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FUČIK SPECTRUM WITH WEIGHTS AND EXISTENCE OF SOLUTIONS FOR NONLINEAR ELLIPTIC EQUATIONS WITH NONLINEAR BOUNDARY CONDITIONS

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In Memory of John W. Neuberger

Abstract. We consider the boundary value problem

$$\begin{aligned} -\Delta u + c(x)u &= \alpha m(x)u^+ - \beta m(x)u^- + f(x,u), \quad x \in \Omega, \\ \frac{\partial u}{\partial \eta} + \sigma(x)u &= \alpha\rho(x)u^+ - \beta\rho(x)u^- + g(x,u), \quad x \in \partial\Omega, \end{aligned}$$

where $(\alpha,\beta) \in \mathbb{R}^2$, $c, m \in L^\infty(\Omega)$, $\sigma, \rho \in L^\infty(\partial\Omega)$, and the nonlinearities f and g are bounded continuous functions. We study the asymmetric (Fučik) spectrum with weights, and prove existence theorems for nonlinear perturbations of this spectrum for both the resonance and non-resonance cases. For the resonance case, we provide a sufficient condition, the so-called generalized Landesman-Lazer condition, for the solvability. The proofs are based on variational methods and rely strongly on the variational characterization of the spectrum.

1. Introduction

We consider the partial differential equation

$$\begin{aligned} -\Delta u + c(x)u &= m(x)[\alpha u^+ - \beta u^-], \quad x \in \Omega, \\ \frac{\partial u}{\partial \eta} + \sigma(x)u &= \rho(x)[\alpha u^+ - \beta u^-], \quad x \in \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Delta z := \nabla \cdot \nabla z$, $\frac{\partial}{\partial \eta}$ is the outward normal derivative, $(\alpha,\beta) \in \mathbb{R}^2$ are parameters, and $c, m \in L^\infty(\Omega)$, $\sigma, \rho \in L^\infty(\partial\Omega)$ with $c(x), m(x) \geq 0$ almost everywhere in Ω, $\sigma(x), \rho(x) \geq 0$ almost everywhere in $\partial\Omega$,

$$\int c(x)\,dx + \oint \sigma(x)\,dx > 0 \quad \text{and} \quad \int m(x)\,dx + \oint \rho(x)\,dx > 0,$$

where $\int$ denotes the (volume) integral on Ω and $\oint$ denotes the (surface) integral on $\partial\Omega$. Throughout this paper we assume that Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with smooth boundary $\partial\Omega$.

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We are interested in the Fučik spectrum, namely,

$$\Sigma := \{(\alpha,\beta)\in\mathbb{R}^2 : (1.1) \text{ has a non-trivial solution}\}$$

and our first main result provides a variational characterization of a curve in Σ.

As an application of the variational characterization we consider

$$\begin{aligned} -\Delta u + c(x)u &= m(x)[\alpha u^+ - \beta u^-] + f(x,u) \quad \text{in } \Omega,\\ \frac{\partial u}{\partial \nu} + \sigma(x)u &= \rho(x)[\alpha u^+ - \beta u^-] + g(x,u) \quad \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

i.e. a nonlinear perturbation of (1.1). We assume nonlinearities of the form $f(x,u) := m(x)\tilde f(u)$ and $g(x,u) := \rho(x)\tilde g(u)$, where $\tilde f, \tilde g : \mathbb{R}\to\mathbb{R}$ are bounded continuous functions. We prove existence theorems for the non-resonance case, $(\alpha,\beta)\notin\Sigma$, and the resonance case, $(\alpha,\beta)\in\Sigma$. For the resonance case we assume a generalized Landesman-Lazer condition as in [4] and [8].

Our methods are built on the results in [7, 4, 8]. Section 2 provides a brief summary of the function spaces and the variational setting. In Section 3, we prove the variational characterization of a curve in Σ using a Hilbert space reduction method as in [2, 4, 8]. Section 4 contains the existence theorem for the non-resonance case. Section 5 contains the existence theorem for the resonance case.

2. Characterization of the Fučik Spectrum

2.1. **Variational preliminaries.** Define the (c,σ)-inner product $\langle\cdot,\cdot\rangle_{(c,\sigma)} : H^1(\Omega)\times H^1(\Omega)\to\mathbb{R}$ by

$$\langle u,v\rangle_{(c,\sigma)} = \int \nabla u\cdot\nabla v + \int c(x)uv + \oint \sigma(x)uv,$$

with the associated norm denoted by $\|u\|_{(c,\sigma)}$. This norm is equivalent to the standard $H^1(\Omega)$-norm. Set

$$\langle u,v\rangle_{(m,\rho)} = \int m(x)uv + \oint \rho(x)uv, \quad \|u\|^2_{(m,\rho)} := \int m(x)u^2 + \oint \rho(x)u^2,$$

for $u,v\in H^1(\Omega)$.

Let $V_{(m,\rho)} = \{u\in H^1(\Omega) : \|u\|_{(m,\rho)} = 0\}$, and let $H^1_{(m,\rho)} = V^{\perp}_{(m,\rho)}$ be the orthogonal complement with respect to the (c,σ) inner product. Then $H^1(\Omega) = H^1_{(m,\rho)}\oplus V_{(m,\rho)}$ (see [7]) and it further follows that $H^1_{(m,\rho)}$ and $V_{(m,\rho)}$ are (m,ρ) orthogonal. We will also make use of the norm $\|\cdot\|_{(c,\sigma)}$ on $H^1(\Omega)$ and $\|\cdot\|_{(m,\rho)}$ on $H^1_{(m,\rho)}$.

We also provide an alternate characterization of $V_{(m,\rho)}$ from [7]: taking $\Omega(m) := \{x\in\Omega : m(x)>0\}$ and $\partial\Omega(\rho) := \{x\in\partial\Omega : \rho(x)>0\}$, we have

$$V_{(m,\rho)} = \{u\in H^1(\Omega) : u=0 \text{ a.e in } \Omega(m) \text{ and } \Gamma u = 0 \text{ a.e in } \partial\Omega(\rho)\}, \tag{2.1}$$

where Γ is the trace operator on $\partial\Omega$.

Consider the functional $J : H^1(\Omega)\to\mathbb{R}$ defined by

$$J_{\alpha,\beta}(u) = \frac{1}{2}[\|u\|^2_{(c,\sigma)} - \alpha\|u^+\|^2_{(m,\rho)} - \beta\|u^-\|^2_{(m,\rho)}]. \tag{2.2}$$

Then

$$J'_{\alpha,\beta}(u)\cdot v = \langle u,v\rangle_{(c,\sigma)} - \alpha\langle u,v\rangle_{(m,\rho)} + (\beta-\alpha)\langle u^-,v\rangle_{(m,\rho)}. \tag{2.3}$$

We note that critical points of $J_{\alpha,\beta}$ are weak solutions of (1.1).

We begin with a lemma on the nature of the Fučik eigenfunctions.

Lemma 2.1. *Every Fučik eigenfunction ψ is contained in $H^1_{(m,\rho)}$.*

Proof. Assume to the contrary that $\psi = u + v$, where $u \in H^1_{(m,\rho)}$, $v \in V_{(m,\rho)}$, and v is nonzero on a set of positive measure. Then

$$\begin{aligned} 0 &= J'_{\alpha,\beta}(\psi)\cdot v \\ &= \langle u+v, v\rangle_{(c,\sigma)} - \alpha\langle u+v, v\rangle_{(m,\rho)} + (\beta-\alpha)\langle (u+v)^-, v\rangle_{(m,\rho)} \\ &= \|v\|^2_{(c,\sigma)}, \end{aligned}$$

because of the alternate characterization of $V_{(m,\rho)}$ in (2.1). Hence, $v = 0$ a.e. which contradicts our assumption. So all Fučik eigenfunctions are in $H^1_{(m,\rho)}$. □

2.2. **Trivial curves.** It is known (see [7]) that the problem

$$\begin{gathered} -\Delta u + c(x)u = \mu m(x)u, \quad x \in \Omega, \\ \frac{\partial u}{\partial \eta} + \sigma(x)u = \mu\rho(x)u, \quad x \in \partial\Omega, \end{gathered}$$

has a simple first eigenvalue $\mu_1 > 0$ with associated eigenfunction ϕ_1 which is of one sign in $\overline{\Omega}$. Therefore $\phi_1^+ = \phi_1$ and $\phi_1^- = 0$, so that

$$-\Delta\phi_1 + c(x)\phi_1 = \mu_1 m(x)\phi_1 = m(x)[\mu_1\phi_1^+ - \beta\phi_1^-]$$

for any $\beta \in \mathbb{R}$, and similarly

$$\frac{\partial \phi_1}{\partial \eta} + \sigma(x)\phi_1 = \rho(x)[\mu_1\phi_1^+ - \beta\phi_1^-]$$

for any $\beta \in \mathbb{R}$. Therefore

$$\mathcal{C}_0 := \{(\mu_1, \beta) : \beta \in \mathbb{R}\} \subset \Sigma.$$

A similar argument will show that

$$\mathcal{C}'_0 := \{(\alpha, \mu_1) : \alpha \in \mathbb{R}\} \subset \Sigma.$$

The curves $\mathcal{C}_0$ and $\mathcal{C}'_0$ are depicted in Figure 1.

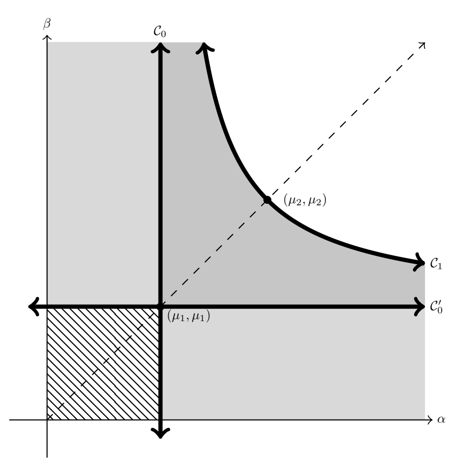

FIGURE 1. Trivial and first Fučik curves

Lemma 2.2.

$$\Sigma \cap \{(\alpha,\beta) \in \mathbb{R}^2 : \alpha < \mu_1 \text{ or } \beta < \mu_1\} \cap \left(\mathcal{C}_0 \cup \mathcal{C}_0'\right)^C = \emptyset$$

Proof. Let $\alpha < \mu_1$ and $\beta \neq \mu_1$. Assume that $(\alpha,\beta) \in \Sigma$ and let $\psi \in H^1_{(m,\rho)}$ be a Fučik eigenfunction associated to (α,β). Then

$$0 = J'_{\alpha,\beta}(\psi) \cdot \psi^+ = \|\psi^+\|^2_{(c,\sigma)} - \alpha\|\psi^+\|^2_{(m,\rho)} \geq (\mu_1 - \alpha)\|\psi^+\|^2_{(m,\rho)}.$$

So, since $\alpha < \mu_1$, it follows that $\|\psi^+\|^2_{(m,\rho)} = 0$, which implies that $\psi^+ = 0$ almost everywhere. Hence, $\psi = -\psi^-$, and hence ψ is a non-positive Steklov eigenfunction. So ψ satisfies

$$\begin{gathered} -\Delta\psi + c(x)\psi = m(x)\beta\psi; \quad x \in \Omega, \\ \frac{\partial\psi}{\partial\eta} + \sigma(x)\psi = \rho(x)\beta\psi; \quad x \in \partial\Omega. \end{gathered}$$

But if ψ is a non-sign-changing solution, then $\beta = \mu_1$, a contradiction. Hence $(\alpha,\beta) \notin \Sigma$.

If $\beta < \mu_1$ and $\alpha \neq \mu_1$, the argument proceeds similarly by examining the expression $J'_{\alpha,\beta}(\psi) \cdot \psi^-$. □

2.3. **Higher curves.** In what follows, we will consider the case $\mu_k < \alpha < \mu_{k+1}$ and $\alpha < \beta$. If $(\alpha,\beta) \in \Sigma$, then $(\beta,\alpha) \in \Sigma$, and therefore, it suffices to only consider the case $\alpha < \beta$. The first curve $\mathcal{C}_1$ is depicted in Figure 1.

We split the space $H^1_{(m,\rho)} = X_k \oplus Y_k$ where $X_k = \operatorname{span}\{\phi_1, \phi_2, \dots, \phi_k\}$ and $Y_k = \operatorname{span}\{\phi_{k+1}, \phi_{k+2}, \dots\}$. We further define $Y = Y_k \oplus V_{(m,\rho)}$ so that $H^1 = X_k \oplus Y$. We begin with an estimate which will be crucial for several lemmas later.

Lemma 2.3. *Let $(\alpha_i, \beta_i) \in \mathbb{R}^2$ for $i = 1,2$ satisfy the previous hypotheses, and let $s_i = \beta_i - \alpha_i$. Let $x_i \in X_k$ and $y_i \in Y$ for $i = 1,2$. Then*

$$\begin{aligned} &\left(J'_{\alpha_2,\beta_2}(x_2+y_2) - J'_{\alpha_1,\beta_1}(x_1+y_1)\right) \cdot (x_2 - x_1) \\ &\leq -\delta\|x_2 - x_1\|^2_{(c,\sigma)} + s_2\left(\|x_2 - x_1\|_{(m,\rho)} + \|y_2 - y_1\|_{(m,\rho)}\right)\|y_2 - y_1\|_{(m,\rho)} \\ &\quad + |\alpha_2 - \alpha_1|\|x_1\|_{(m,\rho)}\|x_2 - x_1\|_{(m,\rho)} + |s_2 - s_1|\|x_1 + x_2\|_{(m,\rho)}\|x_2 - x_1\|_{(m,\rho)}, \end{aligned}$$

where $\delta = \frac{\alpha_2}{\mu_k} - 1$.

Proof. First we show that

$$\begin{aligned} &J'_{\alpha_i,\beta_i}(x_i + y_i)(x_2 - x_1) \\ &= \langle x_i + y_i, x_2 - x_1\rangle_{(c,\sigma)} - \alpha_i\langle x_i + y_i, x_2 - x_1\rangle_{(m,\rho)} + s_i\langle (x_i + y_i)^-, x_2 - x_1\rangle_{(m,\rho)} \\ &= \langle x_i, x_2 - x_1\rangle_{(c,\sigma)} - \alpha_i\langle x_i, x_2 - x_1\rangle_{(m,\rho)} \\ &\quad + s_i\langle (x_i + y_i)^-, x_2 - x_1\rangle_{(m,\rho)}, \end{aligned}$$

by the (c, σ)- and (m, ρ)-orthogonality of X_k and Y. Then utilizing the previous expression, we have

$$\begin{aligned}
&\big(J'_{\alpha_2,\beta_2}(x_2+y_2) - J'_{\alpha_1,\beta_1}(x_1+y_1)\big)\cdot(x_2-x_1)\\
&= \|x_2-x_1\|^2_{(c,\sigma)} - \langle \alpha_2 x_2 - \alpha_1 x_1, x_2 - x_1\rangle_{(m,\rho)}\\
&\quad + \langle s_2(x_2+y_2)^- - s_1(x_1+y_1)^-, x_2-x_1\rangle_{(m,\rho)}\\
&= \|x_2-x_1\|^2_{(c,\sigma)} - \alpha_2\|x_2-x_1\|^2_{(m,\rho)} - (\alpha_2-\alpha_1)\langle x_1, x_2-x_1\rangle_{(m,\rho)}\\
&\quad + s_2\langle (x_2+y_2)^- - (x_1+y_1)^-, x_2-x_1\rangle_{(m,\rho)}\\
&\quad + (s_2-s_1)\langle (x_1+y_1)^-, x_2-x_1\rangle_{(m,\rho)}
\end{aligned} \tag{2.4}$$

By the variational characterization of μ_k and the definition of X_k, we have that

$$\|x_2-x_1\|^2_{(c,\sigma)} - \alpha_2\|x_2-x_1\|^2_{(m,\rho)} \le \Big(1-\frac{\alpha_2}{\mu_k}\Big)\|x_2-x_1\|^2_{(c,\sigma)} = -\delta\|x_2-x_1\|^2_{(c,\sigma)}.$$

Since $f(t) = t^-$ is non-increasing, we have that $v_1^- - v_2^-$ and $v_1 - v_2$ have opposite sign for all $v_1, v_2 \in H^1$. Furthermore, $|f(t_2) - f(t_1)| \le |t_2 - t_1|$. Hence,

$$\begin{aligned}
&s_2\langle (x_2+y_2)^- - (x_1+y_1)^-, x_2-x_1\rangle_{(m,\rho)}\\
&= s_2\langle (x_2+y_2)^- - (x_1+y_1)^-, (x_2+y_2)-(x_1+y_1)\rangle_{(m,\rho)}\\
&\quad + s_2\langle (x_2+y_2)^- - (x_1+y_1)^-, y_1-y_2\rangle_{(m,\rho)}\\
&\le s_2\langle |(x_2+y_2)^- - (x_1+y_1)^-|, |y_1-y_2|\rangle_{(m,\rho)}\\
&\le s_2\langle |(x_2+y_2) - (x_1+y_1)|, |y_1-y_2|\rangle_{(m,\rho)}\\
&= s_2\langle |(x_2-x_1) - (y_2-y_1)|, |y_1-y_2|\rangle_{(m,\rho)}\\
&\le s_2\big(\|x_2-x_1\|_{(m,\rho)} + \|y_2-y_1\|_{(m,\rho)}\big)\|y_2-y_1\|_{(m,\rho)}.
\end{aligned}$$

Using Hölder's inequality, we estimate the remaining two terms as

$$\big|(\alpha_2-\alpha_1)\langle x, x_2-x_1\rangle_{(m,\rho)}\big| \le |\alpha_2-\alpha_1|\|x_1\|_{(m,\rho)}\|x_2-x_1\|_{(m,\rho)}$$

and

$$\big|(s_2-s_1)\langle (x_1+y_1)^-, x_2-x_1\rangle_{(m,\rho)}\big| \le |s_2-s_1|\|x_1+x_2\|_{(m,\rho)}\|x_2-x_1\|_{(m,\rho)}.$$

Combining the previous estimates into (2.4) yields the desired result. □

Lemma 2.4. *For a fixed $y \in Y$, $J_{\alpha,\beta}(x+y)$ is concave on X_k and moreover, for any $x_1, x_2 \in X_k$,*

$$\big(J'_{\alpha,\beta}(x_2+y) - J'_{\alpha,\beta}(x_1+y)\big)\cdot(x_2-x_1) \le -\delta\|x_2-x_1\|^2_{(c,\sigma)}.$$

Proof. Take $y_1 = y_2 = y$, $\alpha_1 = \alpha_2 = \alpha$, and $\beta_1 = \beta_2 = \beta$ in Lemma 2.3. Then $s_1 = s_2 = \beta - \alpha$, and the inequality reduces to

$$\big(J'_{\alpha,\beta}(x_2+y) - J'_{\alpha,\beta}(x_1+y)\big)\cdot(x_2-x_1) \le -\delta\|x_2-x_1\|^2_{(c,\sigma)},$$

as desired. If we further set $x_1 = 0$ and $x_2 = x$, we observe that

$$\big(J'_{\alpha,\beta}(x+y) - J'_{\alpha,\beta}(y)\big)\cdot x \le -\delta\|x\|^2_{(c,\sigma)},$$

and hence $J_{\alpha,\beta}(x+y)$ is concave on X_k. □

Since $J_{\alpha,\beta}$ is concave on X_k, for any fixed $y \in Y$, we define $r_{\alpha,\beta}(y) \in X_k$ to be the unique maximizer of $J_{\alpha,\beta}$ restricted to $X_k + y$, namely

$$J_{\alpha,\beta}(r_{\alpha,\beta}(y) + y) = \max_{x \in X_k} J_{\alpha,\beta}(x + y). \tag{2.5}$$

We now establish several properties of the function $r_{\alpha,\beta}(y)$ which will be helpful later.

Lemma 2.5. *The function $r_{\alpha,\beta}(y)$ is homogeneous (i.e., $r_{\alpha,\beta}(ty) = tr_{\alpha,\beta}(y)$ for all $t \geq 0$.)*

Proof. For any $t > 0$, we have that $J_{\alpha,\beta}(r_{\alpha,\beta}(ty)+ty) \geq J_{\alpha,\beta}(x+ty)$ for all $x \in X_k$. By the homogeneity of $J_{\alpha,\beta}$, we therefore have $J_{\alpha,\beta}\left(\frac{r_{\alpha,\beta}(ty)}{t} + y\right) \geq J_{\alpha,\beta}\left(\frac{x}{t} + y\right)$ for all $x \in X_k$. But this implies that $\frac{r_{\alpha,\beta}(ty)}{t} = r_{\alpha,\beta}(y)$, and therefore $r_{\alpha,\beta}$ is homogeneous.

For $t = 0$, we need only to show $r_{\alpha,\beta}(0) = 0$. Clearly $J_{\alpha,\beta}(0) = 0$. We will show that $J_{\alpha,\beta}(x) < 0$ for all $x \in X_k\backslash\{0\}$, and therefore, $0 = \max_{x\in X_k} J_{\alpha,\beta}(x) = r_{\alpha,\beta}(0)$.

Since $\|x\|^2_{(c,\sigma)} \leq \mu_k\|x\|^2_{(m,\rho)}$ (see [7, Corollary 2.2]), we observe that

$$\begin{aligned}
J_{\alpha,\beta}(x) &= \frac{1}{2}\left(\|x\|^2_{(c,\sigma)} - \alpha\|x^+\|^2_{(m,\rho)} - \beta\|x^-\|^2_{(m,\rho)}\right) \\
&\leq \frac{1}{2}\left(\mu_k\|x\|^2_{(m,\rho)} - \alpha\|x^+\|^2_{(m,\rho)} - \beta\|x^-\|^2_{(m,\rho)}\right) \\
&\leq \frac{1}{2}\left(\mu_k\|x\|^2_{(m,\rho)} - \alpha\|x^+\|^2_{(m,\rho)} - \alpha\|x^-\|^2_{(m,\rho)}\right) \\
&\leq \frac{1}{2}(\mu_k - \alpha)\|x\|^2_{(m,\rho)} < 0,
\end{aligned}$$

for all $x \in X_k\backslash\{0\}$. Hence, $r_{\alpha,\beta}(0) = 0$, and therefore $r_{\alpha,\beta}(ty) = tr_{\alpha,\beta}(y)$ for all $t \geq 0$ and $y \in Y$. □

Lemma 2.6. *For each $y \neq 0$, $r_{\alpha,\beta}(y) + y$ changes sign.*

Proof. Suppose to the contrary that $u = r_{\alpha,\beta}(y) + y$ is nonnegative and strictly positive on some set of positive measure, say Ω_1. Since $u \in H^1$, $u = v + \Sigma k_n\phi_n$ for $k_n = \langle u, \phi_n\rangle_{(m,\rho)}$ and some $v \in V_{(m,\rho)}$. We note that $k_1 = \langle u, \phi_1\rangle_{(m,\rho)} > 0$ since $\phi_1 > 0$ on Ω_1 and $u \notin V_{(m,\rho)}$.

Since $\phi_1 \in X_k$ $\forall k \geq 1$ and $r_{\alpha,\beta}(y)$ maximizes $J_{\alpha,\beta}$ on X_k, we have

$$\begin{aligned}
0 &= J'_{\alpha,\beta}(u) \cdot \phi_1 \\
&= \langle u, \phi_1\rangle_{(c,\sigma)} - \alpha\langle u, \phi_1\rangle_{(m,\rho)} + (\beta - \alpha)\langle u^-, \phi_1\rangle_{(m,\rho)} \\
&= \langle u, \phi_1\rangle_{(c,\sigma)} - \alpha\langle u, \phi_1\rangle_{(m,\rho)} \\
&= k_1\|\phi_1\|^2_{(c,\sigma)} - \alpha k_1\|\phi_1\|^2_{(m,\rho)} \\
&= k_1(\mu_1 - \alpha)\|\phi_1\|^2_{(m,\rho)} < 0,
\end{aligned}$$

which is a contradiction. An identical contradiction can be reached in the case that we assume u is nonpositive and strictly negative on some set of positive measure. Hence, $r_{\alpha,\beta}(y) + y$ must change sign for $y \neq 0$. □

To be precise about the result of the following lemma, let us consider the space $\tilde{Y}$, which is the set of points in Y endowed with the topology generated by $\|\cdot\|_{(m,\rho)}$.

Lemma 2.7. *$r_{\alpha,\beta}(y)$ is locally Lipschitz continuous as a function of $\mathbb{R}^2 \times \tilde{Y}$ into X_k.*

Proof. Take $x_i = r_{\alpha_i,\beta_i}(y_i)$. By the definition of $r_{\alpha_i,\beta_i}(y_i)$, we have that

$$\big(J'_{\alpha_2,\beta_2}(r_{\alpha_2,\beta_2}(y_2) + y_2) - J'_{\alpha_1,\beta_1}(r_{\alpha_1,\beta_1}(y_1)\big) \cdot (r_{\alpha_2,\beta_2}(y_2) - r_{\alpha_1,\beta_1}(y_1)) = 0,$$

and hence by Lemma 2.3, we have that

$$\begin{aligned}
&\delta\|r_{\alpha_2,\beta_2}(y_2) - r_{\alpha_1,\beta_1}(y_1)\|^2_{(c,\sigma)} \\
&\le s_2 \left(\|r_{\alpha_2,\beta_2}(y_2) - r_{\alpha_1,\beta_1}(y_1)\|_{(m,\rho)} + \|y_2 - y_1\|_{(m,\rho)}\right) \|y_2 - y_1\|_{(m,\rho)} \\
&\quad + |\alpha_2 - \alpha_1| \|r_{\alpha_1,\beta_1}(y_1)\|_{(m,\rho)} \|r_{\alpha_2,\beta_2}(y_2) - r_{\alpha_1,\beta_1}(y_1)\|_{(m,\rho)} \\
&\quad + |s_2 - s_1| \|r_{\alpha_1,\beta_1}(y_1) + r_{\alpha_2,\beta_2}(y_2)\|_{(m,\rho)} \|r_{\alpha_2,\beta_2}(y_2) - r_{\alpha_1,\beta_1}(y_1)\|_{(m,\rho)},
\end{aligned}$$

Applying a Poincare-type inequality (see Corollary 2.2 in [7]), we obtain

$$\begin{aligned}
&\delta\|r_{\alpha_2,\beta_2}(y_2) - r_{\alpha_1,\beta_1}(y_1)\|^2_{(c,\sigma)} \\
&\le s_2\Big(\frac{1}{\mu_1}\|r_{\alpha_2,\beta_2}(y_2) - r_{\alpha_1,\beta_1}(y_1)\|_{(c,\sigma)} + \|y_2 - y_1\|_{(m,\rho)}\Big)\|y_2 - y_1\|_{(m,\rho)} \\
&\quad + |\alpha_2 - \alpha_1| \|r_{\alpha_1,\beta_1}(y_1)\|_{(m,\rho)} \frac{1}{\mu_1} \|r_{\alpha_2,\beta_2}(y_2) - r_{\alpha_1,\beta_1}(y_1)\|_{(c,\sigma)} \\
&\quad + |s_2 - s_1| \|r_{\alpha_1,\beta_1}(y_1) + r_{\alpha_2,\beta_2}(y_2)\|_{(m,\rho)} \frac{1}{\mu_1} \|r_{\alpha_2,\beta_2}(y_2) - r_{\alpha_1,\beta_1}(y_1)\|_{(c,\sigma)},
\end{aligned} \tag{2.6}$$

Now, for a given y_1, let $c_1 = \|r_{\alpha_1,\beta_1}(y_1)\|_{(m,\rho)}$, $c_2 = \|r_{\alpha_1,\beta_1}(y_1) + y_1\|_{(m,\rho)}$, and $z = \|r_{\alpha_2,\beta_2}(y_2) - r_{\alpha_1,\beta_1}(y_1)\|_{(c,\sigma)}$. It follows from (2.6) that

$$\delta z^2 \le \left(\|y_2 - y_1\|_{(m,\rho)} + c_1|\alpha_2 - \alpha_1| + c_2|s_2 - s_1|\right) \frac{1}{\mu_1} z + \|y_2 - y_1\|^2_{(m,\rho)}$$

Taking $\gamma := \left(\|y_2 - y_1\|_{(m,\rho)} + c_1|\alpha_2 - \alpha_1| + c_2|s_2 - s_1|\right)$, we observe that $\|y_2 - y_1\|_{(m,\rho)} \le \gamma$, and therefore,

$$\delta z^2 \le \frac{\gamma}{\mu_1} z + \gamma^2.$$

Therefore, $z \le C(\delta)\gamma$, and the lemma is proven. □

Note that in the case $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = \beta$, γ is independent of c_1 and c_2. Therefore, since $C(\delta)$ is also independent of y_1 and y_2, we have the following corollary.

Corollary 2.8. *For a given α and β, $r_{\alpha,\beta} : \tilde{Y} \to X_k$ is globally Lipschitz continuous.*

Lemma 2.9. *There exists a $C > 0$ such that $\|r_{\alpha,\beta}(y)\|_{(c,\sigma)} \le C\|y\|_{(m,\rho)}$.*

Proof. Suppose $y_2 = y$ and $y_1 = 0$ are fixed and further suppose that $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = \beta$. Then $x_2 = r_{\alpha_2,\beta_2}(y_2) = r_{\alpha,\beta}(y)$ and $x_1 = r_{\alpha_1,\beta_1}(y_1) = r_{\alpha,\beta}(0) = 0$. Then (2.6) reduces to

$$\delta\|r_{\alpha,\beta}(y)\|^2_{c,\sigma} \le \Big(\frac{1}{\mu_1}\|r_{\alpha,\beta}(y)\|_{(c,\sigma)} + \|y\|_{(m,\rho)}\Big)\|y\|_{(m,\rho)}.$$

We may solve this inequality to observe that $\delta\|r_{\alpha,\beta}(y)\|_{c,\sigma} \le C(\delta)\|y\|_{(m,\rho)}$ where $C(\delta) = \frac{1}{2\mu_1\delta} + \sqrt{\frac{1}{\delta} + \frac{1}{4\mu_1^2\delta^2}} > 0$. Note that C is a decreasing function of δ, and therefore, if $\alpha - \mu_k = \epsilon > 0$, then we can choose $\overline{\delta} = \frac{\epsilon}{\mu_k} < \frac{\alpha}{\mu_k} - 1 = \delta$ such that

$$\overline{\delta}\|r_{\alpha,\beta}(y)\|_{c,\sigma} \le \delta\|r_{\alpha,\beta}(y)\|_{c,\sigma} \le C(\delta)\|y\|_{(m,\rho)} \le C(\overline{\delta})\|y\|_{(m,\rho)}\,.$$

□

The function $r_{\alpha,\beta}(y)$ also satisfies a compactness condition, namely:

Lemma 2.10. *Let $\{(\alpha_n, \beta_n)\}$ be a bounded sequence in $\mathbb{R}^2$ satisfying $\mu_k < \alpha_n < \mu_{k+1}$ and $\alpha_n < \beta_n$ and let $\{y_n\}$ be a bounded sequence in Y. Then there exist subsequences, again called, $\{(\alpha_n, \beta_n)\}$ and $\{y_n\}$ such that $(\alpha_n, \beta_n) \to (\alpha, \beta)$ in $\mathbb{R}^2$, $y_n \rightharpoonup y$ in Y, $y_n \to y$ in $\tilde{Y}$, and $r_{\alpha_n,\beta_n}(y_n) \to r_{\alpha,\beta}(y)$ in X_k.*

Proof. There exists a subsequence of $\{(\alpha_n, \beta_n)\}$ converging to (α, β) in $\mathbb{R}^2$ by the Bolzano-Weierstrauss Theorem, call it again $\{(\alpha_n, \beta_n)\}$. Then there exists a subsequence of $\{y_n\}$ converging weakly to y in Y by the fact that $H^1(\Omega)$ is reflexive. We again call that subsequence $\{y_n\}$. Finally, by the Rellich-Kondrachov Theorem and the compactness of the trace operator given $m \in L^\infty(\Omega)$ and $\rho \in L^\infty(\partial\Omega)$, there exists a subsequence of $\{y_n\}$ converging strongly to y in $\tilde{Y}$, called again $\{y_n\}$. Hence, by the continuity of $r_{\alpha,\beta}$ established in Lemma 2.7, we have $r_{\alpha_n,\beta_n}(y_n) \to r_{\alpha,\beta}(y)$ in X. □

Finally, we observe the following property of $r_{\alpha,\beta}$.

Lemma 2.11. *If $u \in H^1(\Omega)$ is a critical point of $J_{\alpha,\beta}$, then $u = r_{\alpha,\beta}(y) + y$ for some $y \in Y$.*

Proof. Since u is a critical point of $J_{\alpha,\beta}$, $J'_{\alpha,\beta}(u) \cdot v = 0$ for all $v \in H^1(\Omega)$. Since $H^1(\Omega) = X_k \oplus Y$, we may write $u = x + y$ where $x \in X_k$ and $y \in Y$. We observe that $0 = J'_{\alpha,\beta}(u) \cdot x = J'_{\alpha,\beta}(x) \cdot x$, showing that x is a critical point of $J_{\alpha,\beta}$ on the set $y + X$. But $J_{\alpha,\beta}$ is strictly concave on $y + X$ and its unique maximizer is defined as $r_{\alpha,\beta}(y)$. So $x = r_{\alpha,\beta}(y)$ and hence $u = r_{\alpha,\beta}(y) + y$. □

3. Reducing the functional

Motivated by Lemma 2.11, we now define the restricted functional $\tilde{J}_{\alpha,\beta} : Y \to \mathbb{R}$ by $\tilde{J}_{\alpha,\beta}(y) := J_{\alpha,\beta}(r_{\alpha,\beta}(y) + y)$. We begin by establishing some properties of this new functional.

Lemma 3.1. *The functional $\tilde{J}_{\alpha,\beta} \in C^1(Y, \mathbb{R})$ and $\tilde{J}'_{\alpha,\beta}(y) = J'_{\alpha,\beta}(r_{\alpha,\beta}(y) + y)$ for all $y \in Y$.*

Proof. We will establish this claim by showing that

$$\tilde{J}_{\alpha,\beta}(y_2) - \tilde{J}_{\alpha,\beta}(y_1) = J'_{\alpha,\beta}(r_{\alpha,\beta}(y_1) + y_1) \cdot (y_2 - y_1) + o\left(\|y_2 - y_1\|_{(m,\rho)}\right).$$

In addition to showing that $\tilde{J}_{\alpha,\beta} \in C^1(Y,\mathbb{R})$, this will also establish that $\tilde{J}'_{\alpha,\beta}(y) = J'_{\alpha,\beta}(r_{\alpha,\beta}(y)+y)$. First, note that

$$\begin{aligned}
&\tilde{J}_{\alpha,\beta}(y_2) - \tilde{J}_{\alpha,\beta}(y_1) \\
&= J_{\alpha,\beta}(r_{\alpha,\beta}(y_2)+y_2) - J_{\alpha,\beta}(r_{\alpha,\beta}(y_1)+y_1) \\
&\le J_{\alpha,\beta}(r_{\alpha,\beta}(y_2)+y_2) - J_{\alpha,\beta}(r_{\alpha,\beta}(y_2)+y_1) \\
&= J'_{\alpha,\beta}(r_{\alpha,\beta}(y_2)+y_1)\cdot(y_2-y_1) + o\left(\|y_2-y_1\|_{(m,\rho)}\right) \\
&= J'_{\alpha,\beta}(r_{\alpha,\beta}(y_1)+y_1)\cdot(y_2-y_1) + \Big(J'_{\alpha,\beta}(r_{\alpha,\beta}(y_2)+y_1) \\
&\quad - J'_{\alpha,\beta}(r_{\alpha,\beta}(y_1)+y_1)\Big)\cdot(y_2-y_1) + o\left(\|y_2-y_1\|_{(m,\rho)}\right)
\end{aligned} \tag{3.1}$$

by the maximizing property of $r_{\alpha,\beta}$, the Lipschitz continuity of $r_{\alpha,\beta}$, and the differentiability of $J_{\alpha,\beta}$. By the continuity of $r_{\alpha,\beta}$ and $J'_{\alpha,\beta}$, we note that

$$\left(J'_{\alpha,\beta}(r_{\alpha,\beta}(y_2)+y_1) - J'_{\alpha,\beta}(r_{\alpha,\beta}(y_1)+y_1)\right)\cdot(y_2-y_1) = o\left(\|y_2-y_1\|_{(m,\rho)}\right),$$

and hence (3.1) reduces to

$$\tilde{J}_{\alpha,\beta}(y_2) - \tilde{J}_{\alpha,\beta}(y_1) \le J'_{\alpha,\beta}(r_{\alpha,\beta}(y_1)+y_1)\cdot(y_2-y_1) + o\left(\|y_2-y_1\|_{(m,\rho)}\right).$$

A similar argument will show that

$$\tilde{J}_{\alpha,\beta}(y_2) - \tilde{J}_{\alpha,\beta}(y_1) \ge J'_{\alpha,\beta}(r_{\alpha,\beta}(y_1)+y_1)\cdot(y_2-y_1) + o\left(\|y_2-y_1\|_{(m,\rho)}\right),$$

and hence the claim is proven. □

Remark 3.2. If we knew $r_{\alpha,\beta}$ to be differentiable, this result would be a simple consequence of the chain rule. However, in general, this is not the case.

Given that we have now established that $\tilde{J}_{\alpha,\beta} \in C^1(Y,\mathbb{R})$, we may improve upon Lemma 2.11.

Lemma 3.3. *The element $y \in Y$ is a critical point of $\tilde{J}_{\alpha,\beta}$ if and only if $r_{\alpha,\beta}(y)+y$ is a critical point of $J_{\alpha,\beta}$.*

Proof. First, assume that $r_{\alpha,\beta}(y)+y$ is a critical point of $J_{\alpha,\beta}$. Then $J'_{\alpha,\beta}(r_{\alpha,\beta}(y)+y)\cdot v = 0$ for all $v \in H^1(\Omega)$ (and in particular, for all $v \in Y$). By Lemma 3.1, this implies that $\tilde{J}'_{\alpha,\beta}(y)\cdot v = 0$ for all $v \in Y$, and y is a critical point of $\tilde{J}_{\alpha,\beta}$.

Now, assume that y is a critical point of $\tilde{J}_{\alpha,\beta}$. As before, we then have that $J'_{\alpha,\beta}(r_{\alpha,\beta}(y)+y)\cdot v = 0$ for all $v \in Y$. However, since $r_{\alpha,\beta}(y)$ maximizes $J_{\alpha,\beta}(x+y)$ for all $x \in X_k$, we also have that $J'_{\alpha,\beta}(r_{\alpha,\beta}(y)+y)\cdot x = 0$ for all $x \in X_k$. Hence, since $H^1(\Omega) = X_k \oplus Y$, we have $J'_{\alpha,\beta}(r_{\alpha,\beta}(y)+y)\cdot w = 0$ for all $w \in H^1(\Omega)$. □

Now, we observe a homogeneity property of $\tilde{J}_{\alpha,\beta}$.

Lemma 3.4. *The functional $\tilde{J}_{\alpha,\beta}(ty) = t^2\tilde{J}_{\alpha,\beta}(y)$ for all $t \ge 0$ and $y \in Y$.*

The result follows immediately from the homogeneity of $J_{\alpha,\beta}$ and the homogeneity of $r_{\alpha,\beta}$ from Lemma 2.5. An important consequence of this lemma easily follows.

Lemma 3.5. *If $y \in Y$ is a critical point of $\tilde{J}_{\alpha,\beta}$, then $\tilde{J}_{\alpha,\beta}(y) = 0$.*

Proof. Differentiating the identity $\tilde{J}_{\alpha,\beta}(ty) = t^2\tilde{J}_{\alpha,\beta}(y)$ with respect to t, we find that $\tilde{J}'_{\alpha,\beta}(ty)\cdot y = 2t\tilde{J}_{\alpha,\beta}(y)$. Setting $t = 1$, the result immediately follows. □

As with $J_{\alpha,\beta}$, it will occasionally be helpful to think of $\tilde{J}_{\alpha,\beta}$ as a function on $\mathbb{R}^2 \times Y$, which we denote $\tilde{J}(\alpha,\beta,y)$.

Lemma 3.6. *For each fixed $y \neq 0$, the functional $\tilde{J}(\alpha,\beta,y) := \tilde{J}_{\alpha,\beta}(y)$ is strictly decreasing in α and β.*

Proof. Assume that $\alpha_1 \le \alpha_2$ and $\beta_1 \le \beta_2$, with at least one of these inequalities strict. Then

$$\begin{aligned} \tilde{J}(\alpha_2,\beta_2,y) &= J(\alpha_2,\beta_2,r(\alpha_2,\beta_2,y)+y) \\ &= \frac{1}{2}\big[\|r(\alpha_2,\beta_2,y)+y\|^2_{(c,\sigma)} - \alpha_2\|\big(r(\alpha_2,\beta_2,y)+y\big)^+\|^2_{(m,\rho)} \\ &\quad - \beta_2\|\big(r(\alpha_2,\beta_2,y)+y\big)^-\|^2_{(m,\rho)}\big]. \end{aligned} \tag{3.2}$$

Since $r(\alpha_2,\beta_2,y)+y$ is sign-changing for $y \neq 0$ by Lemma 2.6, it follows that

$$\|\,(r(\alpha_2,\beta_2,y)+y)^+\,\|_{(m,\rho)}\|\,(r(\alpha_2,\beta_2,y)+y)^-\,\|_{(m,\rho)} > 0,$$

and hence, since at least one of the inequalities $\alpha_1 \le \alpha_2$ and $\beta_1 \le \beta_2$ is strict, we have from (3.2) that

$$\begin{aligned} \tilde{J}(\alpha_2,\beta_2,y) &< \frac{1}{2}\big[\|r(\alpha_2,\beta_2,y)+y\|^2_{(c,\sigma)} - \alpha_1\|\,(r(\alpha_2,\beta_2,y)+y)^+\,\|^2_{(m,\rho)} \\ &\quad - \beta_1\|\,(r(\alpha_2,\beta_2,y)+y)^-\,\|^2_{(m,\rho)}\big] \\ &= J(\alpha_1,\beta_1,r(\alpha_2,\beta_2,y)+y). \end{aligned} \tag{3.3}$$

But recalling the maximizing property of $r_{\alpha,\beta}$ (see (2.5)), we must have that

$$J(\alpha_1,\beta_1,r(\alpha_2,\beta_2,y)+y) \le J(\alpha_1,\beta_1,r(\alpha_1,\beta_1,y)+y) = \tilde{J}(\alpha_1,\beta_1,y). \tag{3.4}$$

Combining (3.3) and (3.4) gives the desired result, that $\tilde{J}(\alpha_2,\beta_2,y) < \tilde{J}(\alpha_1,\beta_1,y)$ for each $y \neq 0$. □

Lemma 3.7. *Given any $K > 0$, there exists $C > 0$ such that*

$$\big|\tilde{J}(\alpha_2,\beta_2,x) - \tilde{J}(\alpha_1,\beta_1,x)\big| \le C\,(|\alpha_2-\alpha_1| + |\beta_2-\beta_1|)$$

on $R(K) := \{(\alpha_1,\alpha_2,\beta_1,\beta_2,y) \in \mathbb{R}^4 \times H^1(\Omega) : \max\{|\alpha_1|,|\alpha_2|,|\beta_1|,|\beta_2|,\|y\|_{(c,\sigma)}\} \le K\}$.

Proof. First, we establish that the functional J is uniformly Lipschitz in α,β, and x. Note that

$$\begin{aligned} |J(\alpha_2,\beta_2,x) - J(\alpha_1,\beta_1,x)| &= \frac{1}{2}\Big|(\alpha_2-\alpha_1)\|x^+\|^2_{(m,\rho)} + (\beta_2-\beta_1)\|x^-\|^2_{(m,\rho)}\Big| \\ &\le \frac{1}{2\mu_1}\|x\|^2_{(c,\sigma)}\,(|\alpha_2-\alpha_1| + |\beta_2-\beta_1|) \\ &\le \frac{1}{2\mu_1}K\,(|\alpha_2-\alpha_1| + |\beta_2-\beta_1|)\,. \end{aligned}$$

Hence J is uniformly Lipschitz in α,β on $R(K)$. Since $J \in C^1(H^1(\Omega);\mathbb{R})$, it is also uniformly Lipschitz in x on $R(K)$

Recall from Lemma 2.7 that $r_{\alpha,\beta}$ is locally Lipschitz in α,β. Therefore, we have that $r_{\alpha,\beta}$ is uniformly Lipschitz in α,β on $R(K)$. Therefore, $\tilde{J}(\alpha,\beta,x) = J(\alpha,\beta,r(\alpha,\beta,x)+x)$ is a composition of uniformly Lipschitz functions, and hence the claim follows. □

3.1. **Minimizing in Y.** By Lemma 3.3, we know that searching for critical points of $J_{\alpha,\beta}$ on H is equivalent to searching for critical points of $\tilde{J}_{\alpha,\beta}$ on Y. Further, since $\tilde{J}_{\alpha,\beta}$ is homogeneous, it is sufficient to search for critical points on the (m,ρ)-unit sphere in Y, namely $S_Y := \{y \in Y : \|y\|_{(m,\rho)} = 1\}$.

Since we assume $m, \rho \in L^\infty(\Omega)$, S_Y is weakly closed in $H^1(\Omega)$; that is, for any sequence $\{y_n\} \subset S_Y$ with $y_n \rightharpoonup y$ in $H^1(\Omega)$, we have $y_n \to y$ in Y and $y \in S_Y$. First we note several properties of $\tilde{J}_{\alpha,\beta}$ when restricted to S_Y.

Lemma 3.8. *$\tilde{J}_{\alpha,\beta}$ attains a global minimum on S_Y.*

Proof. First, note that

$$
\begin{aligned}
2\tilde{J}_{\alpha,\beta}(y) &= 2J_{\alpha,\beta}(r_{\alpha,\beta}(y) + y) \\
&= \|r_{\alpha,\beta}(y) + y\|^2_{(c,\sigma)} - \alpha\|(r_{\alpha,\beta}(y) + y)^+\|^2_{(m,\rho)} - \beta\|(r_{\alpha,\beta}(y) + y)^-\|^2_{(m,\rho)} \\
&> -\alpha\|r_{\alpha,\beta}(y) + y\|^2_{(m,\rho)} - \beta\|r_{\alpha,\beta}(y) + y\|^2_{(m,\rho)} \\
&> -2\beta\|r_{\alpha,\beta}(y) + y\|^2_{(m,\rho)}.
\end{aligned}
$$

Since $r_{\alpha,\beta}(y) \in X$ and $y \in Y$, $\langle r_{\alpha,\beta}(y), y\rangle_{(m,\rho)} = 0$ and hence

$$
\begin{aligned}
2\tilde{J}_{\alpha,\beta}(y) &> -2\beta\|r_{\alpha,\beta}(y) + y\|^2_{(m,\rho)} \\
&= -2\beta(\|r(y)\|^2_{(m,\rho)} + \|y\|^2_{(m,\rho)}) \\
&> -2\beta(\frac{1}{\mu_1}\|r(y)\|^2_{(c,\sigma)} + \|y\|^2_{(m,\rho)}) \\
&> -2\beta(\frac{C^2}{\mu_1}\|y\|^2_{(m,\rho)} + \|y\|^2_{(m,\rho)}) \\
&\geq -k(\|y\|^2_{(m,\rho)})
\end{aligned}
$$

where $k = 2\beta(\frac{C^2}{\mu_1} + 1)$ by Corollary 2.2 (a) in [7] and Lemma 2.9.

Now, take $M = \inf_{S_Y} \tilde{J}_{\alpha,\beta}(y) > -\infty$ (since $\|y\|_{(m,\rho)} = 1$ on S_Y) and choose $\{y_n\} \subset S_Y$ to be a minimizing sequence, that is, $\tilde{J}_{\alpha,\beta}(y_n) \to M$. So $\tilde{J}_{\alpha,\beta}(y_n)$ is bounded. We wish to show that $\|y_n\|_{(c,\sigma)}$ is also bounded. Note first that since $\tilde{J}_{\alpha,\beta}(y_n)$ is bounded and

$$
\begin{aligned}
2\tilde{J}_{\alpha,\beta}(y_n) &= 2J_{\alpha,\beta}(r_{\alpha,\beta}(y_n) + y_n) \\
&= \|r_{\alpha,\beta}(y_n) + y_n\|^2_{(c,\sigma)} - \alpha\|(r_{\alpha,\beta}(y_n) + y_n)^+\|^2_{(m,\rho)} \\
&\quad - \beta\|(r_{\alpha,\beta}(y_n) + y_n)^-\|^2_{(m,\rho)} \\
&= \|r_{\alpha,\beta}(y_n)\|^2_{(c,\sigma)} + \|y_n\|^2_{(c,\sigma)} - \alpha\|(r_{\alpha,\beta}(y_n) + y_n)^+\|^2_{(m,\rho)} \\
&\quad - \beta\|(r_{\alpha,\beta}(y_n) + y_n)^-\|^2_{(m,\rho)}.
\end{aligned} \tag{3.5}
$$

We wish to show that all terms other than $\|y_n\|^2_{(c,\sigma)}$ in (3.5) are bounded, and hence $\|y_n\|_{(c,\sigma)}$ must also be bounded.

We recall that $\|r_{\alpha,\beta}(y_n)\|^2_{(c,\sigma)} < C^2\|y_n\|^2_{(m,\rho)} = C^2$ by Lemma 2.9 and by the fact that $\{y_n\} \subset S_Y$. We also note that

$$
\begin{aligned}
\|(r_{\alpha,\beta}(y_n) + y_n)^+\|^2_{(m,\rho)} &\leq \|r_{\alpha,\beta}(y_n) + y_n\|^2_{(m,\rho)} \\
&\leq \|r_{\alpha,\beta}(y_n)\|^2_{(m,\rho)} + \|y_n\|^2_{(m,\rho)}
\end{aligned}
$$

$$\begin{aligned}
&\le \frac{1}{\mu_1}\|r_{\alpha,\beta}(y_n)\|^2_{(c,\sigma)} + \|y_n\|^2_{(m,\rho)} \\
&\le \frac{C^2}{\mu_1}\|y_n\|^2_{(m,\rho)} + \|y_n\|^2_{(m,\rho)} \\
&= \frac{C^2}{\mu_1} + 1,
\end{aligned}$$

by [7, Corollary 2.2(a)], Lemma 2.9, and the fact that $\{y_n\} \subset S_Y$. An identical argument will show that $\|(r_{\alpha,\beta}(y_n) + y_n)^-\|^2_{(m,\rho)} \le \frac{C^2}{\mu_1} + 1$. Hence, we have shown via equation (3.5) that $\|y_n\|_{(c,\sigma)}$ is also bounded.

Hence, by Lemma 2.10, we may choose a subsequence, call it again $\{y_n\}$, with $y_n \xrightarrow{(c,\sigma)} y_0$, $y_n \xrightarrow{(m,\rho)} y_0$ with $\|y_0\|_{(m,\rho)} = 1$, and $r_{\alpha,\beta}(y_n) \xrightarrow{(c,\sigma)} r_{\alpha,\beta}(y_0)$. So taking the limit inferior of both sides of (3.5) as $n \to \infty$, we see that

$$\begin{aligned}
2M &= \liminf_{n\to\infty} 2\tilde{J}_{\alpha,\beta}(y_n) \\
&= \|r_{\alpha,\beta}(y_0)\|^2_{(c,\sigma)} + \liminf_{n\to\infty} \|y_n\|^2_{(c,\sigma)} \\
&\quad - \alpha\|(r_{\alpha,\beta}(y_0) + y_0)^+\|^2_{(m,\rho)} - \beta\|(r_{\alpha,\beta}(y_0) + y_0)^-\|^2_{(m,\rho)} \\
&\ge \|r_{\alpha,\beta}(y_0)\|^2_{(c,\sigma)} + \|y_0\|^2_{(c,\sigma)} - \alpha\|(r_{\alpha,\beta}(y_0) + y_0)^+\|^2_{(m,\rho)} \\
&\quad - \beta\|(r_{\alpha,\beta}(y_0) + y_0)^-\|^2_{(m,\rho)} \\
&= 2\tilde{J}_{\alpha,\beta}(y_0),
\end{aligned}$$

by the weak lower semicontinuity of the (c,σ) norm. But then $M \ge \tilde{J}_{\alpha,\beta}(y_0)$ with $y_0 \in S_Y$ and hence we must have $\tilde{J}_{\alpha,\beta}(y_0) = M$ as desired. ☐

Lemma 3.9. *y_0 is a nontrivial critical point of $\tilde{J}_{\alpha,\beta}$ if and only if $\frac{y_0}{\|y_0\|_{(m,\rho)}}$ is a critical point of $\tilde{J}_{\alpha,\beta}$ restricted to S_Y and $\tilde{J}_{\alpha,\beta}(y_0) = 0$.*

Proof. If y_0 is a nontrivial critical point of $\tilde{J}_{\alpha,\beta}$, then by Lemma 3.5, $\tilde{J}_{\alpha,\beta}(y_0) = 0$. Furthermore, since y_0 is a critical point of $\tilde{J}_{\alpha,\beta}$, we may differentiate both sides of the equation in Lemma 3.4 with respect to y and set $t = 1/\|y_0\|_{(m,\rho)}$ to see that

$$0 = \frac{1}{\|y_0\|_{(m,\rho)}}\tilde{J}'_{\alpha,\beta}(y_0)\cdot y = \tilde{J}'_{\alpha,\beta}\Big(\frac{y_0}{\|y_0\|_{(m,\rho)}}\Big)\cdot y$$

holds for all $y \in Y$. So in particular, it holds for $y \in S_Y$ and the forward direction is established.

Now, let $y_0/\|y_0\|_{(m,\rho)}$ be a critical point of $\tilde{J}_{\alpha,\beta}$ restricted to S_Y and let $\tilde{J}_{\alpha,\beta}(y_0) = 0$. Then as in the previous case, we have

$$0 = \tilde{J}'_{\alpha,\beta}\Big(\frac{y_0}{\|y_0\|_{(m,\rho)}}\Big)\cdot y = \frac{1}{\|y_0\|_{(m,\rho)}}\tilde{J}'_{\alpha,\beta}(y_0)\cdot y$$

for all $y \in S_Y$. But note that, for any $\hat{y} \in Y$, we may write $\hat{y} = ty$ for some $y \in S_Y$. So,

$$\tilde{J}'_{\alpha,\beta}(y_0)\cdot\hat{y} = \tilde{J}'_{\alpha,\beta}(y_0)\cdot(ty) = t\tilde{J}'_{\alpha,\beta}(y_0)\cdot y = 0.$$

So y_0 is a critical point of $\tilde{J}_{\alpha,\beta}$ as desired. ☐

Lemma 3.10. *A function $u \in H^1(\Omega)$ is a nontrivial critical point of $J_{\alpha,\beta}$ if and only if $u = r_{\alpha,\beta}(y_0) + y_0$ where $y_0/\|y_0\|_{(m,\rho)}$ is a critical point of $\tilde{J}_{\alpha,\beta}$ restricted to S_Y and $\tilde{J}_{\alpha,\beta}(y_0) = 0$.*

The above lemma follows from combining Lemma 3.3 and Lemma 3.9. We now define $M(\alpha,\beta) = \min_{y\in S_Y} J_{\alpha,\beta}(y)$.

Lemma 3.11. *The function $M(\alpha,\beta)$ is Lipschitz continuous and is strictly decreasing as a function of both α and β. Moreover, $M(\alpha,\alpha) > 0$.*

Proof. Let (α_1,β_1) and (α_2,β_2) be points in the plane, and let y_1 and y_2 be the corresponding minimizers on S_Y (i.e. $M(\alpha_k,\beta_k) = \tilde{J}_{\alpha_k,\beta_k}(y_k)$ for $k = 1,2$). Let $u_{ij} = r_{\alpha_i,\beta_i}(y_j) + y_j$ for $i,j = 1,2$. Then

$$
\begin{aligned}
&M(\alpha_i,\beta_i)\\
&= \tilde{J}_{\alpha_i,\beta_i}(y_i)\\
&\le \tilde{J}_{\alpha_i,\beta_i}(y_j)\\
&= J_{\alpha_i,\beta_i}(r_{\alpha_i,\beta_i}(y_j)+y_j)\\
&= J_{\alpha_j,\beta_j}(r_{\alpha_i,\beta_i}(y_j)+y_j) + \frac{1}{2}(\alpha_j-\alpha_i)\|u_{ij}^+\|^2_{(m,\rho)} + \frac{1}{2}(\beta_j-\beta_i)\|u_{ij}^-\|^2_{(m,\rho)}\\
&\le J_{\alpha_j,\beta_j}(r_{\alpha_j,\beta_j}(y_j)+y_j) + \frac{1}{2}(\alpha_j-\alpha_i)\|u_{ij}^+\|^2_{(m,\rho)} + \frac{1}{2}(\beta_j-\beta_i)\|u_{ij}^-\|^2_{(m,\rho)}\\
&= M(\alpha_j,\beta_j) + \frac{1}{2}(\alpha_j-\alpha_i)\|u_{ij}^+\|^2_{(m,\rho)} + \frac{1}{2}(\beta_j-\beta_i)\|u_{ij}^-\|^2_{(m,\rho)}
\end{aligned}
\tag{3.6}
$$

by the minimizing property of y_i and the maximizing property of r_{α_j,β_j}. This inequality holds in the case $i = 1$ and $j = 2$, as well as the case $i = 2$ and $j = 1$. Hence

$$|M(\alpha_2,\beta_2) - M(\alpha_1,\beta_1)| \le c(|\alpha_2-\alpha_1| + |\beta_2-\beta_1|)$$

where $c = \frac{1}{2}\max\{\|u_{12}\|^2_{(m,\rho)}, \|u_{21}\|^2_{(m,\rho)}\}$. Note that if $\alpha_2 \ge \alpha_1$ and $\beta_2 \ge \beta_1$ with at least one of the inequalities strict, then $M(\alpha_2,\beta_2) < M(\alpha_1,\beta_1)$ by taking $i = 2$ and $j = 1$ in (3.6). This follows from the fact that u_{ij} must be sign-changing by Lemma 2.6.

In the case $\alpha = \beta$, for every $w \in H^1(\Omega)$, we may write $w = u + v$ where $u \in H^1_{(m,\rho)}$ and $v \in V_{(m,\rho)}$ have

$$
\begin{aligned}
2J_{\alpha,\beta}(w) &= 2J_{\alpha,\alpha}(w)\\
&= \|u+v\|^2_{(c,\sigma)} - \alpha\|u+v\|^2_{(m,\rho)}\\
&= \|u\|^2_{(c,\sigma)} + \|v\|^2_{(c,\sigma)} - \alpha\|u\|^2_{(m,\rho)} + \|v\|^2_{(m,\rho)}\\
&= \|u\|^2_{(c,\sigma)} - \alpha\|u\|^2_{(m,\rho)} + \|v\|^2_{(c,\sigma)}\\
&= \sum_{i=1}^{\infty}(\mu_i-\alpha)|c_i|^2 + \|v\|^2_{(c,\sigma)}
\end{aligned}
$$

by [7, Theorem 2.1(iii)] where

$$c_i = \frac{1}{\mu_i}\langle u,\phi_i\rangle_{(c,\sigma)} = \langle u,\phi_i\rangle_{(m,\rho)}.$$

Since $\mu_k < \alpha < \mu_{k+1}$, we note that the coefficients $(\mu_i - \alpha)$ are negative for $i \le k$, positive for $i \ge k+1$, and are increasing in i. Writing $u = x + y$ where $x \in X_k$ and $y \in Y_k$, we note that, if we maximize in the X_k direction, the maximum occurs when $c_i = 0$ for all $i \le k$ since $(\mu_i - \alpha)$ are negative for $i \le k$. In other words,

$r_{\alpha,\alpha}(y) \equiv 0$. Therefore,

$$2\tilde{J}_{\alpha,\alpha}(y) = 2J_{\alpha,\alpha}(y) = \sum_{i=k+1}^{\infty} (\mu_i - \alpha)|c_i|^2 \text{ for all } y \in Y_k. \tag{3.7}$$

We now wish to show that $M(\alpha,\alpha) = \inf_{y\in S_Y} \tilde{J}_{\alpha,\alpha}(y) > 0$. Taking

$$f(c_{k+1}, c_{k+2}, \dots) = \sum_{i=k+1}^{\infty} (\mu_i - \alpha)|c_i|^2, \quad g(c_{k+1}, c_{k+2}, \dots) = \sum_{i=k+1}^{\infty} |c_i|^2,$$

we apply the method of Lagrange multipliers to find the critical points of f subject to the constraint $g(c_{k+1}, c_{k+2}, \dots) = 1$. Setting $\nabla f = \lambda \nabla g$, we obtain $2(\mu_i - \alpha)c_i = 2\lambda c_i$ for $i \geq k+1$. Hence, critical points occur when $c_j = \pm 1$ for some $j \geq k+1$ and $c_i = 0$ for all $i \neq j$ (corresponding to the Lagrange multiplier $\lambda = \mu_j - \alpha$). Since the coefficients $(\mu_i - \alpha)$ are positive for $i \geq k+1$ and increasing, the minimizing choice occurs when $c_{k+1} = \pm 1$ and $c_i = 0$ for all $i > k+1$. Hence, the minimizer is $y = \pm\phi_{k+1}$ and $M(\alpha,\alpha) = \tilde{J}_{\alpha,\alpha}(\phi_{k+1}) + \|v\|^2_{(c,\sigma)} = \frac{1}{2}(\mu_{k+1} - \alpha) + \|v\|^2_{(c,\sigma)} > 0$. □

Not only $M(\alpha,\alpha) > 0$, we can also make an additional estimate which will later help in establishing bounds for the Fučik spectrum.

Lemma 3.12. *$M(\alpha, \mu_{k+1}) > 0$.*

Proof. Let $y \in S_Y$ and let $y = z + v$ where $z \in Y_k$ and $v \in V_{(m,\rho)}$. Then, by the maximizing property

$$\begin{aligned}
\tilde{J}_{\alpha,\mu_{k+1}} &= J_{\alpha,\mu_{k+1}}(r_{\alpha,\mu_{k+1}}(y) + y) \\
&\geq J_{\alpha,\mu_{k+1}}(y) \\
&= \frac{1}{2}\Big(\|y\|^2_{(c,\sigma)} - \alpha\|y^+\|^2_{(m,\rho)} - \mu_{k+1}\|y^-\|^2_{(m,\rho)}\Big) \\
&\geq \frac{1}{2}\Big(\|y\|^2_{(c,\sigma)} - \mu_{k+1}\|y\|^2_{(m,\rho)}\Big) \\
&= \frac{1}{2}\Big(\|z\|^2_{(c,\sigma)} - \mu_{k+1}\|z\|^2_{(m,\rho)} + \|v\|^2_{(c,\sigma)}\Big) \\
&= \frac{1}{2}\sum_{i=k+1}^{\infty} (\mu_i - \mu_{k+1})|c_i|^2 + \|v\|^2_{(c,\sigma)} \geq 0.
\end{aligned}$$

We note that, of the last two inequalities above, at least one must be strict. If the last inequality is in fact an equality, then $c_{k+1} = 1$ and $c_i = 0$ for all $i > k+1$. But this would imply that $y = \pm\phi_{k+1}$, in which case y^+ is nontrivial, and the previous inequality was strict. So $M(\alpha, \mu_{k+1}) > 0$. □

3.2. **Variational characterization of the Fučik spectrum.** All of the previous lemmas lead to the following theorem.

Theorem 3.13. *Let $\mu_k < \alpha < \mu_{k+1}$. Then one of the following is true:*

1. *$M(\alpha,\beta) > 0$ for all $\beta \geq \alpha$, which implies that $(\alpha,\beta) \notin \Sigma$.*
2. *There is a unique $\beta(\alpha) > \mu_{k+1}$ such that $M(\alpha,\beta(\alpha)) = 0$, which implies that $(\alpha,\beta(\alpha)) \in \Sigma$ but $(\alpha,\beta) \notin \Sigma$ for all $\alpha \leq \beta < \beta(\alpha)$.*

Lemma 3.14. *The curve $(\alpha,\beta(\alpha))$ is Lipschitz continuous, strictly decreasing, and contains the point (μ_{k+1}, μ_{k+1}).*

Proof. Consider two points $(\alpha_1,\beta_1),(\alpha_2,\beta_2)\in\Sigma$ with $\alpha_2>\alpha_1$. Let $y_i\in S_Y$ be a minimizer of $\tilde J_{\alpha_i,\beta_i}(y)$. Then $\tilde J_{\alpha_i,\beta_i}(y_i)=0$ and $\tilde J_{\alpha_i,\beta_i}(y)\ge 0$ for all $y\in S_Y$, and therefore by the homogeneity of $\tilde J_{\alpha_i,\beta_i}$, $\tilde J_{\alpha_i,\beta_i}(y)\ge 0$ for all $y\in Y$. Let $u_i=r_{\alpha_i,\beta_i}(y_i)+y_i$. Since $(\alpha_1,\beta_1)\in\Sigma$, we have

$$\begin{aligned}
0&=M(\alpha_1,\beta_1)\\
&=2\tilde J_{\alpha_1,\beta_1}(y_1)\\
&=2J_{\alpha_1,\beta_1}(u_1)\\
&=\|u_1\|^2_{(c,\sigma)}-\alpha_1\|u_1^+\|^2_{(m,\rho)}-\beta_1\|u_1^-\|^2_{(m,\rho)}\\
&>\|u_1\|^2_{(c,\sigma)}-\alpha_2\|u_1^+\|^2_{(m,\rho)}-\beta_1\|u_1^-\|^2_{(m,\rho)}\\
&=2J_{\alpha_2,\beta_1}\big(r_{\alpha_1,\beta_1}(y_1)\big)\\
&\le M(\alpha_2,\beta_1)
\end{aligned}$$

where we note that the last inequality is strict since $\alpha_2>\alpha_1$ and u_1^- is nontrivial by Lemma 2.6. Since $M(\alpha,\beta)$ is strictly decreasing in β by Lemma 3.11 and $M(\alpha_2,\beta_2)=0$, we must have $\beta_2<\beta_1$, which shows that $\beta(\alpha)$ is strictly decreasing as desired.

Now, we consider

$$M(\alpha_2,\beta_1)\le J_{\alpha_2,\beta_1}(u_2)=J_{\alpha_2,\beta_1}(u_2)-J_{\alpha_2,\beta_2}(u_2)=\frac{1}{2}(\beta_2-\beta_1)\|u_2^-\|^2_{(m,\rho)},$$

since $J_{\alpha_2,\beta_2}(u_2)=0$. Hence $M(\alpha_2,\beta_1)\le\frac12(\beta_2-\beta_1)\|u_2^-\|^2_{(m,\rho)}<0$ since $\beta_1>\beta_2$. Thus, we may rearrange the inequality to observe that

$$\begin{aligned}
|\beta_2-\beta_1|&=\beta_1-\beta_2\\
&\le 2\frac{1}{\|u_2^-\|^2_{(m,\rho)}}(-M(\alpha_2,\beta_1))\\
&=2\frac{1}{\|u_2^-\|^2_{(m,\rho)}}|M(\alpha_2,\beta_1)|\\
&=2\frac{1}{\|u_2^-\|^2_{(m,\rho)}}|M(\alpha_2,\beta_1)-M(\alpha_1,\beta_1)|\\
&\le 2c\frac{1}{\|u_2^-\|^2_{(m,\rho)}}|\alpha_2-\alpha_1|,
\end{aligned}$$

by the fact that $M(\alpha_1,\beta_1)=0$ and the Lipschitz estimate for $M(\alpha,\beta)$ from Lemma 3.11. Hence, $\beta(\alpha)$ is Lipschitz continuous as desired. □

4. Nonresonance problem

We are interested in the existence of weak solutions of (1.2) where $(\alpha,\beta)\in\mathbb{R}^2$ is such that $\mu_k<\alpha<\mu_{k+1}$ and $\alpha\le\beta<\beta(\alpha)$. Since we consider a fixed k in this section, we set $X=X_k$ for notational convenience. By the characterization of the Fučik Spectrum, we know that $(\alpha,\beta)\notin\Sigma$. All properties of f,g,m,ρ,c, and σ are as outlined in Section 1.

Consider the functional associated with (1.2) defined by $I_{\alpha,\beta}:=I:H^1(\Omega)\to\mathbb{R}$; with

$$I(u)=J_{\alpha,\beta}(u)-\Big[\int F(x,u)+\oint G(x,u)\Big], \tag{4.1}$$

where $\int$ denotes the (volume) integral on Ω, $\oint$ denotes the (surface) integral on $\partial\Omega$, and $F(x,u) = \int_0^u m(x)\tilde{f}(\xi)d\xi$, $G(x,u) = \int_0^u \rho(x)\tilde{g}(\xi)d\xi$, and $J_{\alpha,\beta}(u)$ is defined in (2.2). Then

$$I'(u)\cdot v = J'_{\alpha,\beta}(u)v - \Big[\int f(x,u)v + \oint g(x,u)v\Big] \quad \text{for all } v \in H^1(\Omega).$$

So, a critical point of I is a weak solution of (1.2).

Theorem 4.1. *Assume that $\mu_k < \alpha < \mu_{k+1}$, $\alpha \le \beta < \beta(\alpha)$, the nonlinearities f and g are bounded continuous functions, and $m \in L^\infty(\Omega)$ and $\rho \in L^\infty(\partial\Omega)$, then problem* (1.2) *has at least one weak solution.*

We will use a variational argument to prove Theorem 4.1. To do so, we first prove some lemmas which will be needed in the sequel. The first lemma shows the last two terms in I have at most linear growth.

Lemma 4.2. *There is a positive constant κ such that*

$$\Big|\int F(x,u) + \oint G(x,u)\Big| \le \kappa\|u\|_{(m,\rho)} \quad \textit{for all } u \in H^1(\Omega), \tag{4.2}$$

where κ is independent of u.

Proof. Since $\tilde{f}$ and $\tilde{g}$ are bounded then there exist constant C_1 and C_2 such that $|\tilde{f}(u)| \le C_1$ and $|\tilde{g}(u)| \le C_2$. Therefore, $|F(x,u)| \le C_1 m(x)|u|$ and $|G(x,u)| \le C_2\rho(x)|u|$. Using these estimates, Hölder inequality, and the fact that m is bounded, we obtain that

$$\Big|\int F(x,u)\Big| \le \int C_1 m(x)|u| \le \tilde{C}_1\Big(\int (m(x)|u|)^2\Big)^{1/2} \le \kappa_1\int m(x)u^2 \le \kappa_1\|u\|_{(m,\rho)},$$

where $\kappa_1 = \tilde{C}_1\|m\|_\infty$. Similarly, $\big|\oint G(x,u)\big| \le \kappa_2\|u\|_{(m,\rho)}$. Thus,

$$\Big|\int F(x,u) + \oint G(x,u)\Big| \le \kappa\|u\|_{(m,\rho)} \quad \text{for all } u \in H^1(\Omega). \qquad \square$$

The next lemma shows the geometry of I.

Lemma 4.3. *The functional I is such that*

(1) *$I(u) \to -\infty$ as $\|u\|_{(c,\sigma)} \to \infty$, for $u \in X$; that is, I is anti-coercive on X.*
(2) *I is bounded below when restricted to $\mathcal{Y}$, where $\mathcal{Y} := \{r_{\alpha,\beta}(y) + y : y \in Y\}$.*

Proof. We shall first prove that I is anti-coercive when restricted to X. Using the fact that $\|x\|^2_{(c,\sigma)} \le \mu_k\|x\|^2_{(m,\rho)}$ for all $x \in X$, and $\alpha \le \beta < \beta(\alpha)$, it follows that

$$\begin{aligned}
J_{\alpha,\beta}(x) &= \frac{1}{2}[\|x\|^2_{(c,\sigma)} - \alpha\|x^+\|^2_{(m,\rho)} - \beta\|x^-\|^2_{(m,\rho)}] \\
&\le \frac{1}{2}[\|x\|^2_{(c,\sigma)} - \alpha\|x\|^2_{(m,\rho)}] \\
&\le \frac{1}{2}[\|x\|^2_{(c,\sigma)} - \frac{\alpha}{\mu_k}\|x\|^2_{(c,\sigma)}] \\
&= \frac{1}{2}(1 - \frac{\alpha}{\mu_k})\|x\|^2_{(c,\sigma)}
\end{aligned}$$

Then using Lemma 4.2, we obtain

$$I(x) \le -\eta\|x\|^2_{(c,\sigma)} + \kappa\|x\|_{(m,\rho)} + C,$$

where $\eta = \frac{1}{2}(\frac{\alpha}{\lambda_k} - 1) > 0$. Since $\mu_1\|x\|_{(m,\rho)} \le \|x\|_{(c,\sigma)}$ (see [7, Corrolary 2.18]), then $I(x) \le -\eta\|x\|^2_{(c,\sigma)} + \frac{\kappa}{\mu_1}\|x\|_{(c,\sigma)} + C$. So, $I(u) \to -\infty$ as $\|u\|_{(c,\sigma)} \to \infty$ for $u \in X$. Thus I is anti-coercive on X.

Now, we shall prove that I is bounded below when restricted to $\mathcal{Y}$. By the assumption $\beta < \beta(\alpha)$ and Theorem 3.13 , it follows that $\min_{y \in S_Y} \tilde{J}_{\alpha,\beta}(y) = M(\alpha,\beta)$ and $M(\alpha,\beta) > 0$. Then for $y \ne 0$ and $y \in Y$, we have that

$$J_{\alpha,\beta}(r_{\alpha,\beta}(y) + y) = \tilde{J}_{\alpha,\beta}(y) = \|y\|^2_{(m,\rho)}\tilde{J}_{\alpha,\beta}\big(\frac{y}{\|y\|_{(m,\rho)}}\big) \ge \epsilon\|y\|^2_{(m,\rho)}$$

where $\epsilon = M(\alpha,\beta)$. Since $r_{\alpha,\beta}$ is Lipschitz continuous, as in Lemma 2.7, we have that $\|r_{\alpha,\beta}(y)\|_{(c,\sigma)} \le C\|y\|_{(m,\rho)}$ for some $C > 0$, and we see that

$$\begin{aligned} I(u) &\ge \epsilon\|y\|^2_{(m,\rho)} - \kappa\|u\|_{(m,\rho)} \\ &= \epsilon\|y\|^2_{(m,\rho)} - \kappa\|r_{\alpha,\beta}(y) + y\|_{(m,\rho)} \\ &\ge \epsilon\|y\|^2_{(m,\rho)} - \kappa\left(\|r_{\alpha,\beta}(y)\|_{(m,\rho)} + \|y\|_{(m,\rho)}\right) \\ &\ge \epsilon\|y\|^2_{(m,\rho)} - \kappa\,(C+1)\,\|y\|_{(m,\rho)} \end{aligned} \tag{4.3}$$

Thus, I is bounded below when restricted to $\mathcal{Y}$. □

As a consequence of the results above, there exists some $R > 0$ sufficiently large such that

$$\sup_{\{x \in X : \|x\|_{(c,\sigma)} = R\}} I(x) < \inf_{u \in \mathcal{Y}} I(u).$$

The next lemma shows the linking property of I. Let $B_R = \{x \in X : \|x\|_{(c,\sigma)} \le R\}$ and $\partial B_R = \{x \in X : \|x\|_{(c,\sigma)} = R\}$.

Lemma 4.4. *Let $\gamma : B_R \subset X \to H^1(\Omega)$ be a continuous function such that $\gamma|_{\partial B_R}(x) = x$. Then $\gamma(B_R) \cap \mathcal{Y} \ne \emptyset$.*

Proof. Let $x \in B_R$ and let write $\gamma(x) = \gamma_X(x) + \gamma_Y(x)$, where $\gamma_X(x) \in X$ and $\gamma_Y(x) \in Y$. One can see that for all $x \in \partial B_R$, $\gamma_X(x) = x$ and $\gamma_Y(x) = 0$. To show that $\gamma(B_R) \cap \mathcal{Y} \ne \emptyset$, it suffices to show that there is an $x \in B_R$ such that $\gamma_X(x) = r_{\alpha,\beta}(\gamma_Y(x))$.

Let $H : B_R \to X$ defined by $H(x) = \gamma_X(x) - r_{\alpha,\beta}(\gamma_Y(x))$. We shall show that there is $x \in B_R$ such that $H(x) = 0$. Notice that H is continuous and for all $x \in \partial B_R$, $H(x) = x \ne 0$. Therefore, the Brouwer degree $deg(H, B_R, 0)$ is well defined. Now, consider the homotopy $h(x,t) = tH(x) + (1-t)x$. Note that for $x \in \partial B_R$ we have $h(x,t) = tx + (1-t)x = x \ne 0$. Hence, $deg(H, B_R, 0) = deg(Id, B_R, 0) = 1$, where Id represents the identity map. Thus $H(x) = 0$ has a solution in B_R. □

To prove Theorem 4.1 using the saddle point theorem of Rabinowitz, it suffices first to show I satisfies the Palais-Smale condition (PS) which builds some compactness into the functional I.

Lemma 4.5. *I satisfies the Palais-Smale condition (PS).*

Proof. Let $\{u_n\}$ be a sequence in $H^1(\Omega)$ such that $\{I(u_n)\}$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$. We will show that $\{u_n\}$ has a convergent subsequence. In view of the assumptions on the nonlinearities f and g, it suffices to first show that the sequence $\{u_n\}$ is bounded with respect to $\|\cdot\|_{(m,\rho)}$, that is, there exists a constant K such that $\|u\|_{(m,\rho)} < K$. Suppose by contradiction that $\|u_n\|_{(m,\rho)} \to \infty$ as $n \to \infty$. Let $v_n = u_n/\|u_n\|_{(m,\rho)}$. Then

$$\frac{I(u_n)}{\|u_n\|^2_{(m,\rho)}} = J_{\alpha,\beta}(v_n) - \frac{1}{\|u_n\|^2_{(m,\rho)}}\Big[\int F(x,u_n) + \oint G(x,u_n)\Big].$$

Taking the limit, we have that $I(u_n)/\|u_n\|^2_{(m,\rho)} \to 0$ since $\{I(u_n)\}$ is bounded, and $\frac{1}{\|u_n\|^2_{(m,\rho)}}[\int F(x,u_n) + \oint G(x,u_n)] \to 0$ because of the estimate (4.2). Hence, $I(u_n)/\|u_n\|^2_{(m,\rho)}$ and $[\int F(x,u_n) + \oint G(x,u_n)]/\|u_n\|^2_{(m,\rho)}$ are bounded. Also note that $\|v_n^{\pm}\|_{(m,\rho)} \le 1$. From the definition of $J_{\alpha,\beta}$ it follows that $\|v_n\|_{(c,\sigma)}$ is bounded. Using the fact that $H^1(\Omega)$ is reflexive, the Sobolev compact embedding, and the continuity of the trace operator, we obtain that there exists a subsequence v_n that converges weakly to v_0 in $H^1(\Omega)$ and that converges strongly to v_0 in $L^2(\Omega)$ (also in $L^2(\partial\Omega)$). Since m and ρ are bounded functions and using the continuity of the norm $\|\cdot\|_{(m,\rho)}$, we obtain that $\|v_n\|_{(m,\rho)} \to \|v_0\|_{(m,\rho)}$. Thus, $\|v_0\|_{(m,\rho)} = 1$ since $\|v_n\|_{(m,\rho)} = 1$.

Now, for any $w \in H^1(\Omega)$,

$$\begin{aligned}\frac{I'(u_n)}{\|u_n\|_{(m,\rho)}}\cdot w &= \langle v_n, w\rangle_{(c,\sigma)} - \alpha\langle v_n^+, w\rangle_{(m,\rho)} + \beta\langle v_n^-, w\rangle_{(m,\rho)} \\ &\quad - \frac{1}{\|u_n\|_{(m,\rho)}}[\int m(x)\tilde f(u_n)w + \oint \rho(x)\tilde g(u_n)w]\end{aligned}$$

Using the boundedness of the nonlinearities f and g, and of the weights m and ρ, we have that

$$\frac{1}{\|u_n\|_{(m,\rho)}}[\int m(x)\tilde f(u_n)w + \oint \rho(x)\tilde g(u_n)w] \to 0.$$

Since $v_n \to v_0$ strongly in $L^2(\Omega)$ and $L^2(\partial\Omega)$, $\langle v_n^+, w\rangle_{(m,\rho)} \to \langle v_0^+, w\rangle_{(m,\rho)}$ and $\langle v_n^-, w\rangle_{(m,\rho)} \to \langle v_0^-, w\rangle_{(m,\rho)}$. By the weak convergence of v_n in $H^1(\Omega)$, we see that $\langle v_n, w\rangle_{(c,\sigma)} \to \langle v_0, w\rangle_{(c,\sigma)}$. We also note that $\frac{I'(u_n)}{\|u_n\|_{(m,\rho)}}\cdot w \to 0$ as $n \to \infty$. Hence,

$$0 = \langle v_0, w\rangle_{(c,\sigma)} - \alpha\langle v_0^+, \rangle_{(m,\rho)} - \beta\langle v_0^-, w\rangle_{(m,\rho)} \quad \text{for all } w \in H^1(\Omega).$$

Thus v_0 is a nontrivial weak solution of (1.1). This leads to a contradiction since $(\alpha,\beta) \notin \Sigma$. Thus, $\{u_n\}$ are bounded with respect to $\|\cdot\|_{(m,\rho)}$.

Let us analyze carefully the functional I.

$$I(u_n) = \frac{1}{2}[\|u_n\|^2_{(c,\sigma)} - \alpha\|u_n^+\|^2_{(m,\rho)} - \beta\|u_n^-\|^2_{(m,\rho)}] - [\int F(x,u_n) + \oint G(x,u_n)]$$

Since $I(u_n)$ is bounded and using the fact that $\{u_n\}$ is bounded with respect to $\|\cdot\|_{(m,\rho)}$ and the estimate (4.2), we have that $\|u_n^+\|_{(m,\rho)}$, $\|u_n^-\|_{(m,\rho)}$, and $[\int F(x,u_n) + \oint G(x,u_n)]$ are all bounded. Thus, $\|u_n\|_{(c,\sigma)}$ must be bounded. Therefore there exists a subsequence u_n that converges weakly to u in $H^1(\Omega)$ and converges strongly to u in $L^2(\Omega)$ (also in $L^2(\partial\Omega)$). Since m and ρ are bounded functions and using the continuity of the norm $\|\cdot\|_{(m,\rho)}$, we obtain that $\|u_n\|_{(m,\rho)} \to \|u\|_{(m,\rho)}$.

Now, consider

$$\begin{aligned}I'(u_n).(u_n - u) &= \langle u_n, (u_n - u)\rangle_{(c,\sigma)} - \alpha\langle u_n^+, (u_n - u)\rangle_{(m,\rho)} + \beta\langle u_n^-, (u_n - u)\rangle_{(m,\rho)} \\ &\quad - \Big[\int m(x)\tilde f(u_n)(u_n - u) + \oint \rho(x)\tilde g(u_n)(u_n - u)\Big]\end{aligned}$$

By the assumption $I'(u_n) \to 0$ in $\big(H^1(\Omega)\big)^*$ it follows that $I'(u_n).(u_n - u) \to 0$. Since $\|u_n^+\|_{(m,\rho)}$, $\|u_n^-\|_{(m,\rho)}$, $\tilde{f}$, and $\tilde{g}$ are all bounded, and $\|u_n\|_{(m,\rho)} \to \|u\|_{(m,\rho)}$, we have that

$$
\begin{aligned}
&-\alpha\langle u_n^+, (u_n - u)\rangle_{(m,\rho)} + \beta\langle u_n^-, (u_n - u)\rangle_{(m,\rho)} \\
&-\Big[\int m(x)f(u_n)(u_n - u) + \oint \rho(x)g(u_n)(u_n - u)\Big] \to 0.
\end{aligned}
$$

Therefore $\langle u_n, (u_n - u)\rangle_{(c,\sigma)} \to 0$. Thus $\|u_n\|^2_{(c,\sigma)} - \langle u_n, u\rangle_{(c,\sigma)} \to 0$.

Since u_n converges weakly to u in $H^1(\Omega)$, we have that $\langle u_n, u\rangle_{(c,\sigma)} \to \|u\|^2_{(c,\sigma)}$. Hence, $\|u_n\|_{(c,\sigma)} \to \|u\|_{(c,\sigma)}$. Thus, $u_n \xrightarrow{(c,\sigma)} u$ in $H^1(\Omega)$. □

Proof of Theorem 4.1. The functional I satisfies the Palais-Smale condition due to Lemma 4.5, and by Lemma 4.4, I satisfies the linking property. Set

$$
c = \inf_{\gamma\in\Gamma} \sup_{u\in B_R\cap X} I(\gamma(u)),
$$

where R is a sufficiently large constant, and $\Gamma = \{\gamma \in C\big(B_R \cap X; H^1(\Omega)\big) : \gamma|_{\partial B_R\cap X}(x) = x\}$. Then by the Saddle Point Theorem [9], it follows that c is a critical value of I. Thus problem (1.2) has a weak solution. □

5. Resonance Problem

In this section, we again assume $(\alpha, \beta) \in \mathbb{R}^2$ with $\mu_k < \alpha < \mu_{k+1}$. However we now assume that $\beta = \beta(\alpha)$ so that $(\alpha, \beta) \in \Sigma$ by the characterization in Theorem 3.13. Again for notational convenience we take $X = X_k$. Most arguments from the previous section still apply, with the exception of Lemmas 4.3 part 2 and 4.5; namely that I is bounded from below and that I satisfies (PS). This is not surprising, as the case that $(\alpha, \beta) \in \Sigma$ corresponds to the case $\mu = \mu_{k+1}$ in the Fredholm alternative. We expect in such cases that solutions only exist when a generalized orthogonality condition is met.

In establishing existence of solutions in the non-resonance cases, we will need a generalized Landesman-Lazer condition, namely

Definition 5.1. If for any sequence $\{u_n\} \subset H^1(\Omega)$ such that $\|u_n\|_{(m,\rho)} \to \infty$ and $\frac{u_n}{\|u_n\|_{(m,\rho)}} \xrightarrow{(m,\rho)} \psi$, where ψ is a Fučik eigenfunction associated with (α, β), we have

$$
\lim_{n\to\infty} \int_\Omega F(x, u_n) + \int_{\partial\Omega} G(x, u_n) = -\infty. \tag{5.1}
$$

Theorem 5.2. *Assume that $\mu_k < \alpha < \mu_{k+1}$, $\beta = \beta(\alpha)$, the nonlinearities f and g are bounded continuous functions, and $m \in L^\infty(\Omega)$ and $\rho \in L^\infty(\partial\Omega)$, then problem* (1.2) *has at least one weak solution provided that condition* (5.1) *holds.*

Lemma 5.3. *If* (5.1) *is satisfied, then I is bounded below on $\mathcal{Y}$.*

Proof. Suppose to the contrary that there exists a sequence $\{u_n\} \subset \mathcal{Y}$ with $I(u_n) \to -\infty$. Since $\{u_n\} \subset \mathcal{Y}$, we may write $u_n = r_{\alpha,\beta}(y_n) + y_n$. Taking inequality (4.3) with $\epsilon = M(\alpha, \beta) = 0$, we observe that since $I(u_n) \to -\infty$, we must have $\|u_n\|_{(m,\rho)} \to \infty$. But since

$$
\begin{aligned}
\|u_n\|^2_{(m,\rho)} &= \|r_{\alpha,\beta}(y_n) + y_n\|^2_{(m,\rho)} \\
&= \|r_{\alpha,\beta}(y_n)\|^2_{(m,\rho)} + \|y_n\|^2_{(m,\rho)}
\end{aligned}
$$

$$\begin{aligned}
&\le \frac{1}{\mu_1}\|r_{\alpha,\beta}(y_n)\|^2_{(c,\sigma)} + \|y_n\|^2_{(m,\rho)}\\
&\le \frac{C^2}{\mu_1}\|y_n\|^2_{(m,\rho)} + \|y_n\|^2_{(m,\rho)}\\
&= \big(\frac{C^2}{\mu_1}+1\big)\|y_n\|^2_{(m,\rho)},
\end{aligned}$$

we observe that $\|y_n\|_{(m,\rho)} \to \infty$. Thus, no subsequence of $\{u_n\}$ lies in a set of the form $\{u \in \mathcal{Y} : u = r_{\alpha,\beta}(y) + y,\ \tilde{J}_{\alpha,\beta}(y) \ge c\|y\|\}$ for some $c > 0$, since if such a subsequence existed, this would imply $I(u) \to \infty$ by (4.3). Therefore, $\tilde{J}_{\alpha,\beta}(y_n) \to 0$ and by the homogeneity of $\tilde{J}_{\alpha,\beta}$, $\tilde{J}_{\alpha,\beta}\Big(\frac{y_n}{\|y_n\|_{(m,\rho)}}\Big) \to 0$. Since $M(\alpha,\beta) = 0$, $\{y_n/\|y_n\|_{(m,\rho)}\} \subset S_Y$ is a minimizing sequence of $\tilde{J}_{\alpha,\beta}$. As in the proof of Lemma 3.8, this implies that $\|y_n/\|y_n\|_{(m,\rho)}\|_{(c,\sigma)}$ is bounded. Therefore, there exists $y \in S_Y$ such that $y_n/\|y_n\|_{(m,\rho)} \xrightarrow{(c,\sigma)} y$ and $y_n/\|y_n\|_{(m,\rho)} \xrightarrow{(m,\rho)} y$. Using the the homogeneity of $r_{\alpha,\beta}$, we have that

$$u_n = \|y_n\|_{(m,\rho)}\Big(r_{\alpha,\beta}\Big(\frac{y_n}{\|y_n\|_{(m,\rho)}}\Big) + \frac{y_n}{\|y_n\|_{(m,\rho)}}\Big),$$

and

$$\|u_n\|_{(m,\rho)} = \|y_n\|_{(m,\rho)}\|r_{\alpha,\beta}\Big(\frac{y_n}{\|y_n\|_{(m,\rho)}}\Big) + \frac{y_n}{\|y_n\|_{(m,\rho)}}\|_{(m,\rho)}.$$

Therefore,

$$\frac{u_n}{\|u_n\|_{(m,\rho)}} = \frac{r_{\alpha,\beta}\big(\frac{y_n}{\|y_n\|_{(m,\rho)}}\big) + \frac{y_n}{\|y_n\|_{(m,\rho)}}}{\|r_{\alpha,\beta}\big(\frac{y_n}{\|y_n\|_{(m,\rho)}}\big) + \frac{y_n}{\|y_n\|_{(m,\rho)}}\|_{(m,\rho)}}\,.$$

Therefore,

$$\begin{gathered}
\frac{u_n}{\|u_n\|_{(m,\rho)}} \xrightarrow{(c,\sigma)} \frac{r_{\alpha,\beta}(y)+y}{\|r_{\alpha,\beta}(y)+y\|_{(m,\rho)}},\\
\frac{u_n}{\|u_n\|_{(m,\rho)}} \xrightarrow{(m,\rho)} \frac{r_{\alpha,\beta}(y)+y}{\|r_{\alpha,\beta}(y)+y\|_{(m,\rho)}}.
\end{gathered}$$

Setting

$$\frac{r_{\alpha,\beta}(y)+y}{\|r_{\alpha,\beta}(y)+y\|_{(m,\rho)}} = \phi\,.$$

we notice that $\|\phi\|_{(m,\rho)} = 1$ and $\tilde{J}_{\alpha,\beta}(\phi) = 0 = M(\alpha,\beta(\alpha))$. Therefore ϕ is a nontrivial eigenfunction associated to $(\alpha,\beta(\alpha))$. Since $\frac{u_n}{\|u_n\|_{(m,\rho)}} \xrightarrow{(m,\rho)} \phi$ and (5.1) is satisfied, we have that $\lim_{n\to\infty}\big(\int F(x,u_n) + \oint G(x,u_n)\big) = -\infty$. It follows that $I(u_n) \to \infty$, a contradiction. The lemma is proved. □

Lemma 5.4. *If* (5.1) *is satisfied, then I satisfies (PS).*

Proof. The first part of the proof is identical to the proof in Lemma 4.5. Suppose $\{u_n\} \subset H^1(\Omega)$ is a sequence such that $I(u_n)$ is bounded, $I'(u_n) \to 0$, and $\|u_n\|_{(m,\rho)} \to \infty$. As before, we take $v_n = \frac{u_n}{\|u_n\|_{(m,\rho)}}$ and by an identical argument we show that $v_n \xrightarrow{(c,\sigma)} v$ and $v_n \xrightarrow{(m,\rho)} v$ with $\|v\|_{(m,\rho)} = 1$ and v a Fučik eigenfunction associated with (α,β). In the previous case this was a contradiction, but since $(\alpha,\beta) \in \Sigma$ in this case, we have not yet reached a contradiction, and further argument is needed.

Write $u_n = x_n + y_n = \tilde{x}_n + r_{\alpha,\beta}(y_n) + y_n$. Then

$$
\begin{aligned}
I'(u_n)\cdot\tilde{x}_n &= J'_{\alpha,\beta}(u_n)\cdot\tilde{x}_n - \int f(x,u_n)\tilde{x}_n + \oint g(x,u_n)\tilde{x}_n \\
&= J'_{\alpha,\beta}(\tilde{x}_n + r_{\alpha,\beta}(y_n) + y_n)\cdot\tilde{x}_n - \int f(x,u_n)\tilde{x}_n + \oint g(x,u_n)\tilde{x}_n \\
&= \big(J'_{\alpha,\beta}(\tilde{x}_n + r_{\alpha,\beta}(y_n) + y_n) - J'_{\alpha,\beta}(r_{\alpha,\beta}(y_n) + y_n)\big)\cdot\tilde{x}_n \\
&\quad - \int f(x,u_n)\tilde{x}_n + \oint g(x,u_n)\tilde{x}_n \\
&\le -\delta\|\tilde{x}_n\|^2_{(c,\sigma)} - \int f(x,u_n)\tilde{x}_n + \oint g(x,u_n)\tilde{x}_n \\
&\le -\delta\mu_1\|\tilde{x}_n\|^2_{(m,\rho)} - \int f(x,u_n)\tilde{x}_n + \oint g(x,u_n)\tilde{x}_n
\end{aligned}
$$

by the fact that $J'_{\alpha,\beta}(r_{\alpha,\beta}(y)+y)\cdot x = 0$ for all $x \in X$ and Lemma 2.3. Dividing the inequality through by $\|\tilde{x}_n\|_{(m,\rho)}$ gives

$$
I'(u_n)\cdot\frac{\tilde{x}_n}{\|\tilde{x}_n\|_{(m,\rho)}} \le -\delta\mu_1\|\tilde{x}_n\|_{(m,\rho)} - \int f(x,u_n)\tilde{x}_n + \oint g(x,u_n)\frac{\tilde{x}_n}{\|\tilde{x}_n\|_{(m,\rho)}},
$$

but since $I'(u_n) \to 0$ and f, g are bounded, we obtain that $\|\tilde{x}_n\|_{(m,\rho)}$ is also bounded. It now follows that

$$
J'_{\alpha,\beta}(u_n)\cdot\tilde{x}_n = I'(u_n)\cdot\tilde{x}_n + \int f(x,u_n)\tilde{x}_n + \oint g(x,u_n)\tilde{x}_n
$$

must also be bounded.

Now, let $h(t) = J_{\alpha,\beta}(r_{\alpha,\beta}(y_n) + y_n + t\tilde{x}_n)$. Then $h'(t) = J'_{\alpha,\beta}(r_{\alpha,\beta}(y_n) + y_n + t\tilde{x}_n)\cdot\tilde{x}$, and we observe that $h'(0) = 0$ (by the definition of $r_{\alpha,\beta}$) and $h'(t)$ is decreasing by the strict concavity of $J_{\alpha,\beta}$ on $y_n + X$. By the Mean Value Theorem, $h(1) - h(0) = h'(c)$ for some $c \in (0,1)$, and hence $h(1) - h(0) \ge h'(1)$ since h' is decreasing. So,

$$
J_{\alpha,\beta}(r_{\alpha,\beta}(y_n) + y_n + \tilde{x}_n) - J_{\alpha,\beta}(r_{\alpha,\beta}(y_n) + y_n) \ge J'_{\alpha,\beta}(r_{\alpha,\beta}(y_n) + y_n + \tilde{x})\cdot\tilde{x}_n.
$$

Since $J_{\alpha,\beta}(r_{\alpha,\beta}(y_n) + y_n) \ge \|y_n\|_{m,\rho}M(\alpha,\beta) = 0$, we then have that $J_{\alpha,\beta}(u_n) \ge J'_{\alpha,\beta}(u_n)\cdot\tilde{x}_n$. Therefore,

$$
\begin{aligned}
I(u_n) &= J_{\alpha,\beta}(u_n) - \Big[\int F(x,u_n) + \oint G(x,u_n)\Big] \\
&\ge J'_{\alpha,\beta}(u_n)\cdot\tilde{x}_n - \Big[\int F(x,u_n) + \oint G(x,u_n)\Big].
\end{aligned}
$$

However, $J'_{\alpha,\beta}(u_n)\cdot\tilde{x}_n$ is bounded and $[\int F(x,u_n) + \oint G(x,u_n)] \to -\infty$ by (5.1), which contradicts the boundedness of $I(u_n)$. Hence $\|u_n\|_{(m,\rho)}$ is bounded, and the proof proceeds as in Lemma 4.5. □

By a straightforward application of the Saddle Point Theorem, we now conclude that there exists a solution to the resonance problem.

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