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FRIEDRICHS EXTENSION OF SINGULAR SYMMETRIC DIFFERENTIAL OPERATORS

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Abstract. For singular even order symmetric differential operators we find the matrices which determine all symmetric extensions of the minimal operator. And for each of these symmetric operators which is bounded below we find the boundary condition of its Friedrichs extension. The operators of regular problems are bounded below and thus each one has a symmetric extension and thus its symmetric extension has a Friedrichs extension.

1. Introduction

Let $I = (a, b)$, $-\infty \le a < b \le \infty$, be an open interval and let n be a positive integer. Let $M_{n,m}(\Omega)$ denote the $n \times m$ matrices with elements from the set Ω for $n, m = 1, 2, 3, 4, \dots$, and for $m = n$ we write $M_n(\Omega)$. Let $\mathbb{C}$ denote the complex numbers and $\mathbb{R}$ the reals.

Let $C = (c_{r,s})_{1 \le r,s \le 2n} \in M_{2n}(\mathbb{C})$ be a skew diagonal complex matrix with the following properties:

$$C^{-1} = -C = C^*, \tag{1.1}$$

and let

$$\begin{gathered}
Z_{2n}(I) := \{(q_{r,s})_{r,s=1}^{2n} \in M_{2n}(L^1_{\rm loc}(I)),\\
q_{r,r+1} \neq 0 \text{ a.e. on } I, \quad q_{r,r+1}^{-1} \in L^1_{\rm loc}(I), \quad 1 \le r \le 2n-1,\\
q_{r,s} = 0 \text{ a.e. on } I, \quad 2 \le r+1 < s \le 2n;\\
q_{r,s} \in L^1_{\rm loc}(I), \quad s \neq r+1, \quad 1 \le r \le 2n-1\}.
\end{gathered}$$

For $Q \in Z_{2n}(I)$, define the quasi-derivatives $y^{[r]}(0 \le r \le 2n)$ below:

$$\begin{gathered}
y^{[0]} := y \quad (y \in V_0),\\
y^{[r]} = q_{r,r+1}^{-1}[(y^{[r-1]})' - \sum_{s=1}^{r} q_{r,s} y^{[s-1]}] \quad (y \in V_r, r = 1, 2, \dots, 2n),
\end{gathered} \tag{1.2}$$

where

$$\begin{gathered}
V_0 := \{y : I \to \mathbb{C},\ y \text{ is measurable }\},\\
V_r := \{y \in V_{r-1} : y^{[r-1]} \in AC_{\rm loc}(I)\},\ r = 1, 2, \dots, 2n,
\end{gathered}$$

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and $q_{2n,2n+1} = c_{2n,1}$. Since the quasi-derivatives depend on Q, we sometimes write $y_Q^{[r]}$ instead of $y^{[r]}$, $r = 1, 2, \dots, 2n$. Finally we set

$$My = y^{[2n]} = \lambda w y, \quad \lambda \in \mathbb{R},\ y \in V_{2n}, \tag{1.3}$$

these expressions $M = M_Q$ are generated by or associated with Q and the element of constant matrix C. For V_{2n} we also use the notations $D(Q)$ and $V(M)$. If

$$Q = -C^{-1}Q^*C, \tag{1.4}$$

i.e.,

$$q_{r,s} = c_{r,2n+1-r}\overline{q}_{2n+1-s,2n+1-r}c_{2n+1-s,s}, \tag{1.5}$$

then Q is said to be a C-symmetric matrix. For this case M_Q is called a C-symmetric quasi-differential expression [2].

Let $w \in L^1_{\rm loc}(I)$ be positive a.e. on I. Consider the Hilbert space $H = L^2_w(I)$ with inner product $(y,z)_w = \int_a^b y\bar{z}w$ and let

$$D_Q = \{y \in H : y \in D(Q) \text{ and } \frac{1}{w}My \in H\}. \tag{1.6}$$

This linear manifold D_Q is called the maximal domain of M and $T_Q y = \frac{1}{w}My = \lambda y$ for $y \in D_Q$, is called the maximal operator of M. It is well known [1, 7] that D_Q is dense in H. Its uniquely defined adjoint $T_0 = T_Q^*$ is called the minimal operator of M and its domain $D_0 := D(T_0)$ is called the minimal domain of M. It is well known that T_0 is a densely defined symmetric operator in H [17, 24], and it is the closure of the pre-minimal operator T_0' defined by

$$D_0' := D(T_0') = \{y \in D_Q : y \text{ has compact support in } I\}.$$

The operator T_0' is also a symmetric operator in H and its Friedrichs extension has the same characterization as the minimal operator T_0. Some properties of $Z_n(I)$ and the definition of singular or regular expression can be seen in [28].

For a symmetric densely defined operator S which bounded below, Friedrichs [8] constructed a self-adjoint extension S_F which preserves the lower bound. He also addressed this question and showed that the Friedrichs extension of the minimal operator is determined by Dirichlet boundary conditions for a special class of regular second order Sturm-Liouville (SL) expressions. This result is now known to hold [14, 18] for all regular SL expressions with minimal conditions on the coefficients and weight function and for very general classical and quasi-differential expressions of arbitrary even order. In particular, for a smaller class of symmetric expressions M Niessen and Zettl [18] presented the boundary conditions of the Friedrichs extensions for some symmetric extensions with separated boundary conditions. Moreover, for any even order regular Lagrange symmetric differential operators, Möller and Zettl [15] gave the characterization of the boundary conditions that determine their Friedrichs extensions.

One can prove that the Friedrichs extension is distinguished in various ways among other self-adjoint extensions, see [3, 23]. For singular differential operators, Rellich [20] realized that the principal solution plays a key role in the boundary conditions of the Friedrichs extension of the second order SL problems. In 1992, Niessen and Zettl [19] based on the work of Rellich [20], Kalf [10], Rosenberger [23]

and other scholars, used the principal solution to characterize the boundary conditions of the Friedrichs extension of singular second order SL problems. In 2000, for a smaller class of symmetric operators,
Martletta and Zettl [16] gave the boundary conditions for Friedrichs extension of some even order singular symmetric differential expressions. In 2015, Yao et al. [27] used the method of functional analysis to characterize the Friedrichs extension of the second order SL operators by using the limit-circle (LC) type square integrable solutions of real parameters and trigonometric functions. In 2018, Zheng and Kong [29] studied singular Hamiltonian operators with intermediate deficiency indices, and gave a complete characterization of Friedrichs extensions of minimal Hamiltonian operator. In 2020, Wang and Zettl [26] characterized the two-point regular and singular boundary conditions which determine the symmetric operators. In 2022, we [4] found, explicitly, the boundary conditions which determine the Friedrichs extension of the class of regular even order C-symmetric differential operators. Furthermore for singular second order SL problems we [5] gave explicit representations of boundary matrices for any symmetric operators which are bounded below and the boundary conditions which determine their Friedrichs extensions.

This article is a follow up of [5] and by using the technique in [5, 16] we present a new characterization of all symmetric operators generated by (1.3) and the Friedrichs extension of these symmetric extensions which are bounded below in terms of boundary conditions for singular or regular general even order C-symmetric differential expressions. Also some specific examples are given for the boundary conditions of the Friedrichs extensions.

The rest of this article is organized as follows. We briefly introduce some properties of Hamiltonian system in Section 2. The relations between the existence of principal solutions and the boundedness below of the minimal operator are presented in Section 3. The characterization of symmetric operators is given in Section 4. The Friedrichs extensions for the symmetric extensions which are bounded below are presented in Section 5 and some special examples of these extensions are given in Section 6.

2. Hamiltonian system

In this section we briefly discuss properties of Hamitonian systems and give several lemmas for proving our main results.

Note that $C = C_{2n}$ satisfying (1.1) has the form

$$C_{2n} = \begin{pmatrix} 0 & \hat{C}_n \\ -\hat{C}_n^* & 0 \end{pmatrix} \tag{2.1}$$

and $\hat{C}_n$ is a skew-diagonal unitary matrix, that is,

$$\begin{gathered} c_{r,s}\bar{c}_{r,s} = 1, \quad \text{for } r+s = 2n+1,\ 1 \le r \le n, \\ c_{r,s} = 0, \quad \text{otherwise}. \end{gathered} \tag{2.2}$$

For every sufficiently smooth function $y \in D_Q$ in I, we give a associate vector function

$$\mathbf{y}(t) = \begin{pmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{pmatrix}, \quad \mathbf{x}(t) = (yy^{[1]} \cdots y^{[n-1]})^T, \quad \mathbf{u}(t) = (y^{[2n-1]} y^{[2n-2]} \cdots y^{[n]})^T.$$

It is shown that each C-symmetric differential equation (1.3) satisfying (1.4) may be written in terms of the $2n$-dimensional vector function $\mathbf{y}$ as

$$L\mathbf{y} := \tilde{J}_{2n}\mathbf{y}'(t) - G_Q(t)\mathbf{y}(t) = \lambda W(t)\mathbf{y}(t), \tag{2.3}$$

where

$$\tilde{J}_{2n} = \begin{pmatrix} 0 & -\tilde{C}_n \\ \tilde{C}_n^* & 0 \end{pmatrix}, \quad \tilde{C}_n = \begin{pmatrix} c_{1,2n} & 0 & \cdots & 0 \\ 0 & c_{2,2n-1} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & c_{n,n+1} \end{pmatrix},$$

and

$$G_Q(t) = \begin{pmatrix} -C(t) & A^*(t) \\ A(t) & B(t) \end{pmatrix}, \quad W(t) = \begin{pmatrix} W_1(t) & 0 \\ 0 & 0 \end{pmatrix},$$

$$C(t) = \begin{pmatrix} c_{1,2n}q_{2n,1} & \bar{c}_{2,2n-1}\bar{q}_{2n-1,1} & \cdots & \bar{c}_{n,n+1}\bar{q}_{n+1,1} \\ c_{2,2n-1}q_{2n-1,1} & c_{2,2n-1}q_{2n-1,2} & \cdots & \bar{c}_{n,n+1}\bar{q}_{n+1,2} \\ \vdots & \ddots & \ddots & \vdots \\ c_{n,n+1}q_{n+1,1} & c_{n,n+1}q_{n+1,2} & \cdots & c_{n,n+1}q_{n+1,n} \end{pmatrix},$$

$$A(t) = \begin{pmatrix} \bar{c}_{1,2n}q_{1,1} & \bar{c}_{1,2n}q_{1,2} & 0 & \cdots & 0 \\ \bar{c}_{2,2n-1}q_{2,1} & \bar{c}_{2,2n-1}q_{2,2} & \bar{c}_{2,2n-1}q_{2,3} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \bar{c}_{n-1,n+2}q_{n-1,1} & \bar{c}_{n-1,n+2}q_{n-1,2} & \cdots & \ddots & \bar{c}_{n-1,n+2}q_{n-1,n} \\ \bar{c}_{n,n+1}q_{n,1} & \bar{c}_{n,n+1}q_{n,2} & \cdots & \cdots & \bar{c}_{n,n+1}q_{n,n} \end{pmatrix},$$

$$B(t) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \bar{c}_{n,n+1}q_{n,n+1} \end{pmatrix}, \quad W_1(t) = \begin{pmatrix} w & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

It is worthwhile to state that the diagonal elements of $\tilde{C}_n$ are $c_{r,2n+1-r}$, $r = 1, 2, \dots, n$ of C_{2n} satisfying (2.1) and $G_Q(t)$ is associated with the matrix function $Q(t)$ satisfying (1.4). Also notice that $B(t) = B^*(t), C(t) = C^*(t)$. The matrix function $G_Q(t) \in M_{2n}(L^1_{\rm loc}(I))$ is Hermitian. Observe that (2.3) can be written as

$$\begin{gathered} L_1\{\mathbf{x}, \mathbf{u}\} := -\tilde{C}_n\mathbf{u}'(t) + (C(t) - A^*(t)\mathbf{u}(t) - \lambda W_1(t))\mathbf{x} = 0, \\ L_2\{\mathbf{x}, \mathbf{u}\} := \tilde{C}_n^*\mathbf{x}' - A(t)\mathbf{x} - B(t)\mathbf{u} = 0. \end{gathered}$$

Because of $\tilde{J}_{2n}^* = -\tilde{J}_{2n} = \tilde{J}_{2n}^{-1}$, and recall that for $\tilde{C}_n = I_n$, the $n \times n$ identity matrix, its occurrence as the canonical form of an accessory system derived from a variational problem, see [22, Chapter V]. The equation (2.3) may be called a "Hamiltonian system", or "Hermitian system".

The property of disconjugacy is important to system (2.3), see [16, 22].

Definition 2.1. System (2.3) is said to be disconjugate on an interval $(\alpha, \beta) \subseteq I = (a, b)$ if for every interval $(\alpha_0, \beta_0) \subseteq (\alpha, \beta)$ whose endpoints are regular points of the differential equation, the boundary value problem

$$L\mathbf{y} = \tilde{J}_{2n}\mathbf{y}'(t) - G_Q(t)\mathbf{y}(t) = \lambda W(t)\mathbf{y}(t), \mathbf{x}(\alpha_0) = \mathbf{0}, \mathbf{x}(\beta_0) = \mathbf{0} \tag{2.4}$$

the only solution is the trivial solution.

Remark 2.2. The points $t_0, t_1 \in I$, $t_0 \neq t_1$, are said to be mutually conjugate relative to (2.3) if there exists a solution $\mathbf{y}(t) = \begin{pmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{pmatrix}$ of (2.3) such that $\mathbf{x}(t_0) = 0, \mathbf{x}(t_1) = 0$ and $\mathbf{x}(t) \neq 0$ on the subinterval with endpoints t_0 and t_1. System (2.3) is disconjugate on a subinterval I_0 of I if no two distinct points of I_0 are mutually conjugate relative to (2.3). It is said to be oscillatory if for every $t_0 \in I$ there exists a $t_1 > t_0$ such that (2.3) is not disconjugate on $[t_0, t_1]$. See [13].

We call the differential equation (1.3) disconjugate if the associated system (2.3) is disconjugate. Next we concentrate on systems which are normal.

Definition 2.3 ([16])**.** A Hamiltonian system is said to be normal on an interval (a, b) if it has no solutions with $\mathbf{x}(t) = 0$ for all $t \in (a, b)$ and $\mathbf{u}(t) \neq 0$ for some $t \in (a, b)$.

From Definition 2.3 we know that the systems (2.3) arising from differential equations (1.3) have this property. In addition if (2.3) is normal on every non-degenerate subinterval of I, then this equation is said to be identically normal (or called complete controllability) on I. Thus in the following discussion of disconjugacy, we automatically restrict our attention to identically normal systems.

Instead of (2.3), it is more convenient to deal with the matrix equation

$$LY := \tilde{J}_{2n} Y'(t) - G_Q(t) Y(t) = \lambda W(t) Y(t). \tag{2.5}$$

Since $\tilde{C}_n^* = \tilde{C}_n^{-1}$, we obtain by $Y(t) = \begin{pmatrix} X(t) \\ U(t) \end{pmatrix}$ such that

$$\begin{aligned} X'(t) &= \tilde{C}_n(A(t)X(t) + B(t)U(t)), U'(t) \\ &= \tilde{C}_n^*(C(t)X(t) - A^*(t)U(t) - \lambda W_1(t)X(t)), \end{aligned} \tag{2.6}$$

where $X(t)$ and $U(t)$ are $n \times r$ matrix valued. Note that $Y(t)$ is a matrix solution of (2.5) if and only if $\mathbf{x}(t) = X(t)\mathbf{c}$, $\mathbf{u}(t) = U(t)\mathbf{c}$ is a vector solution of (2.3)for every constant vector $\mathbf{c} \in \mathbb{C}^r$, $r \geq 1$. Thus all solutions of (2.3) are determined if we know two solutions $Y_j(t) = \begin{pmatrix} X_j(t) \\ U_j(t) \end{pmatrix} \in M_{2n, r_j}(L^1_{\rm loc}(I))$, $j = 1, 2$, of (2.5) such that the $2n \times (r_1 + r_2)$ matrix $\begin{pmatrix} X_1(t) & X_2(t) \\ U_1(t) & U_2(t) \end{pmatrix}$ is nonsingular.

If $Y_1(t) = \begin{pmatrix} X_1(t) \\ U_1(t) \end{pmatrix}$ and $Y_2(t) = \begin{pmatrix} X_2(t) \\ U_2(t) \end{pmatrix}$ are solutions of (2.5), then we have

$$\{Y_1, Y_2\} := X_2^*(t)\tilde{C}_n U_1(t) - U_2^*(t)\tilde{C}_n^* X_1(t) = K_0, \tag{2.7}$$

where $K_0 \in M_{r_2, r_1}(\mathbb{C})$ is a constant matrix. This is readily verified by differentiating (2.7). If $K_0 = 0$ then the solutions $Y_1(t), Y_2(t)$ are called conjugate solutions of (2.5). Correspondingly the solutions $\mathbf{y}_1(t), \mathbf{y}_2(t)$ are called conjugate solutions of (2.3). Alternate terminologies are mutually conjoined, or isotropic solutions or prepared solutions, see [6, 9]. In particular if $Y_2(t) = Y_1(t)$ then

$$X_1^*\tilde{C}_n U_1 = U_1^*\tilde{C}_n^* X_1; \text{i.e.}, X_1^*\tilde{C}_n U_1 = (X_1^*\tilde{C}_n U_1)^* \text{ is Hermitian.} \tag{2.8}$$

In this case, the solution $Y_1 = \begin{pmatrix} X_1(t) \\ U_1(t) \end{pmatrix}$ (or $\mathbf{y}_1(t) = \begin{pmatrix} \mathbf{x}_1(t) \\ \mathbf{u}_1(t) \end{pmatrix}$) is called self-conjugate. Note that for Y_1 if its column vectors are linearly independent solutions of (2.3) then these solutions form a basis for a conjoined family of solutions of dimension $r = r_1$ and the maximal dimension of a conjoined family of (2.3) is n. Thus if Y_1 satisfies (2.8) with $\operatorname{rank}(Y_1) = n$, then for brevity it is referred to as a conjoined basis for (2.3). A conjoined basis $Y(t) = \begin{pmatrix} X(t) \\ U(t) \end{pmatrix}$ for (2.3) is said to be oscillatory on I, if $\det(X(t))$ has arbitrarily large zeros; otherwise, $Y(t) = \begin{pmatrix} X(t) \\ U(t) \end{pmatrix}$ is called non-oscillatory. System (2.3) is said to be oscillatory if every conjoined basis for (2.3) is oscillatory.

Lemma 2.4. *If* $Y = \begin{pmatrix} X(t) \\ U(t) \end{pmatrix} \in M_{2n,n}(L^1_{\rm loc}(I))$ *is a conjoined basis for* (2.3) *with* $\operatorname{rank}(X(t)) = n$ *on* $[\alpha,\beta] \subset I$*, and* $t_0 \in [\alpha,\beta] \subset I$*, then* $Y_0 = \begin{pmatrix} X_0(t) \\ U_0(t) \end{pmatrix} \in M_{2n,n}(L^1_{\rm loc}(I))$ *is a solution of* (2.5) *on* $[\alpha,\beta] \subset I$ *if and only if*

$$X_0(t) = X(t)H(t), \quad H(t) = K_1 - \Big(\int_{t_0}^t X^{-1}(r)B(r)X^{*-1}(r)dr\Big)K_0. \tag{2.9}$$

In particular for (2.5) *the associated function is*

$$U_0(t) = U(t)H(t) + \tilde{C}_n^* X^{*-1}(t)K_0,$$

where $K_1 = X^{-1}(t_0)X_0(t_0)$ *and* $K_0 = \{Y_0, Y\}$*.*

Furthermore for $K_1 \neq 0$*,* Y_0 *is also a conjoined basis for* (2.3) *with* $\operatorname{rank}(X_0(t)) = n$ *if and only if* $K_1^*K_0 = K_0^*K_1$*. And in this case*

$$X(t) = X_0(t)\tilde{H}(t), \tilde{H}(t) = K_1^{-1} + \Big(\int_{t_0}^t X_0^{-1}(r)B(r)X_0^{*-1}(r)dr\Big)K_0^*. \tag{2.10}$$

Proof. Assume that $Y = \begin{pmatrix} X(t) \\ U(t) \end{pmatrix}$ is a conjoined basis for (2.3) on I. Let (2.9) hold, where $K_0 = \{Y_0, Y\}$ and $K_1 = X^{-1}(t_0)X_0(t_0)$. It is easily verified by differentiation that $Y_0 = \begin{pmatrix} X_0(t) \\ U_0(t) \end{pmatrix}$ is a solution of (2.5) on $[\alpha,\beta] \subset I$. By using the assumption $X^*\tilde{C}_nU = U^*\tilde{C}_n^*X$, we obtain

$$\begin{aligned} U_0(t) &= \tilde{C}_n^{-1}X^{*-1}(t)K_0 + \tilde{C}_n^{-1}X^{*-1}(t)U^*(t)\tilde{C}_n^*X_0(t) \\ &= \tilde{C}_n^{-1}X^{*-1}(t)K_0 + U(t)X^{-1}(t)X_0(t) \\ &= \tilde{C}_n^{-1}X^{*-1}(t)K_0 + U(t)H(t). \end{aligned}$$

Note that Y_0 is a conjoined basis for (2.3) if and only if $X_0^*\tilde{C}_nU_0 = U_0^*\tilde{C}_n^*X_0$ and $\operatorname{rank}(Y_0(t)) = n$. Since $\{Y, Y_0\}$ is independent of t, we obtain

$$X_0(t_0) = X(t_0)K_1, U_0(t_0) = \tilde{C}_n^*X^{*-1}(t_0)K_0 + U(t_0)K_1, \quad t_0 \in [\alpha,\beta].$$

Furthermore for $K_1 \neq 0$, since

$$\begin{aligned} &X_0^*\tilde{C}_nU_0 - U_0^*\tilde{C}_n^*X_0 \\ &= K_1^*X^*(t_0)\tilde{C}_n(\tilde{C}_n^*X^{*-1}(t_0)K_0 + U(t_0)K_1) \\ &\quad - (K_0^*X^{-1}(t_0)\tilde{C}_n + K_1^*U^*(t_0))\tilde{C}_n^*X(t_0)K_1 \\ &= K_1^*K_0 - K_0^*K_1 + K_1^*X^*(t_0)\tilde{C}_nU(t_0)K_1 - K_1^*U^*(t_0)\tilde{C}_n^*X(t_0)K_1, \end{aligned}$$

we obtain that Y_0 is a conjoined basis for (2.3) with $\operatorname{rank}(X_0(t)) = n$ if and only if $K_1^* K_0 = K_0^* K_1$. Therefore for $K_1 \neq 0$, if Y_0 is a conjoined basis for (2.3) with $\operatorname{rank}(X_0(t)) = n$, then from (2.9), $Y(t)$ can be written by $Y_0(t)$ as follows:

$$X(t) = X_0(t)\tilde{H}(t), \quad \tilde{H}(t) = \tilde{K}_1 - \Big(\int_{t_0}^t X_0^{-1}(r)B(r)X_0^{*-1}(r)dr\Big)\tilde{K}_0,$$
$$U(t) = U_0(t)\tilde{H}(t) + \tilde{C}_n^* X^{*-1}(t)\tilde{K}_0,$$

where $\tilde{K}_1 = X_0^{-1}(t_0)X(t_0)$ and $\tilde{K}_0 = \{Y, Y_0\}$.

Note that $\tilde{H}(t)H(t) = I_n$. For $t = t_0$ we have $\tilde{K}_1 K_1 = I_n$. Also from $\{Y_0, Y\} = -\{Y, Y_0\}^*$, it follows that $\tilde{K}_0 = -K_0^*$. Thus (2.10) holds. This completes the proof. □

Lemma 2.5. *Suppose that $B(t)$ is semi-positive definite a.e. on $[\alpha, \beta]$. Then (2.3) is disconjugate or non-oscillatory on $[\alpha, \beta] \subset I$ if and only if there exists a conjoined basis $Y = \begin{pmatrix} X(t) \\ U(t) \end{pmatrix} \in M_{2n,n}(L^1_{\rm loc}(I))$ for (2.3) with $\operatorname{rank}(X(t)) = n$ on $[\alpha, \beta] \subset I$.*

Proof. Let $Y_j = \begin{pmatrix} X_j \\ U_j \end{pmatrix} \in M_{2n,n}(L^1_{\rm loc}(I)), j = 1, 2$ are solutions of (2.5) satisfying $X_1(\alpha) = 0 = X_2(\beta)$ and $U_1(\alpha) = I_n, U_2(\beta) = \tilde{C}_n^* X_1^{*-1}(\beta)$. From Definition 2.1 we know that (2.3) is disconjugate on $[\alpha, \beta]$ (or on (α, β)) if and only if $\operatorname{rank}(X_1(t)) = n$ for $t \in (\alpha, \beta]$. Also (2.3) is disconjugate on $[\alpha, \beta]$ (or on (α, β)) if and only if $\operatorname{rank}(X_2(t)) = n$ on $[\alpha, \beta)$. Thus $\{Y_1, Y_2\} = -I_n$ with $X_1 \not\equiv 0$ for subinterval of (α, β). Note that matrix solutions Y_1, Y_2 both are self-conjugate. By (2.9) with $K_1 = X_2^{-1}(\alpha)X_1(\alpha) = 0$, $K_0 = \{Y_1, Y_2\} = -I_n$ in Lemma 2.4, we have the relation

$$X_1(t) = X_2(t)\int_\alpha^t X_2^{-1}(r)B(r)X_2^{*-1}(r)dr, \quad t \in [\alpha, \beta).$$

Hence we know that X_1, X_2 have the same algebraic sign on $[\alpha, \beta)$. Set

$$Y = \begin{pmatrix} X(t) \\ U(t) \end{pmatrix} = \begin{pmatrix} X_1(t) + X_2(t) \\ U_1(t) + U_2(t) \end{pmatrix}.$$

By taking $\beta = 0$ we find that Y is a self-conjugate solution of (2.5) and $X(\alpha) = X_2(\alpha)$. Moreover note that $\{Y_2, Y\} = I_n$. By Lemma 2.4 we have

$$X(t) = X_2(t)(I_n + \int_\alpha^t X_2^{-1}(r)B(r)X_2^{*-1}(r)dr), \quad t \in [\alpha, \beta).$$

Since $B(t)$ is semi-positive definite a.e. on $[\alpha, \beta]$, it follows that $X(t)$ is nonsingular for $t \in [\alpha, \beta)$. Also note that $X(\beta) = X_1(\beta)$. We obtain $\operatorname{rank}(X(t)) = n$ on $[\alpha, \beta] \subset I$. This completes the proof. □

Note that the property of disconjugacy is λ-dependent. See also the next section.

3. Boundedness below and principal solutions

In this section we discuss the essential relation between the boundedness below of minimal operators associated with the differential equation (1.3) and the existence of principal solutions of system (2.5). For an excellent presentation of the relation between the associated quadratic form and boundedness below also see [21, 22].

We now consider the functional for $[\alpha,\beta]\subset I$

$$J(\mathbf{x}_1,\mathbf{x}_2;\alpha,\beta):=\int_\alpha^\beta \mathbf{u}_2^*B(t)\mathbf{u}_1dt+\int_\alpha^\beta \mathbf{x}_2^*C(t)\mathbf{x}_1dt-\int_\alpha^\beta \mathbf{x}_2^*\lambda W_1(t)\mathbf{x}_1dt, \tag{3.1}$$

where n-dimensional vector functions $\mathbf{x}_j,\mathbf{u}_j, j=1,2$ belong locally to class $L^\infty(I)$. If these vector functions satisfy

$$\tilde{C}_n^*\mathbf{x}_j'-A(t)\mathbf{x}_j-B(t)\mathbf{u}_j=0,\quad j=1,2, \tag{3.2}$$

then we have the relation

$$J(\mathbf{x}_1,\mathbf{x}_2;\alpha,\beta)=\mathbf{x}_2^*\tilde{C}_n\mathbf{u}_1|_\alpha^\beta+\int_\alpha^\beta \mathbf{x}_2^*L_1\{\mathbf{x}_1,\mathbf{u}_1\}dt. \tag{3.3}$$

Moreover note that for $J(\mathbf{x}_1,\mathbf{x}_2;\alpha,\beta)=J^*(\mathbf{x}_2,\mathbf{x}_1;\alpha,\beta)$ we have

$$\begin{aligned}&\int_\alpha^\beta \overline{y}_2My_1dt-\int_\alpha^\beta y_1\overline{My_2}dt\\&=\mathbf{x}_2^*(\beta)\tilde{C}_n\mathbf{u}_1(\beta)-\mathbf{u}_2^*(\beta)\tilde{C}_n^*\mathbf{x}_1(\beta)-\mathbf{x}_2^*(\alpha)\tilde{C}_n\mathbf{u}_1(\alpha)+\mathbf{u}_2^*(\alpha)\tilde{C}_n^*\mathbf{x}_1(\alpha),\end{aligned} \tag{3.4}$$

where

$$\mathbf{x}_j(t)=(y_jy_j^{[1]}\cdots y_j^{[n-1]})^T,\quad \mathbf{u}_j(t)=(y_j^{[2n-1]}y_j^{[2n-2]}\cdots y_j^{[n]})^T,\quad j=1,2.$$

Thus we obtain

$$[y_1,y_2]_\alpha^\beta=\mathbf{x}_2^*(\beta)\tilde{C}_n\mathbf{u}_1(\beta)-\mathbf{u}_2^*(\beta)\tilde{C}_n^*\mathbf{x}_1(\beta)-\mathbf{x}_2^*(\alpha)\tilde{C}_n\mathbf{u}_1(\alpha)+\mathbf{u}_2^*(\alpha)\tilde{C}_n^*\mathbf{x}_1(\alpha),$$

where

$$[y_1,y_2](t)=\mathbf{y}_2^*\tilde{J}_{2n}\mathbf{y}_1, y_1,y_2\in D_Q \tag{3.5}$$

is a Lagrange sesquilinear form.

We abbreviate $J(\mathbf{x}_1,\mathbf{x}_1;\alpha,\beta)$ to $J(\mathbf{x}_1;\alpha,\beta)$ in the following assertions. Define $X^{-1}\mathbf{x}_j:=\mathbf{z}_j, j=1,2$, where $\mathbf{z}_j$ belongs to locally class of $L^\infty(I)$ and $Y(t)=\begin{pmatrix}X(t)\\U(t)\end{pmatrix}$ is a conjoined basis for (2.3) with $\operatorname{rank}(X)=n$. In this case we have

$$\begin{aligned}&J(\mathbf{x}_1,\mathbf{x}_2;\alpha,\beta)\\
&=\int_\alpha^\beta(\mathbf{u}_2-U\mathbf{z}_2)^*B(t)(\mathbf{u}_1-U\mathbf{z}_1)dt+\int_\alpha^\beta \mathbf{x}_2^*(C(t)-\lambda W_1(t))\mathbf{x}_1dt\\
&\quad+\int_\alpha^\beta \mathbf{u}_2^*B(t)U\mathbf{z}_1dt+\int_\alpha^\beta \mathbf{z}_2^*U^*B(t)\mathbf{u}_1dt-\int_\alpha^\beta \mathbf{z}_2^*U^*B(t)U\mathbf{z}_1dt\\
&=\int_\alpha^\beta(\mathbf{u}_2-U\mathbf{z}_2)^*B(t)(\mathbf{u}_1-U\mathbf{z}_1)dt+\int_\alpha^\beta \mathbf{x}_2^*(C(t)-\lambda W_1(t))\mathbf{x}_1dt\\
&\quad+\int_\alpha^\beta(\tilde{C}_n\mathbf{x}_2'-A(t)\mathbf{x}_2)^*U\mathbf{z}_1dt+\int_\alpha^\beta \mathbf{z}_2^*U^*(\tilde{C}_n\mathbf{x}_1'-A(t)\mathbf{x}_1)dt-\int_\alpha^\beta \mathbf{z}_2^*U^*BU\mathbf{z}_1dt\\
&=\int_\alpha^\beta(\mathbf{u}_2-U\mathbf{z}_2)^*B(t)(\mathbf{u}_1-U\mathbf{z}_1)dt+\int_\alpha^\beta \mathbf{x}_2^*(C(t)-\lambda W_1(t))\mathbf{x}_1dt\\
&\quad+\int_\alpha^\beta \mathbf{z}_2^{*\prime}X^*\tilde{C}_nU\mathbf{z}_1dt+\int_\alpha^\beta \mathbf{z}_2^*U^*\tilde{C}_n^*X\mathbf{z}_1'dt+\int_\alpha^\beta \mathbf{z}_2^*U^*BU\mathbf{z}_1dt\\
&=\int_\alpha^\beta(\mathbf{u}_2-U\mathbf{z}_2)^*B(t)(\mathbf{u}_1-U\mathbf{z}_1)dt+\int_\alpha^\beta \mathbf{x}_2^*(C(t)-\lambda W_1(t))\mathbf{x}_1dt\end{aligned}$$

$$
\begin{aligned}
&+\int_\alpha^\beta \mathbf{z}_2^{*\prime} X^*\tilde{C}_n U\mathbf{z}_1 dt+\int_\alpha^\beta \mathbf{z}_2^* U^*\tilde{C}_n^* X\mathbf{z}_1' dt+\int_\alpha^\beta \mathbf{z}_2^*(\tilde{C}_n^* X'^* U\mathbf{z}_1 dt\\
&=\mathbf{z}_2^* X^*\tilde{C}_n U\mathbf{z}_1|_\alpha^\beta+\int_\alpha^\beta (\mathbf{u}_2-U\mathbf{z}_2)^* B(t)(\mathbf{u}_1-U\mathbf{z}_1)dt,
\end{aligned}
$$

where we used the condition $X^*\tilde{C}_n U = U^*\tilde{C}_n^* X$, (2.6) and (3.2).

Therefore if $y \in D(T_0')$, then for $\mathbf{x}(t) = (yy^{[1]}\cdots y^{[n-1]})^T$, we obtain

$$
J(\mathbf{x};\alpha,\beta)=\int_\alpha^\beta (\mathbf{u}-U\mathbf{z})^* B(t)(\mathbf{u}-U\mathbf{z})dt, \quad \mathbf{x}=X\mathbf{z}. \tag{3.6}
$$

We can easily obtain the following Theorem.

Theorem 3.1. *Suppose that $B(t)$ is semi-positive definite a.e. on $[\alpha,\beta]\subset I$. Then the following statements are equivalent:*

(i) *The pre-minimal operator T_0' is bounded below with a bound γ.*

(ii) *there exists a value $\hat{\lambda}$ such that*

$$
J(\mathbf{x};\alpha,\beta)=\int_\alpha^\beta (My-\hat{\lambda}wy)\overline{y}>0, \mathbf{x}(t)=(yy^{[1]}\cdots y^{[n-1]})^T, y\in D(T_0'). \tag{3.7}
$$

(iii) *there exists a self-conjugate solution $\mathbf{y}(t)=\begin{pmatrix}\mathbf{x}(t)\\ \mathbf{u}(t)\end{pmatrix}$ of (2.3) with $\mathbf{x}(t)\not\equiv 0$ for all $t\in[\alpha,\beta]\subset I$.*

(iv) *(1.3) is disconjugate or non-oscillatory on $[\alpha,\beta]$ for all $\lambda<\gamma$, where γ is a lower bound of T_0'.*

Proof. (i)⇒(ii). If (i) holds, i.e., $(T_0'y,y)_w \ge \gamma(y,y), y\in D(T_0')$ for some $\gamma\in\mathbb{R}$, then $((\frac{1}{w}M-\gamma)y,y)_w\ge 0$ for all $y\in D(T_0')$. Hence for $\lambda<\gamma$ we have $((\frac{1}{w}M-\lambda)y,y)_w > ((\frac{1}{w}M-\gamma)y,y)_w \ge 0$. Then there exists a value $\hat{\lambda}$, for $\hat{\lambda}<\lambda$ we know that $J(\mathbf{x};\alpha,\beta)=\int_\alpha^\beta \mathbf{x}^* L_1\{\mathbf{x},\mathbf{u}\}=\int_\alpha^\beta (My-\hat{\lambda}wy)\overline{y}>0$ holds. Therefore (ii) holds.

(ii)⇒(iii). Let (ii) hold. Assume that $\mathbf{x}=\mathbf{x}_1+\mathbf{x}_2$, $\mathbf{u}=\mathbf{u}_1+\mathbf{u}_2$, where $\mathbf{y}_j=\begin{pmatrix}\mathbf{x}_j\\ \mathbf{u}_j\end{pmatrix}$, $j=1,2$ are solutions of (2.3) satisfying $\mathbf{x}_1(\alpha)=\mathbf{x}_2(\beta)=0$ and associated function $\mathbf{u}_1(\alpha)=1$, $\mathbf{u}_2(\beta)=-\tilde{C}_n^*\mathbf{x}_1^{*-1}(\beta)$. Clearly $\{\mathbf{y}_1,\mathbf{y}_2\}=1$ for subinterval of (α,β) throughout $\mathbf{x}_1\not\equiv 0$. Since $\mathbf{y}_1,\mathbf{y}_2$ are self-conjugate solutions, $\mathbf{y}=\begin{pmatrix}\mathbf{x}\\ \mathbf{u}\end{pmatrix}=\begin{pmatrix}\mathbf{x}_1+\mathbf{x}_1\\ \mathbf{u}_1+\mathbf{u}_2\end{pmatrix}$ is a self-conjugate solution of (2.3). Also note that $Y(t)=\begin{pmatrix}X(t)\\ U(t)\end{pmatrix}\in M_{2n,n}(L^1_{\rm loc}(I))$ is a matrix solution of (2.5) if and only if $\mathbf{x}(t)=X\mathbf{c},\mathbf{u}(t)=U\mathbf{c}$ is a vector solution of (2.3) for every nonzero constant vector $\mathbf{c}\in\mathbb{C}^n$. By Lemma 2.5 we know that $\mathbf{x}(t)=X\mathbf{c}$ is invertible on $[\alpha,\beta]\subset I$. Thus we have $\mathbf{x}(t)\not\equiv 0$ for $t\in[\alpha,\beta]$. Therefore (iii) holds.

(iii)⇒(iv). Let (iii) hold. It is clear from Lemma 2.5 that system (2.3) is disconjugate. Hence (1.3) is disconjugate. Therefore (iv) holds.

(iv)⇒(i). Let (iv) hold. Since (1.3) can be transformed into the Hamiltonian system (2.3), there exists a conjoined basis for (2.3) with $\operatorname{rank}(X)=n$ and thus (3.6) holds. By the assumption $B(t)\ge 0$ a.e. on $[\alpha,\beta]\subset I$ we have $J(\mathbf{x};\alpha,\beta)>0$, i.e., (ii) holds. Suppose now that $((\frac{1}{w}M-\lambda)y,y)_w<0$ for $y\in D(T_0')$, i.e., $((\frac{1}{w}M-\gamma)y,y)_w<0$. From (3.3) we have $J(\mathbf{x};\alpha,\beta)=\int_\alpha^\beta \mathbf{x}^* L_1\{\mathbf{x},\mathbf{u}\}=\int_\alpha^\beta \overline{y}(My-\lambda wy)<$

0 for $y \in D(T_0')$ on $[\alpha, \beta]$. This reaches a contradiction. Therefore (i) holds. This completes the proof. $\square$

Remark 3.2. It is easy to show that if $\mathbf{y}_1 = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{u}_1 \end{pmatrix}$ is a solution of (2.3) and $y_2 \in D(T_0')$ on $[\alpha, \beta]$, then $J(\mathbf{x}_1, \mathbf{x}_2; \alpha, \beta) = 0$. From Theorem 3.1 and Definition 2.1 we know that if (2.3) is disconjugate on I, then

$$J(\mathbf{x}; a, b) \geq 0, \quad \mathbf{x}(t) = (yy^{[1]} \cdots y^{[n-1]})^T, \quad y \in D(T_0'),$$

and the equality sign holds if and only if there exists a n-dimensional vector function $\mathbf{u}_1$ such that $\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{u}_1 \end{pmatrix}$ is a solution of (2.3) and $\mathbf{x} = k\mathbf{x}_1, k \in \mathbb{C}, \mathbf{x}_1(t) = (y_1 y_1^{[1]} \cdots y_1{}^{[n-1]})^T$, $y_1(a) = 0 = y_1(b)$, and $y_1(t) \not\equiv 0$, $t \in I$.

Let $B(t) \geq 0$ a.e. on I. Suppose that $Y = \begin{pmatrix} X(t) \\ U(t) \end{pmatrix}$ is a conjoined basis for (2.3) with $\operatorname{rank}(X) = n$ on $s \leq t < b$, $s \in I$. Clearly

$$S_s(t) := \int_s^t X^{-1}(r) B(r) X^{*-1}(r) dr \tag{3.8}$$

is positive for $s < t < b$ and

$$\lim_{t \to b} S_s^{-1}(t) = \Phi \quad \text{exists}, \tag{3.9}$$

where Φ depends on s and the matrix function $X(t)$.

If $\Phi = 0$, then $Y(t)$ is called a principal solution (or recessive solution) of (2.5). See [6]. In this case $\Phi = 0$ can be expressed as

$$\int_s^t X^{-1}(r) B(r) X^{*-1}(r) dr \to +\infty \quad \text{as } t \to b, \tag{3.10}$$

in the sense that

$$\| \int_s^t X^{-1}(r) B(r) X^{*-1}(r) dr \mathbf{c} \| \to +\infty \quad \text{as } t \to b,$$

uniformly for all nonzero constant vectors $\mathbf{c} \in \mathbb{C}^n$. For $n = 1$ this is corresponding to [5, Lemma 4] with $B = -\bar{c}_{12} q_{12}$ and [18, Thereom 2.2] with $B = p^{-1}$.

Similar with Reid's concept [22] on the principal solution of Hamiltonian system, we have the following Theorem.

Theorem 3.3. *Suppose that for some $\lambda \in \mathbb{R}$ and some $e \in (a, b)$ system (2.5) is disconjugate on an interval (e, b). Then for each $s \in (e, b)$, let $Y_s = \begin{pmatrix} X_s \\ U_s \end{pmatrix} \in M_{2n,n}(L^1_{\mathrm{loc}}(I))$ be a matrix solution of (2.5) satisfying the boundary condition $X_s(e) = \tilde{C}_n, X_s(s) = 0$. Moreover*

$$Y_b(t) := \lim_{s \to b} Y_s(t) := \begin{pmatrix} X_b \\ U_b \end{pmatrix} \tag{3.11}$$

exists, uniformly for t in compact subset of $[e, b)$, and is a principal solution of system (2.5) at b. In particular, $Y_b(t)$ is a conjoined basis for (2.3) on (e, b). The principal solution at a is described in a similar manner.

It is readily known that the following Theorem holds.

Theorem 3.4. *Suppose that $B(t)$ is semi-positive definite a.e. on $I = (a, b)$, and (2.3) is disconjugate on I. Then*

(i) *(2.5) possesses a principal solution $Y = \begin{pmatrix} X(t) \\ U(t) \end{pmatrix}$ at b.*

(ii) *Another solution $Y_0 = \begin{pmatrix} X_0(t) \\ U_0(t) \end{pmatrix}$ is also a principal solution at b if and only if*
$$X_0 = XK_1, \quad U_0 = UK_1,$$
where K_1 is a constant nonsingular matrix.

(iii) *Let $Y = \begin{pmatrix} X(t) \\ U(t) \end{pmatrix}$ be a conjoined basis for (2.3) with $\det(X(t)) \neq 0$ for t near b. Assume that $Y_1 = \begin{pmatrix} X_1(t) \\ U_1(t) \end{pmatrix}$ is any solution of (2.5), which is not multiple of Y and $\det(X_1(t)) \neq 0$ for t near b. Then Y is a principal solution at b if and only if $K_0 = \{Y_1, Y\}$ is nonsingular and*
$$X_1^{-1}(t)X(t) \to 0, \quad \text{as } t \to b. \tag{3.12}$$

Samiolar argument holds for the principal solution at a.

Proof. (i). Since $B(t)$ is semi-positive definite a.e. on $I = (a, b)$ and (2.3) is disconjugate on I, from Lemma 2.5 we know that there exists a conjoined basis $Y_0 = \begin{pmatrix} X_0(t) \\ U_0(t) \end{pmatrix}$ such that $\operatorname{rank}(X_0(t)) = n$ on I and the symmetric matrix
$$S_0(t) := \int_s^t X_0^{-1}(r)B(r)X_0^{*-1}(r)dr$$
is positive increasing function, and hence invertible for $t > s$. Thus it follows that its inverse is a decreasing function of t and there exists a symmetric matrix $\Phi_0 \geq 0$ such that
$$S_0^{-1}(t) \to \Phi_0, \quad s \to b.$$
Set
$$X(t) = X_0(t)(I_n - S_0(t)\Phi_0),$$
$$U(t) = U_0(t)(I_n - S_0(t)\Phi_0) - \tilde{C}_n^* X_0^{*-1}(t)\Phi_0.$$
From Lemma 2.4 we know that $Y = \begin{pmatrix} X(t) \\ U(t) \end{pmatrix}$ is a conjoined basis for (2.3). Moreover $\operatorname{rank}(X) = n$ for every $t \geq s$, since $X(s) = X_0(s)$ and for $t > s$,
$$X(t) = X_0(t)S_0(t)(S_0^{-1}(t) - \Phi_0)$$
is a product of nonsingular matrices. Note that
$$S_s^{-1}(t) = S_0^{-1}(t) - \Phi_0.$$
We obtain that as $s \to b$, $S_s^{-1}(t) \to 0$.

From (3.10) we know that $Y(t)$ is a principal solution at b, i.e., (i) holds. And for any constant matrix K, note that
$$(S_s(t) + K)^{-1} = S_s^{-1}(t)(I_n + KS_s^{-1}(t))^{-1} \to 0.$$
We know that there still exists a principal solution y at b if we change the lower limit integration in $\int_s^t X^{-1}(r)B(r)X^{*-1}(r)dr$ from s to $t_0 \in I$.

(ii) Suppose that

$$S_0^{-1}(t) = \Big(\int_s^t X_0^{-1}(r)B(r)X_0^{*-1}(r)dr\Big)^{-1} \to \Phi_0, \quad s \to b.$$

In accord with (3.10), Y_0 is a principal solution if and only if $\Phi_0 = 0$. From (2.9) in Lemma 2.4 we know that

$$X_0(t) = X(t)H(t), H(t) = K_1 - S_s(t)K_0,$$
$$U_0(t) = \tilde{C}_n^* X^{*-1}K_0 + U(t)H(t),$$

where $Y = \begin{pmatrix} X(t) \\ U(t) \end{pmatrix}$ is a principal solution, $S_s(t)$ is described by (3.8) and the constants K_1, K_0 are the same as in Lemma 2.4. If Y_0 is a principal solution, then Y is also given by Y_0, i.e., (2.10)) holds. Thus we have

$$(K_1 - S_s(t)K_0)(K_1^{-1} + S_0(t)K_0^*) = I_n,$$

i.e.,

$$(K_1 - S_s(t)K_0)S_0(t)K_0^* = S_s(t)K_0K_1^{-1}.$$

Also from Lemma 2.4 we have $K_0^*K_1 = K_0K_1^*$, i.e., $K_1K_0^{-1} = K_0^{*-1}K_1^*$. Hence

$$(K_1 - S_s(t)K_0)S_0(t) = S_s(t)K_1^{*-1},$$

viz.,

$$S_0^{-1}(t) = K_1^*(S_s^{-1}(t)K_1 - K_0).$$

From this and the fact that $S_s^{-1}(t) \to 0, s \to b$, we obtain $\Phi_0 = -K_1^*K_0$. Note that K_1 is nonsingular. $\Phi_0 = 0$ if and only if $K_0 = 0$, i.e., if and only if

$$X_0(t) = X(t)K_1, U_0(t) = U(t)K_1.$$

So result (ii) follows.

(iii) Assume that $Y_1 = \begin{pmatrix} X_1(t) \\ U_1(t) \end{pmatrix}$ is any solution of (2.5), which is not multiple of Y and $\det(X_1(t)) \neq 0$ for t near b. Suppose that (3.12) holds. From (2.9) in Lemma 2.4 we know that

$$(K_1 - S_s(t)K_0)^{-1} = X_1^{-1}(t)X(t) \to 0, \quad s \to b,$$

where $K_0 = \{Y_1, Y\}$ and $K_1 = X^{-1}(s)X_1(s)$ is invertible if s is sufficiently near b. Together with (3.10) we obtain that Y is a principal solution at b. Conversely, suppose that Y is a principal solution at b and $K_0 \neq 0$. From (3.10) it follows that

$$\lim_{s\to b} S_s^{-1}(t) = 0.$$

This and (2.9) in Lemma 2.4 it yield

$$X_1^{-1}(t)X(t) = (K_1 - S_s(t)K_0)^{-1} = (S_s^{-1}(t)K_1 - K_0)^{-1}S_s^{-1}(t) \to 0, \quad s \to b.$$

The same argument holds for principal solution at a. This completes the proof. □

We can now assert that there are precisely n principal solutions y_j, $j = 1, \dots, n$, at the endpoints for the differential equation (1.3) by the Hamiltonian system (2.3). For example, the j-th principal solution y_j at b of (1.3) is the j-th column of principal solution Y_b for (2.5), it is given by

$$y_j(t) = \lim_{s\to b} y_{js}(t), \tag{3.13}$$

and $y_{js}(t)$ is the solution of boundary value problem

$$My=\lambda wy,\quad t\in(e,s),\quad \mathbf{x}(e)=(0\cdots 0\underbrace{c_{j,2n+1-j}}_{j-\text{th}}0\cdots 0)^T,\quad \mathbf{x}(s)=\mathbf{0},$$

where $c_{j,2n+1-j}$ is the j row $2n+1-j$ column element of $C_{2n}=(c_{r,s})_{r,s=1}^{2n}$ and $\mathbf{x}$ belongs to $\mathbf{y}$, i.e., $\mathbf{y}=\begin{pmatrix}\mathbf{x}\\ \mathbf{u}\end{pmatrix}$, which is a solution of Hamiltonian system (2.3).

4. Singular symmetric differential operators

In this section we present the explicit characterization of symmetric operators generated by the differential expression (1.3). Assume that both endpoints a,b of I are singular for (1.3). Assume that d_a,d_b denote the deficiency indices of the minimal operator generated by the equation (1.3) on (a,e) and (s,b), respectively. Clearly d_a is also equivalent to the deficiency indices of the minimal operator generated by the equation (1.3) on (a,s). Suppose that d denotes the deficiency indices of the minimal operator generated by the equation (1.3) on I. Recall that $d=d_a+d_b-2n$. Now we fix a number $\hat\lambda\in\mathbb{R}$ and assume that there exists a $\lambda_a<\hat\lambda\in\mathbb{R}$ such that (1.3) has d_a linearly independent solutions near a in H. We denote them as

$$y_{11},y_{12},\dots,y_{1,d_a-n},y_{1,d_a-n+1},\dots,y_{1,n},y_{1,n+1},\dots,y_{1,d_a}. \tag{4.1}$$

Now suppose that in (4.1) the functions $y_{11},y_{12},\dots,y_{1,d_a-n},y_{1,n+1},\dots,y_{1,d_a}$ are LC type solutions on (a,e) and these $m_a:=2d_a-2n$ solutions can contribute to the symmetric and self-adjoint boundary conditions. For the details of LC type solutions of differential equations, see [11, 25, 26, 30]. Assume that

$$\begin{pmatrix}[y_{11},y_{11}](a) & \cdots & [y_{11},y_{1,m_a}](a)\\ \vdots & \dots & \vdots\\ [y_{1,m_a},y_{11}](a) & \cdots & [y_{1,m_a},y_{1,m_a}](a)\end{pmatrix}=C_{m_a}, \tag{4.2}$$

where $y_{1,d_a-n+j}=y_{1,n+j}$, $1\le j\le d_a-n$, and

$$C_{m_a}=\begin{pmatrix}0 & \hat C_{d_a-n}\\ -\hat C^*_{d_a-n} & 0\end{pmatrix} \tag{4.3}$$

satisfies the properties (1.1), that is,

$$\begin{gathered}c_{rs}\bar c_{rs}=1,\quad \text{for } r+s=m_a+1, 1\le r\le d_a-n,\\ c_{rs}=0,\quad \text{otherwise.}\end{gathered} \tag{4.4}$$

Therefore the matrix $([y_{1i},y_{1j}](a))_{1\le i,j\le m_a}$ is nonsingular.

Similarly, assume that there exists a $\lambda_b<\hat\lambda\in\mathbb{R}$ such that (1.3) has d_b linearly independent solutions near b in H. We denote them as

$$y_{21},y_{22},\dots,y_{2,d_b-n},y_{2,d_b-n+1},\dots,y_{2,n},y_{2,n+1},\dots,y_{2,d_b}. \tag{4.5}$$

Suppose that in (4.5) the functions $y_{21},y_{22},\dots,y_{2,d_b-n},y_{2,n+1},\dots,y_{2,d_b}$ are LC type solutions on (e,b) and these $m_b:=2d_b-2n$ LC type solutions at b satisfy

$$\begin{pmatrix}[y_{21},y_{21}](b) & \cdots & [y_{21},y_{2,m_b}](b)\\ \vdots & \dots & \vdots\\ [y_{2,m_b},y_{21}](b) & \cdots & [y_{2,m_b},y_{2,m_b}](b)\end{pmatrix}=C_{m_b}, \tag{4.6}$$

where $y_{2,d_b-n+j} = y_{2,n+j}, 1 \le j \le d_b - n$, and

$$C_{m_b} = \begin{pmatrix} 0 & \hat{C}_{d_b-n} \\ -\hat{C}^*_{d_b-n} & 0 \end{pmatrix}. \tag{4.7}$$

Here $\hat{C}_{d_b-n} \in M_{d_b-n}(\mathbb{C})$ is a skew-diagonal unitary matrix defined by (4.4) in similar way, i.e., C_{m_b} satisfies properties (1.1), and thus the matrix $([y_{2i}, y_{2j}](b))_{1\le i,j\le m_b}$ is nonsingular. Therefore these m_b solutions can contribute to the symmetric and self-adjoint boundary conditions at b.

Notice that we have the decomposition

$$\begin{aligned} D_Q = D_0 &\oplus \operatorname{span}\{y_{11}, y_{12}, \dots, y_{1,d_a-n}, y_{1,n+1}, \dots, y_{1,d_a}\} \\ &\oplus \operatorname{span}\{y_{21}, y_{22}, \dots, y_{2,d_b-n}, y_{2,n+1}, \dots, y_{2,d_b}\}. \end{aligned}$$

We introduce the following notation:

$$\begin{gathered} Y_{d_a-n}(a) = ([y, y_{11}](a) \cdots [y, y_{1,d_a-n}](a))^T \in \mathbb{C}^{d_a-n}, \\ \hat{Y}_{d_a-n}(a) = ([y, y_{1,d_a-n+1}](a) \cdots [y, y_{1,m_a}](a))^T \in \mathbb{C}^{d_a-n}, \\ Y_{d_b-n}(b) = ([y, y_{21}](b) \cdots [y, y_{2,d_b-n}](b))^T \in \mathbb{C}^{d_b-n}, \\ \hat{Y}_{d_b-n}(b) = ([y, y_{2,d_b-n+1}](b) \cdots [y, y_{2,m_b}](b))^T \in \mathbb{C}^{d_b-n}. \end{gathered}$$

Assume that

$$Y_{a,b} = \begin{pmatrix} Y_{d_a-n}(a) \\ \hat{Y}_{d_a-n}(a) \\ Y_{d_b-n}(b) \\ \hat{Y}_{d_b-n}(b) \end{pmatrix} = \begin{pmatrix} Y_a \\ Y_b \end{pmatrix}, \quad \hat{Y}_{a,b} = \begin{pmatrix} Y_{d_a-n}(a) \\ Y_{d_b-n}(b) \\ \hat{Y}_{d_a-n}(a) \\ \hat{Y}_{d_b-n}(b) \end{pmatrix} \in \mathbb{C}^{2d}. \tag{4.8}$$

It is known [26] that $Y_{a,b}$ runs through the entire space $\mathbb{C}^{2d}$ as y runs through the maximal domain D_Q. For a given matrix $U \in M_{l,2d}(\mathbb{C})$ with $\operatorname{rank}(U) = l, 0 \le l \le 2d$, if an operator $T = T(U)$ is defined by

$$\begin{gathered} T(U)y = T_Q y, \\ y \in D(T(U)) = \{y \in D_Q : UY_{a,b} = 0\}, \end{gathered} \tag{4.9}$$

then U is a boundary matrix of the operator T and $UY_{a,b} = 0$ is its boundary condition. It is obvious that when $l = 0$ we have $\operatorname{rank}(U) = 0$ and $T = T_Q$ and when $l = 2d$ we have $\operatorname{rank}(U) = 2d$ and $T = T_0$. Also it is clear that, for any nonsingular matrix G of order l, the operators $T(M)$ and $T(GM)$, corresponding to the boundary matrices M and GM respectively, are the same.

For which matrices U is $T = T(U)$ a symmetric operator in $L^2_w(I)$? This question is answered by the next Theorem.

Theorem 4.1. *Suppose that $M = M_Q$, $Q \in Z_{2n}(I)$, $I = (a, b)$, $-\infty \le a < b \le \infty$, is C-symmetric, w is a weight function. Let $U = (AB)$, $A \in M_{l,m_a}(\mathbb{C})$, $B \in M_{l,m_b}(\mathbb{C})$ be a boundary condition matrix with* $\operatorname{rank}(U) = l$, $0 \le l \le 2d$ *of* (4.7)*) and let*

$$R = AC_{m_a}A^* - BC_{m_b}B^*, \quad r = \operatorname{rank}(R).$$

Then we have

1. *If $l < d$, then $T(U)$ is not symmetric.*
2. *If $l = d$, then $T(U)$ is self-adjoint (and hence also symmetric) if and only if $r = 0$.*
3. *Let $l = d + s$, $0 < s \le d$. Then $T(U)$ is symmetric if and only if $r = 2s$.*

For a proof of the above theorem see [26, Theorems 6 and 11]. From the above Theorem if U satisfies $\operatorname{rank}(UGU^*) = 2(l-d)$, $d < l \le 2d$ with

$$G = \begin{pmatrix} C_{m_a} & 0 \\ 0 & -C_{m_b} \end{pmatrix}, \tag{4.10}$$

then the operator $T = T(U)$ defined by (4.9) is symmetric. Now we write

$$U := VJ, \quad V = (V_1 V_2), \quad V_j \in M_{l,d}(\mathbb{C}), \quad j = 1, 2,$$

in (4.9), and set

$$\hat{G} = JGJ^*, \tag{4.11}$$

with

$$J = \begin{pmatrix} I_{d_a-n} & 0 & 0 & 0 \\ 0 & 0 & I_{d_b-n} & 0 \\ 0 & I_{d_a-n} & 0 & 0 \\ 0 & 0 & 0 & I_{d_b-n} \end{pmatrix},$$

where I_{d-n} denotes the $(d-n)\times(d-n)$ identity matrix. Also $\mathcal{N}(V)$ denotes the null space of the matrix V and $\mathcal{R}(V)$ denotes the range of the matrix V.

Note that C_{m_a} and C_{m_b} have the form (4.3) and (4.7), respectively. We have

$$\hat{G} = \begin{pmatrix} 0 & G_1 \\ -G_1^* & 0 \end{pmatrix}, \quad G_1 = \begin{pmatrix} \hat{C}_{d_a-n} & 0 \\ 0 & -\hat{C}_{d_b-n} \end{pmatrix}.$$

Remark 4.2. Here $\hat{G}$ satisfies $\hat{G}^{-1} = -\hat{G} = \hat{G}^*$.

Following are our new characterizations of symmetric operators for symmetric differential equation (1.3) we have the following result.

Lemma 4.3. *Suppose $d \le l \le 2d$, where d denotes the deficiency indices of the minimal operator on $I = (a, b)$. Then the operator T which is defined on*

$$D(T) = \{y \in D_Q : V\hat{Y}_{a,b} = 0, V \in M_{l,2d}(\mathbb{C})\}, \tag{4.12}$$

is a symmetric operator with l dimensional restriction of T_Q if and only if there exists a matrix $N \in M_{(2d-l),2d}(\mathbb{C})$ satisfying

$$\operatorname{rank}(N) = 2d - l, \quad N\hat{G}N^* = 0, \tag{4.13}$$

and V is a complete solution of the matrix equation

$$NV^* = 0, \tag{4.14}$$

i.e., V satisfies the equation (4.14) with $\operatorname{rank}(V) = l$. Moreover, the domain of its adjoint operator T^* is characterized by

$$D(T^*) = \{y \in D_Q : N\hat{G}\hat{Y}_{a,b} = 0\}, \tag{4.15}$$

where $\hat{G} \in M_{2d}(\mathbb{C})$ is defined by (4.11).

Proof. Assume that $N \in M_{(2d-l),2d}(\mathbb{C})$ satisfies (4.13) and (4.14). Now we show that T defined by (4.12) is symmetric. Note that $T(U) = T(VJ)$ with $U = VJ$ and $J\mathcal{N}(U) = \mathcal{N}(V)$, from [26, Lemma 14] and Theorem 4.1 we only need to prove that $\mathcal{N}(V) \subset \mathcal{R}(\hat{G}V^*)$, i.e.,

$$\hat{Z}_{a,b}^* \hat{G}\hat{Y}_{a,b} = 0$$

for all $y, z \in D(T)$ with

$$\hat{Y}_{a,b} = \begin{pmatrix} Y_{d_a-n}(a) \\ Y_{d_b-n}(b) \\ \hat{Y}_{d_a-n}(a) \\ \hat{Y}_{d_b-n}(b) \end{pmatrix}, \quad \hat{Z}_{a,b} = \begin{pmatrix} Z_{d_a-n}(a) \\ Z_{d_b-n}(b) \\ \hat{Z}_{d_a-n}(a) \\ \hat{Z}_{d_b-n}(b) \end{pmatrix}.$$

If $y, z \in D(T)$, in view of (4.14) there exists a column vector $\mathbf{c} \in \mathbb{C}^{2d-l}$ such that $\hat{Z}_{a,b} = N^*\mathbf{c}$ and a column vector $\hat{\mathbf{c}} \in \mathbb{C}^{2d-l}$ such that $\hat{Y}_{a,b} = N^*\hat{\mathbf{c}}$. This yields

$$\hat{Z}^*_{a,b}\hat{G}\hat{Y}_{a,b} = \mathbf{c}^*(N\hat{G}N^*)\hat{\mathbf{c}} = 0. \tag{4.16}$$

Since V is a complete solution of matrix equation (4.14), it follows that $\operatorname{rank}(V) = l$. From Von Neumann Theorem in [26] we also see that T is a l dimensional restriction of the maximal operator T_Q. Clearly the converse also holds. In fact, if T defined by (4.12) is symmetric, then from Theorem 4.1 we obtain that $\operatorname{rank}(UGU^*) = 2(l-d)$, $d \le l \le 2d$, with $U = VJ$. From Naimark Patching Lemma [26, Lemma 6] and (4.16) we know that there exists a matrix $N \in M_{(2d-l),2d}(\mathbb{C})$ such that (4.13) is establish. Moreover 4.14 also holds.

Next we prove that (4.15) holds. Note that

$$D_0 \subseteq D(T) \subseteq D(T^*) \subseteq D_Q.$$

since T is an l dimensional restriction of the maximal operator T_Q, this shows that the deficiency index of T is $(l-d)$ and, therefore, T^* is a $2d-l$ dimensional restriction of the maximal operator T_Q. On the other hand, we obtain

$$0 = (Ty, z)_w - (y, T^*z)_w = Z^*_{a,b}GY_{a,b} = \hat{Z}^*_{a,b}\hat{G}\hat{Y}_{a,b},$$

where $y \in D(T)$ and $z \in D(T^*)$. It should be noted that, for any $\mathbf{c} \in \mathbb{C}^{2d-l}$, there exists a function $y \in D(T)$ such that $\hat{Y}_{a,b} = N^*\mathbf{c}$. It leads to $(N\hat{G}^*)\hat{Z}^*_{a,b} = 0$ if $z \in D(T^*)$. By the fact $\operatorname{rank}(N\hat{G}^*) = 2d-l$, we know that the dimension of the space solutions of equation $(N\hat{G}^*)\hat{Z}^*_{a,b} = 0$ is l. Therefore, combining this with $\hat{G}^* = -\hat{G}$, we obtain that (4.15) holds. The proof is complete. □

Remark 4.4. Suppose that $B(t)$ is semi-positive definite a.e. on I and system (2.3) is disconjugate on I for some $\lambda < \gamma$, where γ is a lower bound of T_0'. Let

$$\mathbf{x}_j(t) = (y_{1j}y_{1j}^{[1]} \cdots y_{1j}^{[n-1]})^T, \quad \mathbf{u}_j(t) = (y_{1j}^{[2n-1]}y_{1j}^{[2n-2]} \cdots y_{1j}^{[n]})^T, \quad 1 \le j \le d_a.$$

Thus system (2.3) has d_a linearly independent solutions $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{d_a}$ near a with $\mathbf{y}_j = \begin{pmatrix} \mathbf{x}_j \\ \mathbf{u}_j \end{pmatrix}$, $1 \le j \le d_a$ in $L^2_W(I)$. Note that a principal solution is an LC type solution of Hamiltonian system. (See also [29] and the references therein.) Set

$$Y(t) = \begin{pmatrix} \tilde{Y}_a(t) & \hat{Y}(t) \end{pmatrix} \in M_{2n,d_a}(L^1_{\rm loc}(I)),$$

where $\tilde{Y}_a(t) = (\mathbf{y}_1\mathbf{y}_2 \cdots \mathbf{y}_n)$ is a principal solution at a of (2.5) such that $\tilde{Y}(s) = \tilde{Y}_e(s) = \begin{pmatrix} \tilde{C}_n \\ 0 \end{pmatrix}$, $\tilde{X}_e(e) = 0$, and $\lim_{e\to a} \tilde{Y}_e(t) = \tilde{Y}_a(t)$. Furthermore $\hat{Y}(t) = (\mathbf{y}_{n+1}\mathbf{y}_{n+2} \cdots \mathbf{y}_{d_a}) \in M_{2n,(d_a-n)}(L^1_{\rm loc}(I))$ satisfies $\hat{Y}(s) = \begin{pmatrix} 0 \\ \hat{U}_n(s) \end{pmatrix}$ with $d_a - n$

rows of $\hat{U}_n(e)$ forming a nonsingular $(d_a-n)\times(d_a-n)$ sub-matrix. We may assume $\hat{U}(s)=\begin{pmatrix}\hat{C}_{d_a-n}\\ \check{U}_n(s)\end{pmatrix}$ with $\check{U}_n$ a $(2n-d_a)\times(d_a-n)$ matrix. Hence we see that for $1\le i\le d_a-n, \mathbf{y}_i(s)=\begin{pmatrix}\mathbf{x}_i(s)\\ 0\end{pmatrix}$ with $\mathbf{x}_i(s)=(0\cdots 0\underbrace{c_{i,2n+1-i}}_{i-\text{th}}0\cdots 0)^T$ and for $1\le j\le d_a-n, \mathbf{y}_{n+j}(s)=\begin{pmatrix}0\\ \mathbf{u}_{n+j}(s)\end{pmatrix}$ with $\mathbf{u}_{n+j}(s)=(0\dots 0\underbrace{c_{j,m_a+1-j}}_{j-\text{th}}0\cdots 0u_{j,1}\cdots u_{j,(2n-d)})^T$. Thus from (3.5) it follows that

$$\mathbf{y}_{n+j}^*\tilde{J}_{2n}\mathbf{y}_i=\mathbf{u}_{n+j}^*\tilde{C}_n^*\mathbf{x}_i=c_{i,m_a+1-i}\delta_{i,d_a-n+1-j},$$

where δ is the Kronecker delta and

$$\mathbf{y}_{n+i}^*\tilde{J}_{2n}\mathbf{y}_{n+j}=0=\mathbf{y}_i^*\tilde{J}_{2n}\mathbf{y}_j.$$

Therefore there are d_a-n principal solutions which are contributed to the symmetric and self-adjoint boundary conditions at a and d_a-n non-principal solutions are contributed to the symmetric and self-adjoint boundary conditions at a. Similar assertions hold at endpoint b. Otherwise the Wronskian matrices (4.2) and (4.6) may be singular.

We decompose $N=(N_1N_2)$ with matrices $N_1,N_2\in M_{(2d-l),d}(\mathbb{C})$. Then

$$N\hat{G}=(-N_2G_1^*N_1G_1).$$

Note that $D(T)\subset D(T^*)$. This implies that $N\hat{G}$ can be represented by a linear combination of row vectors of V. By transformation of rows of V, we can rewrite V as

$$V=\begin{pmatrix}V_{11} & V_{12}\\ -N_2G_1^* & N_1G_1\end{pmatrix}, \tag{4.17}$$

where $V_{11},V_{12}\in M_{(2l-2d),d}(\mathbb{C})$.

Theorem 4.5. *Let T be a symmetric operator as stated in Lemma* (4.3)*. Then V can be represented as*

$$V=\begin{pmatrix}\hat{V}_{11} & \hat{V}_{12}\\ \hat{V}_{21} & 0\\ -\hat{N}_{12}G_1^* & 0\\ -\hat{N}_{22}G_1^* & \hat{N}_{21}G_1\end{pmatrix},\quad \hat{V}_{rs}\in M_{(l-d),d}(\mathbb{C}),\ r,s=1,2; \tag{4.18}$$

with $\operatorname{rank}(\hat{V}_{12})=\operatorname{rank}(\hat{V}_{21})=l-d$, *and* $\hat{N}_{21},\hat{N}_{22}\in M_{r_1,d}(\mathbb{C})$ *with* $\operatorname{rank}(\hat{N}_{21})=r_1=\operatorname{rank}(\hat{N}_{22})$, $r_1\le 2n-l$. *Here* 0 *is a* $(2d-l-r_1)\times d$ *zero matrix and* $\hat{N}_{12}\in M_{(2d-l-r_1),d}(\mathbb{C})$.

Proof. For $N=(N_1N_2)$ with $N_1,N_2\in M_{(2d-l),d}(\mathbb{C})$ being given, if $\operatorname{rank}(N_1)=r_1(\le(2d-l))$, and $B\in M_{(d-r_1),d}(\mathbb{C})$ is a complete solution of the matrix equation $N_1B^*=0$, it is easy to see that $\operatorname{rank}(B)=d-r_1$. Since V is a complete solution of matrix equation $NV^*=0$, it follows that the row vectors of $(B0)$ can be linear expressed by the row vectors of V. This implies that there exists a matrix $(\tilde{V}_{11}\tilde{V}_{12})$ and a nonsingular matrix of order l such that

$$V \to \begin{pmatrix} \tilde{V}_{11} & \tilde{V}_{12} \\ B & 0 \end{pmatrix}, \tag{4.19}$$

with $0 \in M_{(d-r_1),d}$ and $\operatorname{rank}(\tilde{V}_{11}) = \operatorname{rank}(\tilde{V}_{12}) = l - d + r_1$. Here the notation "$\to$" denotes the left multiplication process of a nonsingular matrix.

On the other hand, in view of $\operatorname{rank}(N_1) = r_1$, we obtain by elementary matrix transformation of the rows that

$$N \to \begin{pmatrix} 0 & \hat{N}_{12} \\ \hat{N}_{21} & \hat{N}_{22} \end{pmatrix} = (N_1 N_2),$$

where $\operatorname{rank}(\hat{N}_{12}) = 2d - l - r_1$, $\hat{N}_{21}, \hat{N}_{22} \in M_{r_1,d}(\mathbb{C})$, 0 is a $(2d - l - r_1) \times d$ zero matrix. Moreover, it is easy to see that

$$(-N_2 G_1^* N_1 G_1) = \begin{pmatrix} -\hat{N}_{12} G_1^* & 0 \\ -\hat{N}_{22} G_1^* & \hat{N}_{21} G_1 \end{pmatrix}.$$

Since $N\hat{G}N^* = 0$ and $\operatorname{rank}(\hat{N}_{21}) = r_1$, we obtain that the row vectors of $(-\hat{N}_{12}G_1^* 0)$ and $(-\hat{N}_{22}G_1^*\hat{N}_{21}G_1)$ can be linear expressed by the row vectors of $(B0)$ and $(\tilde{V}_{11}\tilde{V}_{12})$ in (4.19), respectively. This together with (4.17) and (4.19) yields (4.18) and $\operatorname{rank}(\hat{V}_{12}) = l - d$. This also shows that $\operatorname{rank}(\hat{V}_{21}) = l - d$. The proof is complete. □

Remark 4.6. Let $l = 2d$ in Theorem 4.5. In this case N is singular, $T = T_0$, and (4.18) can be reduced to

$$V = \begin{pmatrix} \hat{V}_{11} & \hat{V}_{12} \\ \hat{V}_{21} & 0 \end{pmatrix}, \quad \hat{V}_{rs} \in M_d(\mathbb{C}),\ r, s = 1, 2; \tag{4.20}$$

with $\operatorname{rank}(\hat{V}_{12}) = \operatorname{rank}(\hat{V}_{21}) = d$. Moreover by transformation of rows (4.20) can be transformed into

$$V \to \begin{pmatrix} I_d & 0 \\ 0 & I_d \end{pmatrix} \to \begin{pmatrix} I_{m_a} & 0 \\ 0 & I_{m_b} \end{pmatrix} \to J.$$

Thus in this case Theorem 4.5 can be reduced to [26, Theorem 4.4.1].

Furthermore if $m_a = 2n = m_b$, i.e., $l = 4n$, then we have

$$D(T_0) = \{y \in D_Q : Y_n(a) = Y_n(b) = 0,\ \hat{Y}_n(a) = \hat{Y}_n(b) = 0\},$$

which is the characterization of domain of the minimal operator of (1.3) when a, b are LC endpoints or regular endpoints. In this case from Remark 4.4 we know that all principal solutions at endpoints a, b can contribute to the characterization of boundary conditions of the minimal symmetric operator.

Note that if $B(t)$ be semi-positive definite a.e. on $I = (a, b)$ and (2.3) is disconjugate on I, then Theorem 3.1 and Remark 3.2 ensure that the minimal operator associated with (1.3) is bounded below. So we infer every symmetric extension T of T_0 is also bounded below and thus the Friedrichs extension T_F of T exists. We fix a value of λ, say $\hat{\lambda}$. Let $\hat{\gamma} > 0$ be a lower bound of symmetric operator T. Then

$$\big((T - \hat{\lambda})y, y\big)_w \ge \hat{\gamma}(y, y)_w, \quad \hat{\gamma} \le \gamma, \tag{4.21}$$

where γ is a lower bound of T_0. Define a semi-bounded sesquilinear form on $D(T)$:

$$l(y,z) = ((T-\hat{\lambda})y, z)_w, \quad y,z \in D(T). \tag{4.22}$$

If

$$\langle\cdot,\cdot\rangle_s = \big((T-\hat{\lambda})y, z\big)_w + (1-\hat{\gamma})(y,z)_w, y,z \in D(T),$$

then $(D(T), \langle\cdot,\cdot\rangle_s)$ is a pre-Hilbert space. Let H_s be a $\|\cdot\|_s$ completion of $D(T)$, with $\|\cdot\|_s$ denotes the norm induced by the inner product $\langle\cdot,\cdot\rangle_s$ on H. Then H_s is a Hilbert space. Let

$$\bar{l}(y,z) = ((T-\hat{\lambda})y, z)_w, \quad y,z \in H_s.$$

Consequently $\bar{l}(y,z) = l(y,z)$ for $y,z \in D(T)$ and $\bar{l}$ is the closure of l. Combined with (4.21) we infer that l and $\bar{l}$ are both bounded below by $\hat{\gamma}$. Thus according to Kato's result in [12, p. 352] we have $y \in D(T_F) \subseteq D(\bar{l})$. In particular this implies that for any $y \in D(T_F)$ there exists a sequence $\{y_m\}_{m=1}^{\infty}$ in $D(T)$ such that

$$\lim_{m\to\infty} \bar{l}(y-y_m, y-y_m) = 0. \tag{4.23}$$

In the next section we characterize explicitly the boundary conditions of Friedrichs extensions T_F for any symmetric operators T which are bounded below for (1.3) applying the property (4.23).

5. Characterization of Friedrichs extensions

In this section we find, explicitly, a boundary condition which determines the Friedrichs extension of every symmetric operator which is bounded below. All the symbols and notation in this section are the same as those in Section 4. Next we state and prove our main results.

Theorem 5.1. *Assume that $B(t)$ is semi-positive definite a.e. on $I=(a,b)$ and system (2.3) is disconjugate on I for some λ. Let $y_{1a}, y_{2a}, \dots, y_{d_a-n,a}$ be the principal solutions at a of (1.3) and $v_{1a}, v_{2a}, \dots, v_{d_a-n,a}$ be the non-principal solutions at a of (1.3); Let $y_{1b}, y_{2b}, \dots, y_{d_b-n,b}$ be the principal solutions at b of (1.3) and $v_{1b}, v_{2b}, \dots, v_{d_b-n,b}$ be the non-principal solutions at b of (1.3). Assume that T defined by*

$$D(T) = \{y \in D_Q : V\hat{Y}_{a,b} = 0,\ V \in M_{l,2d}(\mathbb{C})\}, quadd < l \le 2d, \tag{5.1}$$

is a symmetric extension of T_0, where $\hat{Y}_{a,b} = (Y_{d_a-n}(a)Y_{d_b-n}(b)\hat{Y}_{d_a-n}(a)\hat{Y}_{d_b-n}(b))^T$ with

$$\begin{aligned} Y_{d_a-n}(a) &= ([y,y_{1a}](a)\cdots[y,y_{d_a-n,a}](a))^T, \\ \hat{Y}_{d_a-n}(a) &= ([y,v_{1a}](a)\cdots[y,v_{d_a-n,a}](a))^T, \\ Y_{d_b-n}(b) &= ([y,y_{1b}](b)\cdots[y,y_{d_b-n,b}](b))^T, \\ \hat{Y}_{d_b-n}(b) &= ([y,v_{1b]}(b)\cdots[y,v_{d_b-n,b}](b))^T. \end{aligned}$$

Then V has the form (4.18) in Theorem 4.5 and T is bounded below. Furthermore the Friedrichs extension T_F of T is characterized by

$$D(T_F) = \Big\{y \in D(T^*) : \hat{V}_{21}\begin{pmatrix} Y_{d_a-n}(a) \\ Y_{d_b-n}(b)\end{pmatrix} = 0\Big\} \tag{5.2}$$

$$= \big\{ y \in D_Q : \hat V_{21} \begin{pmatrix} Y_{d_a-n}(a) \\ Y_{d_b-n}(b) \end{pmatrix} = 0,\ N\hat G \hat Y_{a,b} = 0 \big\}, \tag{5.3}$$

where $N \in M_{(2d-l),2d}(\mathbb{C})$ satisfying $N\hat G N^* = 0$ is a complete solution of the matrix equation $NV^* = 0$.

Proof. It is clear that T is bounded below from Theorem 3.1. From Theorem 4.5 we obtain the boundary matrix V of T can be rewritten as

$$V = \begin{pmatrix} \hat V_{11} & \hat V_{12} \\ \hat V_{21} & 0 \\ -\hat N_{12} G_1^* & 0 \\ -\hat N_{22} G_1^* & \hat N_{21} G_1 \end{pmatrix}, \quad \hat V_{rs} \in M_{(l-d),d}(\mathbb{C}), \quad r,s = 1,2.$$

with $\operatorname{rank}(\hat V_{12}) = \operatorname{rank}(\hat V_{21}) = l-d$, and

$$\begin{pmatrix} 0 & \hat N_{12} \\ \hat N_{21} & \hat N_{22} \end{pmatrix} = (N_1 N_2) = N,$$

where $\operatorname{rank}(\hat N_{12}) = 2d-l-r_1$, $\hat N_{21}, \hat N_{22} \in M_{r_1,d}(\mathbb{C})$, 0 is a $(2d-l-r_1) \times d$ zero matrix. We define an operator T_s with domain $D(T_s)$:

$$D_s := \big\{ y \in D(T^*) : \hat V_{21} \begin{pmatrix} Y_{d_a-n}(a) \\ Y_{d_b-n}(b) \end{pmatrix} = 0 \big\}.$$

It is easy to ensure that $D(T) \subset D_s \subset D_Q$. Denote by

$$A = \begin{pmatrix} \hat V_{21} & 0 \\ -N_2 G_1^* & N_1 G_1 \end{pmatrix}.$$

Thus $D_s = \{ y \in D_Q : A\hat Y_{a,b} = 0\}$. Since T is symmetric, from Lemma 4.3 we have $NV^* = 0$. It follows from (4.18) that $N_1 \hat V_{21}^* = 0$. Furthermore, by Theorem 4.5 we obtain $\operatorname{rank}(A) = d$. Together with $G_1 G_1^* = I_d$ we have

$$\begin{aligned}
A\hat G A^* &= \begin{pmatrix} \hat V_{21} & 0 \\ -N_2 G_1^* & N_1 G_1 \end{pmatrix} \begin{pmatrix} 0 & G_1 \\ -G_1^* & 0 \end{pmatrix} \begin{pmatrix} \hat V_{21}^* & -G_1 N_2^* \\ 0 & G_1^* N_1^* \end{pmatrix} \\
&= \begin{pmatrix} 0 & \hat V_{21} G_1 \\ -N_1 G_1 G_1^* & -N_2 G_1^* G_1 \end{pmatrix} \begin{pmatrix} \hat V_{21}^* & -G_1 N_2^* \\ 0 & G_1^* N_1^* \end{pmatrix} \\
&= \begin{pmatrix} 0 & \hat V_{21} G_1 G_1^* N_1^* \\ -N_1 G_1 G_1^* \hat V_{21}^* & N_1 G_1 G_1^* G_1 N_2^* - N_2 G_1^* G_1 G_1^* N_1^* \end{pmatrix} \\
&= \begin{pmatrix} 0 & \hat V_{21} N_1^* \\ -N_1 \hat V_{21}^* & N\hat G N^* \end{pmatrix} = 0.
\end{aligned}$$

Thus we know that the operator T_s is a self-adjoint extension of T.

Next we prove that T_s is the Friedrichs extension of T. Let $y \in D(T_F)$. From result (4.23) with [24, Theorem 5.38] we see that $y \in D(T^*)$ and there exists a sequence $\{y_m\} \subset D(T)$ such that

$$\lim_{m\to\infty} \bar l(y-y_m, y-y_m) = 0. \tag{5.4}$$

Since all the functions y_m belong to $D(T)$, it follows that $V\hat Y_{m,a,b} = 0$. This implies

$$0 = (\hat V_{21} 0)\hat Y_{m,a,b} = \hat V_{21} \begin{pmatrix} Y_{m,d_a-n}(a) \\ Y_{m,d_b-n}(b) \end{pmatrix}.$$

Note that on $[\alpha,\beta]\subset I$ we have

$$\bar{l}(y,y) = -\mathbf{x}^*\tilde{C}_n\mathbf{u}|_\alpha^\beta + \int_\alpha^\beta \mathbf{u}^*B(t)\mathbf{u}dt + \int_\alpha^\beta \mathbf{x}^*C(t)\mathbf{x}dt - \int_\alpha^\beta \mathbf{x}^*\lambda W_1(t)\mathbf{x}dt,$$

with $\mathbf{u} = B^{-1}(t)(\tilde{C}_n^*\mathbf{x}' - A(t)\mathbf{x})$, where

$$\mathbf{x}(t) = (yy^{[1]}\cdots y^{[n-1]})^T, \quad \mathbf{u}(t) = (y^{[2n-1]}y^{[2n-2]}\cdots y^{[n]})^T.$$

We infer that

$$\int_\alpha^\beta \mathbf{u}^*B(t)\mathbf{u}dt \le C_0((\bar{l}+K)y,y), \tag{5.5}$$

where $C_0, K\in\mathbb{C}$. By applying (5.5) to $y_m - y$ instead of y, we obtain

$$\int_\alpha^\beta (\mathbf{u}_m-\mathbf{u})^*B(t)(\mathbf{u}_m-\mathbf{u})dt \le C_0((\bar{l}+K)(y_m-y), y_m-y),$$

where $\mathbf{u}_m = B^{-1}(t)(\tilde{C}_n^*\mathbf{x}_m' - A(t)\mathbf{x}_m)$.

Since $B(t)\ge 0$ a.e. on I, from (5.4) we obtain that

$$\lim_{m\to\infty}\int_\alpha^\beta (\mathbf{u}_m-\mathbf{u})^*B(t)(\mathbf{u}_m-\mathbf{u})dt = 0.$$

Thus we have $\mathbf{u}_m\to\mathbf{u}$ as $m\to\infty$ for all $t\in[\alpha,\beta]\subset I$ on $D(T^*)\subset D_Q$. Therefore $\lim_{m\to\infty}\mathbf{u}_m(\alpha) = \mathbf{u}(\alpha)$ on $D(T^*)\subset D_Q$. Moreover

$$\begin{aligned}[y_m-y, y_{1i}](\alpha) &= \{\mathbf{y}_m-\mathbf{y}, \mathbf{y}_{1i}\}(\alpha)\\ &= \mathbf{x}_{1i}^*(\alpha)\tilde{C}_n(\mathbf{u}_m-\mathbf{u})(\alpha) - (\mathbf{u}_m-\mathbf{u})^*(\alpha)\tilde{C}_n^*\mathbf{x}_{1i}(\alpha).\end{aligned}$$

Thus we infer that

$$\lim_{m\to\infty}[y_m-y, y_{1i}](\alpha) = 0.$$

Hence we have $\lim_{m\to\infty}[y-y_m, y_{1i}](a) = 0$, $i=1,2,\dots,d_a-n$. Similarly, we obtain $\lim_{m\to\infty}[y-y_m, y_{2j}](b) = 0$, $j=1,2,\dots,d_b-n$. Therefore

$$\lim_{m\to\infty}\begin{pmatrix}Y_{m,d_a-n}(a)\\ Y_{m,d_b-n}(b)\end{pmatrix}\begin{pmatrix}Y_{d_a-n}(a)\\ Y_{d_b-n}(b)\end{pmatrix}.$$

Then $\hat{V}_{21}\begin{pmatrix}Y_{d_a-n}(a)\\ Y_{d_b-n}(b)\end{pmatrix} = 0$ shows that $y\in D_s$ and so $D(T_F)\subset D_s$. On the other hand, we have proved that D_s is the domain of a self-adjoint extension T_s of T. Consequently, the self-adjointness of T_F leads to $T_s = T_F$. The proof is complete. □

Remark 5.2. From Theorem 5.1 we know that there are d_a-n principal solutions at a and d_b-n principal solutions at b which contribute to the Friedrichs extension of any symmetric operator which is bounded below. Moreover, if the differential equation (1.3) is regular, then $d_a = 2n = d_b$ in Theorem 5.1 i.e., $d = 2n$. In this case the result of Theorem 5.1 is reduced to the Friedrichs extension of any even order regular symmetric differential operators, see [4, Theorem 4.7].

Corllary 5.3. *Let $B(t)$ be semi-positive definite a.e. on $I=(a,b)$. Suppose that system (2.3) is disconjugate on I for some $\lambda<\gamma$, where γ is a lower bound of T_0'. Then the Friedrichs extension $T_{0,F}$ of the minimal operator T_0 is characterized by*

$$D(T_{0,F}) = \{y\in D_Q : Y_{d_a-n}(a) = 0 = Y_{d_b-n}(b)\}, \tag{5.6}$$

where

$$Y_{d_a-n}(a) = ([y, y_{1a}](a) \cdots [y, y_{d_a-n,a}](a))^T,$$
$$Y_{d_b-n}(b) = ([y, y_{1b}](b) \cdots [y, y_{d_b-n,b}](b))^T$$

and $y_{i,a}$*,* $i = 1, \dots, d_a - n$*, and* $y_{j,b}$*,* $j = 1, \dots, d_b - n$ *are principal solutions at* a *and* b *respectively.*

Proof. Let $l = 2d$ in Theorem 5.1. In this case the operator T defined by (5.1) is the minimal operator T_0, and the domain of the Friedrichs extension $T_{0,F}$ of T_0 is obtained easily by (5.2), i.e. (5.6) holds. This completes the proof. □

Remark 5.4. Here the principal solutions $y_{i,a}$, $i = 1, \dots, d_a - n$ define independent boundary conditions at $t = a$ and the principal solutions $y_{j,b}$, $j = 1, \dots, d_b - n$ define independent boundary conditions at $t = b$. For instance, if a is regular and b is singular, i.e., $d_a = 2n$ and $d = d_b$ in Corollary 5.3, then (5.6) can be reduced to

$$D(T_{0,F}) = \{y \in D_Q : y^{[n-i]}(a) = 0, Y_{d-n}(b) = 0, i = 1, 2, \dots, n\}. \tag{5.7}$$

For $C_{2n} = E_{2n} = ((-1)^r \delta_{r,2n+1-s})_{r,s=1}^{2n}, C_{m_b} = E_{m_b} = ((-1)^r \delta_{r,m_b+1-s})_{r,s=1}^{m_b}$ we know that (5.7) can be reduced to the result in [16, Theorem 12]. However, for $d < 2n$, the characterization of the Friedrichs extension of the minimal operator in [16, Theorem 12] did not define independent boundary conditions at $t = b$. Furthermore if both a, b are regular, then $d_a = 2n = d_b$, and (5.7) can be reduced to

$$D(T_{0,F}) = \{y \in D_Q : y^{[n-i]}(a) = 0 = y^{[n-i]}(b) = 0, i = 1, 2, \dots, n\}. \tag{5.8}$$

This is an extension of a result of [14, Theorem 8.1] from E_{2n} to C_{2n}.

Theorem 5.5. *Let the assumptions in Theorem 5.1 hold and let the symmetric operator* T *be given by* (5.1)*. Let* $V = (V_1 V_2)$*,* $V_j \in M_{l,d}(\mathbb{C})$*,* $j = 1, 2, d < l \le 2d$*. Then the domain* $D(T_F)$ *of its Friedrichs extension* T_F *is characterized by*

$$D(T_F) = \big\{y \in D(T_Q) : \hat{Y}_{a,b} \in \mathcal{R}(\hat{G}V^*), \begin{pmatrix} Y_{d_a-n}(a) \\ Y_{d_b-n}(b) \end{pmatrix} \in G_1 V_2^*(\mathcal{N}(V\hat{G}V^*))\big\} \tag{5.9}$$

Proof. Observe that $V = UJ$ in (4.7) and $JGU^*(\mathcal{N}(UGU^*)) = \hat{G}V^*(\mathcal{N}(V\hat{G}V^*))$, Then (5.9) is obtained easily from [26, Lemma 14 property (7)] and the proof of Theorem 5.1. □

Remark 5.6. Under the assumptions of Theorem 5.5 we have

$$D(T_F) = \big\{y \in D(T_Q) : \hat{Y}_{a,b} \in \mathcal{R}(\hat{G}V^*), \begin{pmatrix} Y_{d_a-n}(a) \\ Y_{d_b-n}(b) \end{pmatrix} \in V_1^{-1}\mathcal{R}(V_2)\big\}.$$

Remark 5.7. Let both endpoints a, b of (1.3) be regular, i.e., $d_a = d_b = 2n$ and $C_{2n} = E_{2n}$ in Theorem 5.5 and Remark 5.6, i.e.,

$$\hat{G} = \begin{pmatrix} 0 & G_1 \\ -G_1^* & 0 \end{pmatrix}, \quad \text{with } G_1 = \begin{pmatrix} -E_n & 0 \\ 0 & E_n \end{pmatrix}.$$

In this case we obtain that the Möller-Zettl results in [15] is a special case. Moreover in this case if $l = 2n$, then we obtain the characterization of Friedrichs extension of the minimal operator in [14, Theorem 8.1]. Also Niessen-Zettl [18, Theorem 2.1] found a special case for E_{2n} and a certain matrix Q.

Remark 5.8. Comparing Theorem 5.1 and Theorem 5.5 although they are equivalent to each other Theorem 5.1 is more explicit than the result in Theorem 5.5. For a better understanding of our main results we give some simple examples for the special cases in the next section.

6. Examples

In this section we consider the Friedrichs extension T_F of the symmetric operator T for some special cases. We consider the real symmetric differential equation

$$My = \sum_{j=0}^{n} (-1)^j (p_j y^{(j)})^{(j)} = \lambda w y, p_n, w > 0 \quad \text{a.e. on } I = (0,b), b \le \infty, \tag{6.1}$$

where $y^{(j)}$ denotes the classical derivatives. In this case, $M = M_Q$ is generated by $Q = (q_{r,s})_{r,s=1}^{2n} \in Z_{2n}(I)$ whose components are

$$q_{r,r+1} = 1, \quad q_{r+n,r+n+1} = -1, \quad r = 1,2,\dots,n-1;$$
$$q_{n,n+1} = p_n^{-1}, \quad q_{r,s} = p_{s-1}, \quad s \ne n,\ r+s = 2n+1,\ 1 \le r \le 2n$$

and Q satisfies $Q = CQ^*C$ with

$$C = \begin{pmatrix} 0 & \hat{C}_n \\ -\hat{C}_n & 0 \end{pmatrix}, \quad \hat{C}_n = \left(\delta_{r,n+1-s}\right)_{r,s=1}^{n}.$$

Thus the quasi-derivatives are as follows:

$$y^{[r]} = y^{(r)}, \quad r = 1,2,\dots,n-1,$$
$$y^{[n]} = p_n y^{(n)},$$
$$y^{[n+r]} = (-1)^r (p_n y^{(n)})^{(r)} + (-1)^{r-1} (p_{n-1} y^{(n-1)})^{(r-1)} + \dots + p_{n-r} y^{(n-r)},$$
$$r = 1,2,\dots,n.$$

Note that (6.1) is equivalent to Hamiltonian system

$$L\mathbf{y} = \tilde{J}_{2n}\mathbf{y}' - G_Q(t)\mathbf{y} = \lambda W \mathbf{y}, \tag{6.2}$$

where

$$\tilde{J}_{2n} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, \quad G_Q(t) = \begin{pmatrix} -C(t) & A^*(t) \\ A(t) & B(t) \end{pmatrix}, \quad W(t) = \begin{pmatrix} W_1(t) & 0 \\ 0 & 0 \end{pmatrix}$$

with

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}, \quad C(t) = \begin{pmatrix} p_0 & 0 & \cdots & 0 \\ 0 & p_1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & p_{n-1} \end{pmatrix},$$

$$B(t) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & p_n^{-1} \end{pmatrix}, \quad W_1(t) = \begin{pmatrix} w & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

Assume that (6.2) is disconjugate on I for some $\lambda < \gamma$, where γ is a lower bound of T_0'. Notice that $B(t) \ge 0$ a.e. on I (For $n = 1$ $B(t) > 0$ a.e. on I). From

Theorem 3.1 we know that the minimal operator T_0 generated by the equation (6.1) is bounded below with $\hat{\gamma} \le \gamma$. Assume that d_0 and d_b denote the deficiency indices of T_0 on $(0,e)$ and (s,b), respectively. Let $y_{10}, \dots, y_{d_0-n,0}$ and $v_{10}, \dots, v_{d_0-n,0}$ be principal and non-principal solutions of (6.1) at 0, respectively; $y_{1b}, \dots, y_{d_b-n,b}$ and $v_{1b}, \dots, v_{d_b-n,b}$ be principal and non-principal solutions of (6.1) at b, respectively.

Suppose that d denotes the deficiency indices of the minimal operator generated by (6.1) on $I=(0,b), b \le \infty$. Recall that $d = d_0 + d_b - 2n$. Let

$$\hat{Y}_{0,b} = \begin{pmatrix} Y_{d_0-n}(0) \\ Y_{d_b-n}(b) \\ \hat{Y}_{d_0-n}(0) \\ \hat{Y}_{d_b-n}(b) \end{pmatrix} \in \mathbb{C}^{2d},$$

where

$$\begin{gathered} Y_{d_0-n}(0) = ([y, y_{10}](0) \cdots [y, y_{d_0-n,0}](0))^T, \\ \hat{Y}_{d_0-n}(0) = ([y, v_{10}](0) \cdots [y, v_{d_0-n,0}](0))^T, \\ Y_{d_b-n}(b) = ([y, y_{1b}](b) \cdots [y, y_{d_b-n,b}](b))^T \\ \hat{Y}_{d_b-n}(b) = ([y, v_{1b}](b) \cdots [y, v_{d_b-n,b}](b))^T. \end{gathered}$$

For a given matrix $N \in M_{(2d-l),2d}(\mathbb{C})$ satisfying

$$\operatorname{rank}(N) = 2d - l, \quad N\hat{G}N^* = 0, \quad d \le l \le 2d, \tag{6.3}$$

where

$$\hat{G} = (-1)^n \begin{pmatrix} 0 & G_1 \\ -G_1 & 0 \end{pmatrix}, \quad G_1 = \begin{pmatrix} \hat{C}_{d_0-n} & 0 \\ 0 & -\hat{C}_{d_b-n} \end{pmatrix} \tag{6.4}$$

with

$$\hat{C}_{d_0-n} = (\delta_{r,d_0-n+1-s})_{r,s=1}^{d_0-n}, \hat{C}_{d_b-n} = (\delta_{r,d_b-n+1-s})_{r,s=1}^{d_b-n}.$$

If $V \in M_{l,2d}(\mathbb{C})$ is a complete solution of the matrix equation

$$NV^* = 0, \tag{6.5}$$

then by Lemma 4.3, T generated by (6.1) is a symmetric operator with an l dimensional restriction of the associated maximal operator T_Q for (6.1) and

$$D(T^*) = \{y \in D_Q : N\hat{G}\hat{Y}_{0,b} = 0\}. \tag{6.6}$$

Moreover by Theorem 4.5 we obtain that the domain of T is equivalent to

$$D(T) = \{y \in D_Q : V\hat{Y}_{0,b} = 0\}, \tag{6.7}$$

with the boundary matrix

$$V = \begin{pmatrix} \hat{V}_{11} & \hat{V}_{12} \\ \hat{V}_{21} & 0 \\ -N_{12}G_1 & 0 \\ -N_{22}G_1 & N_{11}G_1 \end{pmatrix}, \quad \hat{V}_{rs} \in M_{(l-d),d}(\mathbb{C}), \quad r,s = 1,2 \tag{6.8}$$

with $\operatorname{rank}(\hat{V}_{12}) = \operatorname{rank}(\hat{V}_{21}) = l - d$ for $d \le l \le 2d$, and

$$\begin{pmatrix} 0 & \hat{N}_{12} \\ \hat{N}_{21} & \hat{N}_{22} \end{pmatrix} = (N_1 N_2) = N.$$

Thus from Theorem 5.1 we infer the following result.

Corllary 6.1. *Assume that both endpoints of* (6.1) *are limit-circle (LC). Let* $2n < l \le 4n$ *and* T *defined by* (6.7) *be a symmetric operator with an* l *dimensional restriction of the associated maximal operator* T_Q *for* (6.1). *Then the boundary conditions of its Friedrichs extensions* T_F *can be characterized by*

$$\begin{gathered}\hat{V}_{21}\begin{pmatrix}Y_n(0)\\ Y_n(b)\end{pmatrix}=0,\\ (-N_2G_1N_1G_1)\hat{Y}_{0,b}=0.\end{gathered}\tag{6.9}$$

Here $\hat{V}_{21}$ *is a submatrix of* (6.8).

Example 6.2. We consider a differential equation

$$My=(-1)^n y^{(2n)}+qy,\quad q\in L^1_{\rm loc}(I,\mathbb{R}),\ I=(0,b),\tag{6.10}$$

where both $0,b$ are LC endpoints. Let $A\in M_{m,2n}(\mathbb{C})$ and $B\in M_{(2n-m),2n}(\mathbb{C})$ satisfy

$$\operatorname{rank}(A)=m, AB^*=0, n\le m\le 2n,$$

with $\operatorname{rank}(B)=2n-m$. Define

$$V=\begin{pmatrix}A & 0\\ 0 & I_{2n}\end{pmatrix}.$$

By setting $N=(B0)$, N satisfies (6.3) and (6.5), where

$$\hat{G}=(-1)^n\begin{pmatrix}0 & G_1\\ -G_1 & 0\end{pmatrix},\quad G_1=\begin{pmatrix}\hat{C}_n & 0\\ 0 & -\hat{C}_n\end{pmatrix},\quad \hat{C}_n=(\delta_{r,n+1-s})^n_{r,s=1}.$$

Accordingly we know that an operator T defined on

$$D(T)=\Big\{y\in D_Q: V\hat{Y}_{0,b}=0\Big\},\tag{6.11}$$

is a symmetric operator with a $2n+m$ dimensional restriction of the associated maximal operator T_Q for (6.10). Note that T is bounded below. By using Corollary 6.1, we obtain that the characterization T_F of T is characterized by

$$D(T_F)=\big\{y\in D_Q: A\begin{pmatrix}Y_n(0)\\ Y_n(b)\end{pmatrix}=0,\ \hat{Y}_{2n-m}(b)=0\big\}.$$

Here when $m=2n$ we obtain $T=T_0$ and $T_F=T_{0,F}$.

Remark 6.3. In Example 6.2 if we set

$$V=\begin{pmatrix}A_1 & 0 & A_2 & 0\\ 0 & I_n & 0 & 0\\ 0 & 0 & 0 & I_n\end{pmatrix},\quad \operatorname{rank}(A_1A_2)=m, A_1,\ A_2\in M_{m,n}(\mathbb{C})$$

in (6.11), then by setting $N=(B_10B_20)$, $\operatorname{rank}(B_1B_2)=2n-m$, $B_1,B_2\in M_{(2n-m),n}(\mathbb{C})$ and

$$\begin{gathered}B_1\hat{C}_nB_2^*-B_2\hat{C}_nB_1^*=0,\\ B_1A_1^*+B_2A_2^*=0\end{gathered}\tag{6.12}$$

we obtain that

$$D(T)=\big\{y\in D_Q:(A_1A_2)\begin{pmatrix}Y(0)\\ \hat{Y}(0)\end{pmatrix}=0,\ \begin{pmatrix}Y(b)\\ \hat{Y}(b)\end{pmatrix}=0\big\}$$

is symmetric. Notice that from (6.12) we have $A_2=-A_1B_1^*B_2(B_2^*B_2)^{-1}$ and

$$-B_2\hat{C}_n(A_1^*A_1)^{-1}A_1^*A_2=B_2\hat{C}_nB_1^*B_2(B_2^*B_2)^{-1}=B_1\hat{C}_n.$$

By transformation of rows V can be rewritten as

$$V \to \begin{pmatrix} 0 & 0 & 0 & I_m \\ 0 & I_n & 0 & 0 \\ I_n & 0 & (A_1^*A_1)^{-1}A_1^*A_2 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 0 & I_m \\ 0 & I_m & 0 & 0 \\ -B_2\hat{C}_n & 0 & B_1\hat{C}_n & 0 \end{pmatrix}.$$

Thus from Corollary 6.1 we infer that its Friedrichs extension is

$$\begin{aligned} D(T_F) &= \Big\{y \in D_Q : \begin{pmatrix} 0 \ I_m \end{pmatrix} \begin{pmatrix} Y(0) \\ Y(b) \end{pmatrix} = 0,\ \begin{pmatrix} -B_2\hat{C}_n \ B_1\hat{C}_n \end{pmatrix} \begin{pmatrix} Y(0) \\ \hat{Y}(0) \end{pmatrix} = 0\Big\} \\ &= \Big\{y \in D_Q : I_mY(b) = 0, \begin{pmatrix} A_1 \ A_2 \end{pmatrix} \begin{pmatrix} Y(0) \\ \hat{Y}(0) \end{pmatrix} = 0\Big\}. \end{aligned}$$

Clearly when $m = n$ and $\operatorname{rank}(A_1) = n$ (6.12) is equivalent to

$$\begin{pmatrix} A_1 & A_2 \end{pmatrix} \begin{pmatrix} 0 & \hat{C}_n \\ -\hat{C}_n & 0 \end{pmatrix} \begin{pmatrix} A_1^* \\ A_2^* \end{pmatrix} = 0.$$

Moreover in this case we have

$$D(T_F) = \Big\{y \in D_Q : (A_1A_2) \begin{pmatrix} Y(0) \\ \hat{Y}(0) \end{pmatrix} = 0, Y(b) = 0\Big\}.$$

This result was obtained in the regular case with $\hat{C}_n = ((-1)^r\delta_{r,n+1-s})_{r,s=1}^n$ in [15, Theorem 4.3].

Since there is no boundary condition is required or allowed at the limit-point endpoints, if the both endpoints $0, b$ for (6.1) are limit-point, then the associated minimal operator is self-adjoint. We now assume that one endpoint of I for (6.1) is limit-point, i.e., $d = n$. We give some special examples in $n = 2$ case.

Example 6.4. Consider the differential equation

$$My = (p_2y'')'' - (p_1y')' + p_0y, p_2 > 0, \quad p_j(t) \in L^1_{\rm loc}(I, \mathbb{R}),\ j = 0, 1, 2, I = (0, b), \tag{6.13}$$

where the endpoint 0 is limit-circle and the endpoint b is limit-point. Assume that T generated by (6.13) is defined on

$$D(T) = \Big\{y \in D_Q : V \begin{pmatrix} [y, y_{10}](0) \\ [y, y_{20}](0) \\ [y, v_{10}](0) \\ [y, v_{20}](0) \end{pmatrix} = 0\Big\}$$

with

$$V = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

Here y_{10}, y_{20} and v_{10}, v_{20} are principal and non-principal solutions of (6.13) at endpoint 0, respectively. Note that there exists a matrix $N = \begin{pmatrix} 1 \ 1 \ 1 \ -2 \end{pmatrix}$ such that

$$NV^* = 0, \quad N\hat{G}_4N^* = 0,$$

where

$$\hat{G}_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Thus T is a symmetric operator with 3 dimensional restriction of T_Q. By elementary matrix transformations of the rows V can be transformed into

$$V = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 2 & -1 & 1 & 1 \end{pmatrix}.$$

Since T is bounded below, from Corollary 6.1 we obtain that the boundary conditions of Friedrichs extension of T are:

$$\begin{gathered} [y, y_{10}](0) - [y, y_{20}](0) = 0, \\ 2[y, y_{10}](0) - [y, y_{20}](0) + [y, v_{10}](0) + [y, v_{20}](0) = 0. \end{gathered}$$

Example 6.5. Consider the differential equation (6.13) with the endpoint 0 is regular and the endpoint b is limit-point. Let T generated by (6.13) be defined on

$$D(T) = \Big\{ y \in D_Q : \begin{pmatrix} 0 & 2 & 3 & 4 \\ 1 & 0 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} y(0) \\ y'(0) \\ (p_2 y'')(0) \\ (p_1 y')(0) - (p_2 y'')'(0) \end{pmatrix} = 0 \Big\}.$$

Let

$$V = \begin{pmatrix} 0 & 2 & 3 & 4 \\ 1 & 0 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

Note that there exists a matrix $N = \begin{pmatrix} 0 & 0 & 4 & -3 \end{pmatrix}$ such that $NV^* = 0$ and $N\hat{G}_4 N^* = 0$, where

$$\hat{G}_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Thus T is a symmetric operator with 3 dimensional restriction of T_Q. By transformations of rows V can be transformed into

$$V = \begin{pmatrix} 0 & 0 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 3 & -4 & 0 & 0 \end{pmatrix}.$$

Note that T is bounded below. From Corollary 6.1 we obtain that the Friedrichs extension of T is characterized as

$$\begin{gathered} y'(0) = 0, \\ 3y(0) - 4y'(0) = 0, \end{gathered}$$

i.e., $y(0) = y'(0) = 0$.

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