studies in Mathematics, Vol. **6**. Pitman Publ., Boston, Mass.-London, 1979. (Translated from the Japanese by N. Mugibayashi and H. Haneda.)

Peter Takáč

Universität Rostock, Institut für Mathematik, Ulmenstrasse 69, Haus 3, D-18057 Rostock, Germany

Email address: `peter.takac@uni-rostock.de`