Special Issue in honor of John W. Neuberger
Electronic Journal of Differential Equations, Special Issue 02 (2023), pp. 101–107.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu or https://ejde.math.unt.edu

REMARKS ON COMPACTNESS CONDITIONS AND THEIR APPLICATIONS

DAVID G. COSTA

In memory of Prof. John W. Neuberger and his legacy to mathematics

ABSTRACT. We review typical compactness conditions used in variational techniques and some of their properties, and the relationships between them. In particular, we provide some new insights into results related to the Palais-Smale and Cerami conditions, and their comparison.

1. INTRODUCTION

Let H be a Hilbert space with inner-product $\langle \cdot, \cdot \rangle$ and $J : H \to \mathbb{R}$ a C^1 functional defined on H. Researchers in variational techniques and their applications to differential equations (ODEs or PDEs) are familiar with the following compactness conditions, where (u_n) is a sequence in H:

- *Palais-Smale condition at level c*, $(PS)_c$: If (u_n) is such that $J(u_n) \to c$ and $J'(u_n) \to 0$, then (u_n) has a convergent subsequence (see [**?**]);
- *Cerami condition at level c*, $(Ce)_c$: If (u_n) is such that $J(u_n) \to c$ and $(1 + \|u_n\|)J'(u_n) \to 0$, then (u_n) has a convergent subsequence (see [**?**]);
- *Brézis-Coron-Nirenberg condition at level c*, $(BCN)_c$: If (u_n) is such that $J(u_n) \to c$ and $J'(u_n) \to 0$, then $c \in \mathbb{R}$ is a critical value of J (see [**?**]).

Researchers familiar with the above conditions know and it is also easy to show that

$$(PS)_c \Rightarrow (Ce)_c \Rightarrow (BCN)_c\,.$$

Indeed, $(PS)_c \Rightarrow (Ce)_c$ *as* $(1 + \|u_n\|)J'(u_n) \geq J'(u_n)$, *and either* $(PS)_c$ *or* $(Ce)_c$ *implies that* $c \in \mathbb{R}$ *is a critical value of* J, since the limit $\bar{u}$ of the convergence subsequence (still denoted (u_n)) satisfies $J(\bar{u}) = c$, $J'(\bar{u}) = 0$.

As a side remark, one should notice that $(BCN)_c$ simply says that c is a critical value of J, a result that might be applicable in situations where J is periodic with period (say) $p > 0$. Indeed, one could use $\widehat{u}_n$ with $J(\widehat{u}_n)$ belonging to the closed interval $[0, p]$ and note that $J(\widehat{u}_n) \to c$ and $J'(\widehat{u}_n) \to 0$.

2020 *Mathematics Subject Classification.* 35-04.
Key words and phrases. Compactness conditions; Palais-Smale condition; Cerami condition, Brézis-Coron-Nirenberg condition.

Published March 27, 2023.

2. Two results

Now we state and prove two new and simple results involving the Palais-Smale as well as the Cerami condition (inspired by results in [?, ?]). Let us start by recalling the notion of *Strong Resonant* problems, as was introduced by Benci-Bartolo-Fortunato [?] in 1983 for Dirichlet problems in bounded domains $\Omega \subset \mathbb{R}^N$, $N \geq 3$ (cf. also [?]). Such problems were also used in the context of unbounded domains (e.g. see [?, ?] and references therein).

In [?] the authors considered the "strong" resonant problem below in a bounded domain Ω, with λ_k denoting the k^{th} eigenvalue of $-\Delta$ under Dirichlet condition on $\partial\Omega$,

$$-\Delta u - \lambda_k u + g(u) = 0, \quad u = 0 \quad \text{on } H_0^1(\Omega)\,, \tag{2.1}$$

and assumed the conditions

(A1) $tg(t) \to 0$ as $|t| \to \infty$;
(A2) $G(t) := \int_{-\infty}^{t} g(s)\,ds$ well-defined and such that $G(t) \to 0$ as $t \to \infty$;
(A3) $G(t) \geq 0$ for all $t \in \mathbb{R}$

Then they proved the following three theorems:

Theorem 2.1. *If* (A1)–(A3) *hold, then problem* (**??**) *has at least one solution.*

Theorem 2.2. *If $g(0) = 0$, $g'(0) = \sup\{g'(t) : t \in \mathbb{R}\}$ and* (A1)–(A3) *hold, then problem* (**??**) *has at least one nontrivial solution.*

Theorem 2.3. *Assume* (A2) *and* (A3) *with g odd and $G(0) \geq 0$. Moreover, suppose that there exists an eigenvalue $\lambda_h \leq \lambda_k$ such that $g'(0) + \lambda_h - \lambda_k > 0$. Then problem* (**??**) *possesses at least*

$$m := \text{dimension}(M_h \oplus \cdots \oplus N_k)$$

distinct pairs of nontrivial solutions, where M_i denotes the eigenspace corresponding to λ_i.

As pointed out by the authors, the definition of "strong" resonant problem applies to the situation in Theorem **??** where the conditions (A1)–(A3) hold (with (A1) weakened to $g(t) \to 0$ as $|t| \to \infty$). In fact, as stated by the authors, condition (A1) is simply a technical condition in case g has a "good" behavior at ∞. In addition, in their approach, the authors show that the Cerami condition $(Ce)_c$ holds for all $c \in (0, \infty)$, by making use of "linking" results.

In our approach, we plan to show that the stronger $(PS)_c$ hods for all $c \in \mathbb{R}$ except for a finite set of values that can be found explicitly. In particular, given that the authors use linking arguments, another alternative one could have is to use the stronger Palais-Smale condition $(PS)_c$ by avoiding the exceptional finite set of values that we shall find in our approach. We may assume, without loss of generality, that the eigenvalues of $-\Delta$ under Dirichlet boundary condition are simple; see Remark **??**.

First result.

Theorem 2.4. *Consider the Dirichlet problem*

$$-\Delta u = \lambda_k u + g(u), \quad u = 0 \quad \textit{on } H_0^1(\Omega)\,, \tag{2.2}$$

and assume the conditions

(A4) *$g(t) \to 0$ as $|t| \to \infty$, with g continuous;*

(A5) *$G(t) := \int_0^t g(s)\,ds$ is such that* $\lim_{t\to\pm\infty} G(t) := G_\pm \in (-\infty, +\infty)$,

where λ_k is a given eigenvalue of $-\Delta$ under Dirichlet boundary condition. Then there exist a finite set $\Gamma_k \subset \mathbb{R}$ such that the functional

$$J(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda_k u^2)\,dx - \int_\Omega G(u)\,dx := Q(u) - \int_\Omega G(u)\,dx\,,$$

for $u \in H_0^1(\Omega)$, satisfies $(PS)_c$ if and only if $c \notin \Gamma_k$, where

$$\Gamma_k := \{-\operatorname{measure}([v > 0])G_+ - \operatorname{measure}([v < 0])G_- : v \in N_k,\ \|v\| = 1\}\,,$$

and $[v > 0]$ (resp. $[v < 0]$) denotes the set $\{x \mid v(x) > 0\}$ (resp. $\{x : v(x) < 0\}$).

Proof. Recall we are denoting $\|u\| = (\int_\Omega |\nabla u|^2\,dx)^{1/2}$ the usual norm in $H_0^1(\Omega)$, and $N_k = \mathbb{R}\phi_k$ is the one-dimensional eigenspace associated with λ_k, with $\|\phi_k\| = 1$. Let us also denote by $\mathcal{X}^+$, $\mathcal{X}^-$ the subspaces of $H_0^1(\Omega)$ where Q is positive definite, negative definite, respectively, and set $\mathcal{X}^0 = N_k$, so that

$$H_0^1(\Omega) = \mathcal{X}^+ \oplus \mathcal{X}^- \oplus \mathcal{X}^0\,.$$

Since g has subcritical growth by (A4), the functional J satisfies $(PS)_c$ if and only if any sequence (u_n) in $H_0^1(\Omega)$ satisfying

(i) $J(u_n) \to c$, and
(ii) $J'(u_n) \to 0$,

must have a bounded subsequence. So, let us assume that J satisfies (i), (ii), but

$$\|u_n\| \to \infty\,,$$

and prove that $c \in \Gamma_k$.

Claim: Assuming (A4) and (A5), the functional J satisfies $(PS)_c$ if and only if $c \notin \Gamma_k$, where we recall that

$$\Gamma_k := \{-\operatorname{measure}([v > 0])G_+ - \operatorname{measure}([v < 0])G_- : v \in N_k,\ \|v\| = 1\}\,,$$

and $[v > 0]$ (resp. $[v < 0]$) denotes the set $\{x \mid v(x) > 0\}$ (resp. $\{x \mid v(x) < 0\}$).

Proof. Since we are denoting by $\mathcal{X}^+$, $\mathcal{X}^-$ the subspaces of $\mathcal{X} := H_0^1(\Omega)$ where Q is positive definite, negative definite, respectively, and $\mathcal{X}^0 = N_k$, we shall write $u \in H_0^1$ as $u_n = u_n^+ + u_n^- + u_n^0$, where $u_n^+ \in \mathcal{X}^+$, $u_n^- \in \mathcal{X}^-$, $u_n^0 \in \mathcal{X}^0 = N_k$. And, since g has subcritical growth, the functional J satisfies $(PS)_c$ if and only if any sequence (u_n) in H_0^1 verifying

(i) $J(u_n) \to c$, and
(ii) $\|J'(u_n)\|_{H^{-1}} \to 0$,

must have a bounded subsequence. So, by negation, let us then assume that J satisfies (i), (ii), but

(iii) $\|u_n\| \to \infty$.

and show in this case that $c \in \Gamma_k$.

Indeed, (ii) implies that

$$|\langle \nabla J(u_n), u_n^+\rangle| = |\|u_n^+\|^2 - \lambda_k \|u_n^+\|_{L^2}^2 - \int_\Omega g(u_n)u_n^+\,dx| \le C\|u_n^+\|\,, \tag{2.3}$$

where $C = \sup_{n\in\mathbb{N}} \|J'(u_n)\|_{H^{-1}}$. Also, in view of Holder's and Sobolev's inequality, we have that $C_2\|u_n^+\|_{L^2} \le \|u_n^+\|$, where we may replace C_2 by a smaller $0 < C_0$ with

$$1 - C_0^2\lambda_k > 0\,. \tag{2.4}$$

On the other hand, note by (A4) that if $q' \leq 2N/(N-2)$, $N \geq 3$ (where $q' = q/(q-1)$ denotes the *conjugate exponent* of q), we can estimate the integral term in (**??**) as

$$\Big|\int_\Omega g(u_n)u_n^+\,dx\Big| \leq \|g(u_n)\|_{L^q}\|u_n^+\|_{L^{q'}} \leq C\|g(u_n)\|_{L^q}\|u_n^+\|\,. \tag{2.5}$$

Therefore, using Holder's inequality and Sobolev's embedding, it follows from (**??**), (**??**), and (**??**), that

$$(1-C_0^2\lambda_k)\|u_n^+\|^2 \leq (C\|g(u_n)\|_{L^q} + \widehat{C})\|u_n^+\|\,, \tag{2.6}$$

which implies the (u_n^+) is bounded in H_0^1. Similarly, we show that (u_n^-) is also bounded. □

Thus, by (iii), we must have that $\|u_n^0\| \to \infty$ and, by setting $\widehat{u}_n = u_n/\|u_n^0\|$ (and recalling that $N_k = \mathbb{R}\phi_k$ with $\|\phi_k\| = 1$), it follows that $\widehat{u}_n \to \phi_k \in N_k$ and we may also assume that $\widehat{u}_n(x) \to v(x)$ a.e. in Ω. Hence,

$$u_n(x) \to +\infty \quad \text{a.e. in } [\phi_k > 0], \tag{2.7}$$

$$u_n(x) \to -\infty \quad \text{a.e. in } [\phi_k < 0]\,. \tag{2.8}$$

Next, in view of (A4), we apply Lebesgue's theorem to the sequence $G(u_n(x))$ to obtain

$$\lim_{n\to\infty}\int_\Omega G(u_n(x))\,dx = \int_{[\phi_k>0]} G_+\,dx + \int_{[\phi_k<0]} G_-\,dx\,,$$

which proves the Claim with $v = \phi_k$.

Therefore, using Holder's inequality and Sobolev's embedding as in (**??**), we obtain (with $q \geq 2N/(N+2)$, $N \geq 3$)

$$\Big|\int_\Omega g(u_n)u_n^+\,dx\Big| \leq C\Big(\int_\Omega |g(u_n(x))|^q\,dx\Big)^{\frac{1}{q}}\|u_n^+\|$$

and, since $g(u_n(x)) \to 0$ a.e. in Ω in view of (A4), an application of Lebesgue's theorem once again implies the desired conclusion that $c \in \Gamma_k$ in case (iii) holds.

In other words, assuming (i), (ii) (i.e., that (u_n) is a Palais-Smale sequence), we have shown through the negation argument (iii) that <u>any</u> Palais-Smale sequence (u_n) has a convergent subsequence. On the other hand, it is clear that if $c \in \Gamma_k$ then $(PS)_c$ does not hold. □

Remark 2.5. Since we are assuming that λ_k is a simple eigenvalue, the set

$$\Gamma_k := \{-\alpha_k.G_+ - \beta_k G_-\,,\, -\beta_k G_+ - \alpha_k G_-\}$$

(where $\alpha_k :=$ measure$([v > 0])$, $\beta_k :=$ measure$([v < 0])$ has either one or two elements.

When λ_k is not a simple eigenvalue the set Γ_k has ν_k or $2\nu_k$ elements, where ν_k is the dimension of the eigenspace (N_k) associated with the eigenvalue λ_k.

Remark 2.6. We should also note that nonlinear resonant problems were originally introduced and studied via different methods by Landesman-Lazer [**?**] in 1970, and by Ahmad-Lazer-Paul [**?**] in 1976. Later, in 1986, Rabinowitz [**?**] published a CBMS monograph (in AMS Conf. Ser. in Math.) introducing Minimax methods in critical point theory with applications to differential equations, where his seminal abstract Saddle-Point Theorem, motivated by the Ahmad-Lazer-Paul paper, provided yet a third different proof for nonlinear resonant problems. It is illustrating to contrast the *resonant* situations in [**?**, **?**, **?**], where $G_\pm$ is infinite with the *strong resonant* situation in [**?**] and in the above result, where $G_\pm$ are *finite* real numbers. We must mention that there is a large literature on both "resonant" and "strong resonant" problems (on bounded and unbounded domains), but we tried to restrict the references to a minimum by only listing those which were related to the very first results on this subject, or that pertain to the results which we wish to address in this short paper.

Second result. The next theorem uses the non-quadratic condition at infinity (A8) that was introduced in [**?**].

Theorem 2.7. *Consider the Dirichlet problem*

$$-\Delta u = f(x,u), \quad u = 0 \quad \text{on } H_0^1(\Omega)\,, \tag{2.9}$$

where again $\Omega \subset \mathbb{R}^N$, $N \geq 3$ is bounded, f is continuous, subcritical, and assume the conditions

- (A6) $\lambda_k = \lim_{|s|\to\infty} \frac{2F(x,s)}{s^2}$, *uniformly for $x \in \Omega$,*
- (A7) $\lambda_k = \liminf_{|s|\to\infty} \frac{2F(x,s)}{s^2} \leq \limsup_{|s|\to\infty} \frac{2F(x,s)}{s^2} = \lambda_l$, *uniformly for $x \in \Omega$,*
- (A8) $\lim_{|s|\to\infty}[f(x,s)s - 2F(x,s)] = +\infty$, *uniformly for $x \in \Omega$,*

where $\lambda_k < \lambda_l$ are two eigenvalues of $-\Delta$ under Dirichlet boundary condition on $\partial\Omega$. Then the functional

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega F(x,u)\,dx := \frac{1}{2}\|u\|^2 - \int_\Omega F(x,u)\,dx$$

satisfies $(Ce)_c$ for all $c \in \mathbb{R}$.

Proof. Recall that the functional J satisfies $(Ce)_c$ if any sequence (u_n) in $H_0^1(\Omega)$ such that

- (i) $J(u_n) \to c$, and
- (ii) $\|J'(u_n)\|\|u_n\| \to 0$,

has a bounded subsequence. Let us assume by negation that J does not satisfy $(Ce)_c$ for some $c \in \mathbb{R}$. Then there exists a sequence (u_n) which satisfies (i) and (ii) above, but

$$\|u_n\| \to \infty\,.$$

It follows that

$$\lim_{n\to\infty} \int_\Omega [f(x,u_n)u_n - 2F(x,u_n)]\,dx = \lim_{n\to\infty}[2J(u_n) - J'(u_n)\cdot u_n] = 2c\,, \tag{2.10}$$

and we shall obtain a contradiction by showing that the left-hand side of (**??**) must go to infinity. Indeed, we make the following claim.

Claim: There exists a subset $\widehat{\Omega} \subset \Omega$ with measure$(\widehat{\Omega}) > 0$ such that $|u_n(x)| \to \infty$ a.e. $x \in \widehat{\Omega}$. Using the *Claim*, the subcritical growth of f and the assumption (A8), we conclude that the left-hand side of (**??**) goes to infinity. In fact, in this case, the subcritical growth of f and (A8) imply that

$$\begin{gathered} f(x,u_n(x))u_n(x) - 2F(x,u_n(x))) \geq -C, \quad \text{for a.e. } x \in \Omega \text{ and some } C \in \mathbb{R}\,, \\ \lim_{n\to\infty}[f(x,u_n(x))u_n(x) - 2F(x,u_n(x)))] = +\infty, \quad \text{for a.e. } x \in \Omega\,, \end{gathered}$$

while Fatou's lemma with $Q_n := f(x,u_n)u_n - 2F(x,u_n)$ gives

$$\int_\Omega \liminf_{n\to\infty} Q_n\,dx \geq \liminf_{n\to\infty} \int_{\widehat{\Omega}} Q_n\,dx - C\,\text{measure}(\Omega\backslash\widehat{\Omega}) = +\infty$$

for some $C \in \mathbb{R}$, which contradicts (**??**).

Now, it remains to prove the claim. To that end, we note that that (A6) and (A7) imply

$$\limsup_{|n| \to \infty} \frac{1}{\|u_n\|^2} \int_\Omega [F(x,u_n) - \frac{1}{2}\lambda_l u_n^2]\,dx \le 0\,. \tag{2.11}$$

And, setting $\widehat{u}_n = u_n/\|u_n\|$, we may assume that $\widehat{u}_n$ converges *weakly* to some $\widehat{u}$ in $H_0^1\Omega$, and *strongly* to $\widehat{u}$ in $L^2(\Omega)$. We shall then define our subset $\widehat{\Omega}$ to complete the proof. Indeed, passing to the limit in the equality

$$\frac{1}{\|\widehat{u}_n\|^2} J(u_n) = \frac{1}{2}(1 - \lambda_l \|\widehat{u}_n\|_{L^2}^2) - \frac{1}{\|u_n\|^2} \int_\Omega [F(x,u_n) - \frac{1}{2}\lambda_l u_n^2]\,dx\,,$$

and using (**??**), we obtain

$$0 \ge \frac{1}{2}(1 - \lambda_l \|\widehat{u}\|_{L^2}^2)$$

which shows that $\widehat{u} \ne 0$. The claim is proved by taking $\widehat{\Omega} = \{x \in \Omega : \widehat{u}(x) \ne 0\}$. □

Remark 2.8. As a final remark, we shall exhibit various possibilities of Γ_k (indicated in Remark **??**) in terms of the measures of the sets $[\phi_k > 0]$ (denoted α_k) and $[\phi_k < 0]$ (denoted β_k), as well as the relative signs of the limits G_+ and G_-. Indeed, let us define the parameters

$$\gamma \in [0,1] \quad \text{and} \quad \delta \in [-1,1]\,,$$

and set $\beta_k = \gamma\alpha_k$, $G_- = \delta G_+$. Then, an easy calculation shows that the finite set Γ_k can be rewritten as

$$\Gamma_k = \{-(1+\gamma\delta)\alpha_k.G_+, -(\gamma+\delta)\alpha_k G_+\}\,. \tag{2.12}$$

Note that the set Γ_k has 2 elements (or 1 element, if the above elements coincide). Indeed, recall that in Remark **??** we assumed λ_k to be a simple eigenvalue. Clearly, when λ_k has multiplicity ν_k (i.e., dimension$(N_k) = \nu_k$), we'll get $2\nu_k$ (or ν_k) elements in Γ_k.

Finally, we consider some special cases of γ and δ (assuming λ_k is a simple eigenvalue) where the situation described in Remark **??** arises by using Γ_k in (**??**)
.

Special cases.

Case 1: If $\gamma = 0$, then $\Gamma_k = \{-\alpha_k.G_+, -\delta\alpha_k G_+\}$, and
- (i) Γ_k has 1 element if $\delta = 1$,
- (ii) Γ_k has 2 elements if $\delta < 1$;

Case 2: If $\gamma > 0$, then Γ_k, and
- (i) Γ_k has has 1 element if $\delta = 0$, and $\gamma = 1$,
- (ii) Γ_k has 2 elements if $\gamma < 1$ [see (**??**)];

Case 3: If $\delta < 1$, then
- (i) Γ_k has (i) 1 element if $\gamma = 1$, and
- (ii) Γ_k has 2 elements if $\gamma < 1$;
  Indeed, $\delta < 1$, $\gamma = 1 \Rightarrow -1 - \delta = -1 - \gamma$, so Γ_k has 1 element,
  whereas $\delta < 1$, $\gamma < 1 \Rightarrow -1 - \delta\gamma \ne -\gamma - \delta$, so Γ_k has 2 elements; on the other hand,

Case 4: If $\delta = 1$ [i.e. $G_+ = G_-$] and $\gamma = 1$ [i.e. $\beta_k = \alpha_k$], then $\beta_k = \alpha_k =$ measure$(\Omega)/2$, which is equivalent to $\gamma = 1$.

References

[1] S. Ahmad, A. C. Lazer, J. L. Paul; *Elementary critical point theory and perturbations of elliptic boundary value problems at resonance*, Indiana Univ. Math. J., 25 (1976), 933–944.

[2] D. Arcoya, D. G. Costa; *Nontrivial solutions for a strong resonant problem*, Diff. Int. Eqs., 8 (1995), 151–159.

[3] V. Benci, B. Bartolo, D. Fortunato; *Abstract Critical Points Theorems and Applications to some Nonlinear Problems with "Strong" Resonance at Infinity*, Nonlinear Analysis, Theory, Methods & Applications 7, (1983), no. 8, 981–1012.

[4] H. Brézis, J. M. Coron, L. Nirenberg; *Free vibrations for a nonlinear wave equation and a theorem of P. Rabinowitz*, Comm. Pure Appl. Math., 33 (1980), 667–689.

[5] G. Cerami; *Un criterio de esistenza per i punti critici su variet'a ilimitate*, Rx. Ist. Lomb. Sci. Lett., 112 (1978), 332–336.

[6] D. G. Costa; *An Invitation to Variational Methods in Differential Equations*, Birkhauser Boston, MA, 2007.

[7] D. G. Costa, C. A. Magalhaes; *Variational elliptic problems which are nonquadratic at infinity*, Nonl. Anal., 23 (1994), 1401–1412.

[8] D. G. Costa, H. Tehrani; *On a class of asymptotically linear elliptic problems in* $\mathbb{R}^N$, J. Diff. Eqs., 173 (2001), 470–494.

[9] E. M. Landesman, A. C. Lazer; *Nonlinear perturbations of linear elliptic boundary value problems at resonance*, J. Math. Mech., 19 (1970), 609–623.

[10] R. S. Palais, S. Smale; *A generalized Morse theory*, Bull. Amer. Math. Soc. 70 (1964), 165–171.

[11] P. H. Rabinowitz; *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Regional Conf. Ser. in Math., 65, AMS, Providence, RI, 1986.

[12] C. A. Stuart, H. S. Zhou; *Applying the mountain-pass theorem to an asymptotically linear elliptic equation on* $\mathbb{R}^N$, Comm. Partial Differential Equations, 24 (1999), 1731–1758.

David G. Costa

Department of Mathematical Sciences, University of Nevada Las Vegas, Box 454020, Las Vegas, NV 89154-4020, USA

Email address: `david.costa@unlv.edu`