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A SEMILINEAR WAVE EQUATION WITH NON-MONOTONE NONLINEARITY

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Abstract. We prove the existence of weak solutions to a semilinear wave with non-monotone asymptotically linear nonlinearity when the forcing is dominated by a trigonometric polynomial.

1. Introduction

Let $\Omega = (0,\pi)\times[0,2\pi]$ and Ψ a trigonometric polynomial of the form

$$\Psi(x,t) = \sum_{i,j=1,N,\, i\neq j} a_{kj}\sin(kx)\cos(jt) + b_{kj}\sin(kx)\cos(jt), \tag{1.1}$$

where N is a positive integer. We study the existence of weak solutions to the *Dirichlet-periodic* problem

$$\begin{gathered} \Box\, u + \tau u + h(u) = f(x,t) := C\Psi(x,t) + g(x,t), \\ u(0,t) = u(\pi,t) = 0, \\ u(x,t) = u(x,t+2\pi), \quad (x,t)\in\ [0,\pi]\times\mathbb{R}, \end{gathered} \tag{1.2}$$

where $\Box$ denotes the D'Alembert operator $\partial_{tt}-\partial_{xx}$, $\tau>0$ and $\tau\notin\sigma(\Box) = \{k^2-j^2 : k=1,2,\dots,,\ j=0,1,2,\dots\}$, $C\in\mathbb{R}$, $g\in L^2(\Omega)$, and

$$\iint_\Omega \Psi(x,t)g(x,t)\,dx\,dt = 0, \tag{1.3}$$

We assume that h is bounded and differentiable, that $h'(u) < -\tau$ for some $u\in\mathbb{R}$, and that

$$\lim_{|u|\to\infty} h'(u) = 0. \tag{1.4}$$

That is, $H(u) := \tau u + h(u)$ is non-monotone and asymptotically linear. We denote by $\|\cdot\|_2$ the norm in $L^2(\Omega)$. Our main result is the following theorem.

Theorem 1.1. *For each $g\in L^2(\Omega)$ satisfying* (1.3)*, there exists $C_0(\|g\|_2)$ such that if $|C| > C_0(\|g\|_2)$ then* (1.2) *has a solution.*

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This result is in the spirit of determining the range of semilinear wave operators with non-monotone nonlinearities which goes back to the results in [9, 12] where the range of such operators was proven to be dense in $L^2(\Omega)$. The reader is referred to [6] for a review in the subject and to [4] for a recent result on (1.2) with Ψ replaced by functions that may be *flat on characteristics*. For earlier results on semilinear wave equations with monotone nonlinearities see [1, 10, 11].

A key piece in our arguments is the Nazarov-Turan lemma which we state next for the sake of completeness in the presentation. For a role of the Nazarov-Turan lemma in bifurcation at infinity, the reader is referred to [8].

Lemma 1.2 (Nazarov-Turan lemma)**.** *If Φ is a trigonometric polynomial with*

$$\int_\Omega \Phi(x,t)\sin(kx)\sin(kt)\,dx\,dt = \int_\Omega \Phi(x,t)\sin(kx)\sin(kt)\,dx\,dt = 0, \tag{1.5}$$

for any positive integer k then there exists $\alpha > 0$ such that for any $\delta \in (0,1)$,

$$m(\{(x,t)\in\Omega; |\Phi(x,t)| < \delta\}) < \delta^\alpha. \tag{1.6}$$

Moreover,

$$m(A_{r,\delta}) := m(\{x \in [0,\pi] : |\Phi(x, r+x)| < \delta\}) < \delta^\alpha, \tag{1.7}$$

uniformly for $r \in [0,2\pi]$.

In the above lemma and in what follows m denotes the Lebesgue measure in one or two dimensions as given by the context.

2. Preliminaries

Let $\mathcal{N}$ denote the closure of the linear subspace of $L^2(\Omega)$ spanned by

$$\{\sin(kx)\cos(kt),\ \sin(kx)\sin(kt),\ k = 1,2,\dots\}. \tag{2.1}$$

That is, $\mathcal{N}$ is the kernel of the wave operator $\Box$ in (1.2). If $v \in \mathcal{N}$, then there exists a unique 2π-periodic function $p : \mathbb{R} \to \mathbb{R}$ such that $p \in L^2([0,2\pi])$, $\int_0^{2\pi} p(t)dt = 0$, and

$$v(x,t) = p(t+x) - p(t-x). \tag{2.2}$$

We let $\mathbf{H}^1$ denote the Sobolev space of functions $u : [0,\pi]\times\mathbb{R} \to \mathbb{R}$ that are 2π-periodic in their second variable, with u and its first order partial derivatives in $L^2(\Omega)$, and vanishing on $\{0,\pi\}\times\mathbb{R}$. The norm in in $\mathbf{H}^1$ by $\|\cdot\|_{1,2}$. We also let $\mathbf{Y} = \mathcal{N}^\perp \cap \mathbf{H}^1$. We say that $u = y + v \in \mathbf{Y}\oplus\mathcal{N}$ is a weak solution to (1.2) if

$$\iint_\Omega \{(y_t\hat{y}_t - y_x\hat{y}_x) - (H(u) - f)(\hat{y}+\hat{v})\}\,dx\,dt = 0, \tag{2.3}$$

for all $\hat{y}+\hat{v} \in \mathbf{Y}\oplus\mathcal{N}$. We let $\Pi_N : L^2(\Omega) \to \mathcal{N}$ and $\Pi_Y : L^2(\Omega) \to \mathcal{N}^\perp$ denote the corresponding orthogonal projections.

For each $f \in L^2(\Omega)$, the equation $\Box u + \tau u = f$ has a unique weak solution $v + y$ which we denote as $(\Box + \tau I)^{-1}(f)$. Moreover, there exists a real number κ such that

$$\begin{gathered} \|(\Box+\tau I)^{-1}(\Pi_Y(f))\|_{1,2} + \|(\Box+\tau I)^{-1}(\Pi_Y(f))\|_{C^{1/2}} \le \kappa\|f\|_2, \\ \|(\Box+\tau I)^{-1}(\Pi_N(f))\|_2 \le \kappa\|f\|_2 \end{gathered} \tag{2.4}$$

where $C^{1/2}$ denotes the Hölder space of continuous functions with exponent $1/2$.

3. Proof of Theorem 1.1

By (1.4), there exists $K_1 \geq \frac{128|h'|_\infty}{\tau\pi}$ such that if $|s| \geq K_1$ then $|h'(s)| < \tau/128$. Let $\phi = (\Box + \tau I)^{-1}(\Psi)$, $\alpha > 0$ be as in Lemma 1.2 applied to $\Phi = \phi$, and

$$\delta = \Big(\frac{\tau^2\pi^2}{(128(1+\tau+|h'|_\infty))^2}\Big)^{1/\alpha}, \tag{3.1}$$

$$C_0 = \frac{(K_1 + 2K_1^2 + \kappa(\sqrt{2}\pi[|h|_\infty + |h'|_\infty] + \|g\|_2 + 1))(\tau+1)}{\sin(\delta)}. \tag{3.2}$$

Since $\delta^\alpha < 1$, we have

$$\delta^\alpha < \frac{\tau\pi}{128(1+\tau+|h'|_\infty)}.$$

From [9] and [12], there exist sequences $\{\phi_n\}, \{u_n\} \subset L^2$ with $u_n = z_n + w_n \in \mathcal{N} \oplus \mathbf{Y}$ such that

$$\Box\, w_n + \tau(z_n + w_n) + h(z_n + w_n) = C\Psi(x,t) + g(x,t) + \phi_n(x,t), \quad \|\phi_n\|_2 \to 0. \tag{3.3}$$

Without loss of generality we may assume that $\|\phi_n\|_2 \leq 1$ for $n = 1, 2, \dots$. Let $v_n \in \mathcal{N}$ and $y_n \in \mathbf{Y}$ be such that

$$\Box\, y_n + \tau(v_n + y_n) = g(x,t) + \phi_n(x,t), \quad \|\phi_n\|_2 \to 0. \tag{3.4}$$

Subtracting (3.3) from (3.4), we obtain

$$\Box(w_n - y_n) + \tau(w_n - y_n + z_n - v_n) + h(z_n + w_n) = C\Psi(x,t) = C(\Box + \tau I)(\psi). \tag{3.5}$$

Letting $W_n = w_n - y_n - C\psi$ and $V_n = z_n - v_n$,

$$\Box\, W_n + \tau(W_n + V_n) + h\big(V_n + v_n + W_n + y_n + C\psi(x,t)\big) = 0. \tag{3.6}$$

Equation (3.6), in turn, is equivalent to the equations

$$W_n = -(\Box + \tau I)^{-1}\Pi_Y\left(h\left(V_n + v_n + W_n + y_n + C\psi\right)\right), \tag{3.7}$$

$$\tau V_n = -\Pi_N\Big(h\Big(V_n + v_n + W_n + y_n + \frac{C}{\tau+1}\Psi(x,t)\Big)\Big). \tag{3.8}$$

Since h is assumed to be bounded, by the continuity of $(\Box + \tau I)^{-1}$: $L^2 \to Y$ and Arzela-Ascoli's theorem we may assume that $\{W_n\}$ converges uniformly in Ω.

By (2.2), there exists $g_1, p_n, P_n \in L^2(0, 2\pi)$ such that

$$\begin{gathered}\Pi_N(g)(x,t) = g_1(t+x) - g_1(t-x), v_n(x,t) = p_n(x,t) - p_n(t-x), \\ V_n(x,t) = P_N(x,t).\end{gathered} \tag{3.9}$$

By (3.4), the sequence $\{\tau p_n\}$ converges to g_1 in $L^2([0, 2\pi])$. From (3.8) and [2],

$$2\pi\tau P_n(r) = -I_{1n}(r) + I_{2n}(r), \qquad \text{a.e. in } [0, 2\pi], \tag{3.10}$$

where

$$\begin{aligned} I_{1n}(r) &= \int_0^\pi h\left((W_n + y_n)(x, r-x) + q_n(r) - q_n(r-2x) + C\psi(x,t)\right) dx \\ I_{2n}(r) &= \int_0^\pi h\left((W_n + y_n)(x, r+x) + q_n(r+2x) - q_n(r) + C\psi(x,t)\right) dx, \end{aligned} \tag{3.11}$$

and

$$q_n(s) = p_n(s) + P_n(s) \quad \text{for all } s \in \mathbb{R}. \tag{3.12}$$

Since h is bounded, from (3.10) and (3.11) we see that the sequence $\{P_n\}$ is bounded in L^∞.

Let us show that the sequence $\{P_n\}$ converges en $L^2([0,2\pi])$. Indeed, let us show that $\{P_n\}$ is a Cauchy sequence in $L^2([0,2\pi])$. Let $Y_j(r,x) = (W_j + y_j)(x, r-x)$, $Q_j(r,x) = q_j(r) - q_j(r-2x)$, $F_{mn}(s,r,x) = C\psi(x,r-x) + (Q_n + s(Y_n - Y_m))(r,x)$, and $G_{mn}(s,r,x) = C\psi(x,r-x) + (Y_m + s(Q_n - Q_m))(r,x)$. Hence

$$\begin{aligned} \big|I_{1n}(r) - I_{1m}(r)\big| \le \int_0^\pi \int_0^1 |h'\left(F_{mn}(s,r,x)\right)| \, ds \cdot |(Y_n - Y_m)(r,x)| \, dx \\ + \int_0^\pi \int_0^1 |h'\left(G_{mn}(s,r,x)\right)| \, ds \cdot |(Q_n - Q_m)(r,x)| dx. \end{aligned} \tag{3.13}$$

Let $|C| \ge C_0$ with C_0 given by (3.2). If $x \notin A_{r,\delta}$ and $|(Q_n - Q_m)(r,x)| \ge 2K_1^2$, then $m[\{s \in [0,1]; |Y_m(r,x) + C\phi(x,r-x) + s(Q_n - Q_m)(r,x)| \le K_1)\}] \le 1/K_1$. Hence

$$\begin{aligned} \Big|\int_0^1 h'\left(Y_m(r,x) + C\phi(x,r-x) + s(Q_n - Q_m)(r,x)\right) \, ds\Big| &\le \frac{|h'|_\infty}{K_1} + \frac{\pi\tau}{128} \\ &\le \frac{\pi\tau}{64}. \end{aligned} \tag{3.14}$$

On the other hand, if $|(Q_n - Q_m)(r,x)| \le 2K_1^2$, then $|Y_m(r,x) + C\psi(x,r-x) + s(Q_n - Q_m)(r,x))| > K_1$. Thus $|h'(Y_m(r,x) + C\psi(x,t) + s(Q_n - Q_m)(r,x))| < \frac{\tau}{128}$. Hence

$$\Big|\int_0^1 h'\left(Y_m(r,x) + C\psi(x,r-x) + s(Q_n - Q_m)(r,x)\right) ds\Big| \le \frac{\tau}{128} < \frac{\pi\tau}{64}. \tag{3.15}$$

From (3.13), (3.14), and (3.15),

$$\begin{aligned} &|I_{1n}(r) - I_{1m}(r)| \\ &\le \int_0^\pi |h'|_\infty |(Y_n - Y_m)(r,x)| dx \\ &\quad + \frac{\tau\pi}{64} \int_{[0,\pi]\setminus A_{r,\delta}} |(Q_n - Q_m)(r,x)| \, dx \\ &\quad + \int_{A_{r,\delta}} |h'|_\infty |Q_n(r,x) - Q_m(r,x)| dx \\ &\le \int_0^\pi |h'|_\infty |(Y_n - Y_m)(r,x)| \, dx + \frac{\tau\pi}{64}\Big(\pi(|(P_n - P_m + p_n - p_m)(r)| \\ &\quad + \int_{[0,\pi]\setminus A_{r,\delta}} |(p_n + P_n - p_m - P_m)(r-2x)| \, dx\Big) \\ &\quad + m(A_{r,\delta}) |h'|_\infty |(P_n - P_m + p_n - p_m)(r)| \\ &\quad + \int_{A_{r,\delta}} |h'|_\infty |(P_n - P_m + p_n - p_m)(r-2x)| dx. \end{aligned} \tag{3.16}$$

Similarly,

$$\begin{aligned}|I_{2n}(r)-I_{2m}(r)| \le \int_0^\pi &|h'|_\infty |(Y_n-Y_m)(r,x)|\,dx \\ &+\frac{\tau\pi}{64}\Big(\pi(|(P_n-P_m+p_n-p_m)(r)| \\ &+\int_{[0,\pi]\setminus A_{r,\delta}} |(p_n+P_n-p_m-P_m)(r+2x)|\,dx\Big) \\ &+m(A_{r,\delta})|h'|_\infty\,|(P_n-P_m+p_n-p_m)(r)| \\ &+\int_{A_{r,\delta}} |h'|_\infty |(P_n-P_m+p_n-p_m)(r+2x)|\,dx.\end{aligned} \tag{3.17}$$

Since

$$\int_{A_{r,\delta}} |P_n(r-2x)-P_m(r-2x)|dx \le \delta^{\alpha/2}\|P_n-P_m\|_2,$$

by (3.10), (3.16) y (3.17) we have

$$\begin{aligned}2\pi\tau|P_n(r)-P_m(r)| \le 2\int_0^\pi &|h'|_\infty |(Y_n-Y_m)(r,x)|\,dx \\ &+\frac{\tau\pi}{32}\Big(\int_{[0,\pi]\setminus A_{r,\delta}} |(p_n+P_n-p_m-P_m)(r-2x)|\,dx \\ &+\pi(|P_n(r)-P_m(r)|+|(p_n-p_m)(r)|)\Big) \\ &+2\delta^\alpha|h'|_\infty(|P_n-P_m+p_n-p_m)(r)|) \\ &+2|h'|_\infty\delta^{\alpha/2}\left(\|P_n-P_m\|_2+\|p_n-p_m\|_2\right).\end{aligned} \tag{3.18}$$

By the definition of δ,

$$64\tau\pi-\tau\pi^2-128\delta^\alpha|h'|_\infty > 64\tau\pi-\tau\pi^2-\tau\pi > 59\tau\pi.$$

Therefore,

$$\begin{aligned}&\frac{59\tau\pi}{32}|P_n(r)-P_m(r)| \\ &\le 2\int_0^\pi |h'|_\infty |(Y_n-Y_m)(r,x)|\,dx+\frac{5\tau\pi}{32}|(p_n-p_m)(r)| \\ &\quad+\Big(\frac{\pi\tau\sqrt{2\pi}}{64}+2\delta^{\alpha/2}|h'|_\infty\Big)(\|p_n-p_m\|_2+\|P_n-P_m\|_2) \\ &\quad+\frac{5\tau\pi}{64}\|(p_n-p_m\|_2.\end{aligned} \tag{3.19}$$

Hence,

$$\begin{aligned}&\frac{59\tau\pi}{32}\|P_n(r)-P_m(r)\|_2 \\ &\le 2\Big(\int_0^{2\pi}\int_0^\pi |h'|_\infty |(Y_n-Y_m)(r,x)|\,dx\,dr\Big)^{1/2} \\ &\quad+\Big(\frac{\pi\tau\sqrt{2\pi}}{64}+|h'|_\infty 2\delta^{\alpha/2}\Big)\sqrt{2\pi}(\|p_n-p_m\|_2+\|P_n-P_m\|_2) \\ &\quad+\frac{5\tau\pi}{64}\|(p_n-p_m\|_2.\end{aligned} \tag{3.20}$$

Since $\{p_n\}$ converges in $L^2([0,2\pi])$, $\{Y_n\}$ converges uniformly and

$$\Big(\frac{\tau\pi\sqrt{2\pi}}{64}+|h'|_\infty 2\delta^{\alpha/2}\Big)\sqrt{2\pi}<\frac{\tau\pi}{4}<\frac{59\tau\pi}{32}. \tag{3.21}$$

Thus $\{P_n\}$ is a Cauchy sequence in $L^2([0,2\pi])$, which proves the theorem.

Obituary. With great sadness, the last three authors report the passing of Professor José Francisco Caicedo on February 23, 2022. He was our friend, teacher, and mentor. The results in this paper were proven prior to his death. He was a leading force in the understanding the solvability of semilinear wave equations.

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