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REMARKS ON PERIODIC RESONANT PROBLEMS WITH NONLINEAR DISSIPATION

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In memory of Professor Alan C. Lazer

Abstract. We consider the periodic problem for a 2nd order ODE with non-invertible linear part, and mild nonlinear dissipation term. The motivation for this study is a paper by Lazer [9]. We add a bounded restoring force $g(u)$ and show that the sufficient condition (of Landesman-Lazer type) given in [9] still implies the existence of a periodic solution in our case. We also comment on some variants of the problem and on the existence of bounded solutions.

1. Introduction

Lazer [9] gave a simple proof of necessary and sufficient conditions for the existence of a 2π-periodic solution to the second order ordinary equation with a nonlinear dissipative term

$$u''(t) + u(t) + \frac{d}{dt}F(u(t)) = e(t) \tag{1.1}$$

where F is a bounded C^1 function, and e is continuous and 2π-periodic function. The conditions include an inequality involving the size of the projection of e onto the kernel of the linear operator $u'' + u$ in the space of 2π-periodic functions, and the gap between the limits of F at $\pm\infty$, namely

$$2(F(\infty) - F(-\infty)) > \sqrt{e_s^2 + e_c^2} \tag{1.2}$$

where

$$e_c = \int_0^{2\pi} \cos x\, e(x)\, dx, \quad e_s = \int_0^{2\pi} \sin(x)\, e(x)\, dx. \tag{1.3}$$

Among other features, the proof in [9] invokes the Brouwer fixed point theorem for a disk in the plane.

Inequality (1.2) is a condition of Landesman-Lazer type; see [7] for the original paper of Landesman and Lazer. As stated in [9], the applicability of that condition to the resonant periodic problem had appeared in articles by Lazer and Leach [10], and Frederickson and Lazer [4].

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In this note we consider the slightly more general problem

$$u''(t) + u(t) + g(u) + \frac{d}{dt}F(u(t)) = e(t) \tag{1.4}$$

where g is continuous and bounded. That is, we are interested in a bounded perturbation of equation (1.1). Basically, we intend to show that (1.2) remains applicable to (1.4) and we propose an alternative, although basically equivalent, method of proof.

We stress that, beyond the papers already mentioned, there exists an extensive and rich literature concerning equations similar to (1.1), covering existence and stability of periodic, almost periodic or bounded solutions, under a variety of assumptions. A handful of material, which is significant and representative of the research about these problems can be found in works by Ahmad [1], Ezeilo [3], Mawhin [11], Ortega [13], Ortega and Tineo [14], Fonda and Zanolin [5], Mawhin and Ward [12], and, of course, in their references.

In this note, as in [9], our arguments use a shooting method and continuity with respect to parameters. They are as simple as possible if we assume that g is a locally Lipschitz function, although weaker conditions may be considered via approximate problems.

On the other hand, in our proof the Brouwer fixed point theorem is naturally replaced with the particular version of the following Brouwer-Bohl existence principle (see e.g. [11]).

Theorem 1.1. *Let $\mathbb{B}^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$. If $V : \mathbb{B}^n \to \mathbb{R}^n$ is a continuous vector field such that for all $x \in \partial\mathbb{B}^n$, $V(x) \cdot x > 0$, then the equation $V(x) = 0$ has a solution in $\mathbb{B}^n$.*

To deal with the perturbation g we need in addition the following property of sequences of oscillatory integrands.

Lemma 1.2. *Let $T = 2n\pi$, $n \in \mathbb{N}$, $\{W_{R,\phi}(t)\}$ ($R > 0$, $\phi \in \mathbb{R}$) be a family of C^1 functions in $[0, T]$, bounded independently of R; ϕ, with their derivatives and g be a continuous bounded function. Then*

$$\int_0^T g(R\sin(t+\phi) + W_{R,\phi}(t))\cos(t+\phi)dt \longrightarrow 0$$

as $R \to \infty$ uniformly with respect to $\phi \in \mathbb{R}$.

Proof. Let G be an antiderivative of g. We have

$$\begin{aligned}
&G(R\sin(\phi) + W_{R,\phi}(2n\pi)) - G(R\sin(\phi) + W_{R,\phi}(0)) \\
&= \int_0^{2n\pi} \frac{d}{dt}(G(R\sin(t+\phi) + W_{R,\phi}(t)))dt \\
&= R\int_0^{2n\pi} g(R\sin(t+\phi) + W_{R,\phi}(t))\cos(t+\phi)dt \\
&\quad + \int_0^{2n\pi} g(R\sin(t+\phi) + W_{R,\phi}(t))W'_{R,\phi}(t)dt
\end{aligned}$$

From our assumptions and the mean-value theorem we can conclude that

$$R\int_0^{2n\pi} g(R\sin(t+\phi) + W_{R,\phi}(t))\cos(t+\phi)dt$$

is bounded, and therefore

$$\lim_{R\to\infty}\int_0^T g(R\sin(t+\phi)+W_{R,\phi}(t))\cos(t+\phi)dt=0.$$

□

In section 2 we present the main result of this note with some remarks on the analogous (simpler) problem where the term u is dropped in (1.4) (that is, the case of resonance at the eigenvalue zero). In the final section, following mainly [4], we refer to the existence of bounded solutions.

2. Main result

Theorem 2.1. *Let $e(t)$ be continuous and 2π-periodic, $F:\mathbb{R}\to\mathbb{R}$ a bounded C^1 function such that $F(\infty)$, $F(-\infty)$ exist, and $g:\mathbb{R}\to\mathbb{R}$ a locally Lipschitz continuous, bounded function. Defining e_c, e_s as in* (1.3)*, the condition* (1.2) *is sufficient for the existence of a 2π-periodic solution of* (1.4)*.*

Proof. We want to prove the existence of a solution to the problem

$$\begin{gathered} u''(t)+u(t)+\frac{d}{dt}F(u(t))+g(u(t))=e(t)\\ u(0)=u(2\pi),\quad u'(0)=u'(2\pi) \end{gathered}\tag{2.1}$$

Let u be a function of class C^2. Consider the decomposition

$$u(x)=A\cos(x)+B\sin(x)+W(x,A,B),$$

where $A=u(0)$ and $B=u'(0)$, so that $W(0)=W'(0)=0$. Hence u is a solution of (2.1) if, and only if W is a solution of

$$W''(t,A,B)+W(t,A,B)+F(u(t))'+g(u(t))=e(t),\tag{2.2}$$

$$W(2\pi)=0,\quad W'(2\pi)=0.\tag{2.3}$$

By an elementary formula, the solution of (2.2) such that $W(0)=W'(0)=0$ is given by

$$W(t,A,B)=\int_0^t \sin(t-x)(e(x)-F(u(x))'-g(u(x))\,dx\tag{2.4}$$

Integrating by parts and using well known results on integral equations with parameters (see e.g. [16]) we see that there exists in fact a unique solution $W(t,A,B)$ of (2.4), continuous with respect to A, B, and (2.4) shows that it is bounded in $[0,2\pi]$ independently of A and B.

We now prove the existence of a point (A,B) in the plane which is a solution of (2.2)-(2.3). We have

$$W'(t,A,B)=\int_0^t \cos(t-x)(e(x)-F(u(x))'-g(u(x))\,dx$$

and we look for a solution of the system

$$\begin{gathered} W(2\pi,A,B)=0,\\ W'(2\pi,A,B)=0 \end{gathered}$$

which is equivalent to

$$\int_0^{2\pi}\sin(2\pi-x)(e(x)-F(u(x))'-g(u(x))\,dx=0,$$

$$\int_0^{2\pi} \cos(2\pi - x)(e(x) - F(u(x))' - g(u(x))\,dx = 0$$

which is equivalent to

$$\begin{gathered}
\int_0^{2\pi} \sin(x)(F(u(x))' + g(u(x))\,dx - e_s = 0,\\
-\int_0^{2\pi} \cos(x)(F(u(x))' + g(u(x))\,dx + e_c = 0\,.
\end{gathered}$$

Using integration by parts we obtain the system

$$\begin{gathered}
e_s + \int_0^{2\pi} \cos(x)F(u(x))\,dx - \int_0^{2\pi} \sin(x)g(u(x)) = 0,\\
e_c - \int_0^{2\pi} \sin(x)F(u(x))\,dx - \int_0^{2\pi} \cos(x)g(u(x))\,dx = 0\,.
\end{gathered} \tag{2.5}$$

We define the vector field $(X(A,B), Y(A,B))$:

$$\begin{gathered}
X(A,B) = e_c - \int_0^{2\pi} \sin(x)F(u(x))\,dx - \int_0^{2\pi} \cos(x)g(u(x))\,dx,\\
Y(A,B) = e_s + \int_0^{2\pi} \cos(x)F(u(x))\,dx - \int_0^{2\pi} \sin(x)g(u(x))\,.
\end{gathered}$$

Let $(A,B) \in \mathbb{R}^2$, $R = \sqrt{A^2+B^2}$ and $\phi \in \mathbb{R}$, such that

$$\cos(\phi) = \frac{B}{\sqrt{A^2+B^2}}, \sin(\phi) = \frac{A}{\sqrt{A^2+B^2}}.$$

Then we consider the dot product

$$\begin{aligned}
&(-Y(A,B), X(A,B)) \cdot (A,B)\\
&= Be_c - Ae_s - \int_0^{2\pi} (A\cos(x) + B\sin(x))F(A\cos(x) + B\sin(x) + W(x,A,B))\,dx\\
&\quad + \int_0^{2\pi} (A\sin(x) - B\cos(x))g(A\cos(x) + B\sin(x) + W(x,A,B))\,dx\\
&= Be_c - Ae_s - \int_0^{2\pi} R\sin(x+\phi)F(R\sin(x+\phi) + W(x,A,B))\,dx\\
&\quad + \int_0^{2\pi} R\cos(x+\phi)g(R\sin(x+\phi) + W(x,A,B))\,dx\\
&\le \|(A,B)\| \cdot \|(e_s,e_c)\| - \int_0^{2\pi} R\sin(x+\phi)F(R\sin(x+\phi) + W(x,A,B))\,dx\\
&\quad + \int_0^{2\pi} R\cos(x+\phi)g(R\sin(x+\phi) + W(x,A,B))\,dx\\
&= R\sqrt{e_c^2+e_s^2} - \int_0^{2\pi} R\sin(x+\phi)F(R\sin(x+\phi) + W(x,A,B))\,dx\\
&\quad + \int_0^{2\pi} R\cos(x+\phi)g(R\sin(x+\phi) + W(x,A,B))\,dx\\
&= R\Big(\sqrt{e_c^2+e_s^2} - \int_0^{2\pi} \sin(x+\phi)F(R\sin(x+\phi) + W(x,A,B))\,dx\Big)
\end{aligned}$$

$$+ R\int_0^{2\pi} \cos(x+\phi)g(R\sin(x+\phi)+W(x,A,B))\,dx$$

Using the assumptions on F and g and using Lemma 1.2, we obtain

$$\begin{aligned}
&\lim_{R\to\infty}\Big(\sqrt{e_c^2+e_s^2} - \int_0^{2\pi} \sin(x+\phi)F(R\sin(x+\phi)+W(x,A,B))\,dx \\
&+ \int_0^{2\pi} \cos(x+\phi)g(R\sin(x+\phi)+W(x,A,B))\,dx\Big) \\
&= \sqrt{e_c^2+e_s^2} - 2(F(\infty)-F(-\infty)) < 0,
\end{aligned}$$

uniformly with respect to $\phi \in \mathbb{R}$. Note that by (2.4) and since F and g are bounded, W is a bounded function in $[0,2\pi]$, independently of A and B.

Therefore there is $R_0 \in \mathbb{R}_+$ such that for all $(A,B) \in \mathbb{R}^2$ with $\|(A,B)\| \ge R_0$, we have $(-Y(A,B),X(A,B))\cdot(A,B) < 0$. So that by Theorem 2.1 there is a $(A^*,B^*) \in \mathbb{R}^2$ with $\|(A^*,B^*)\| < R_0$ such that $X(A^*,B^*) = Y(A^*,B^*) = 0$. Hence we have a solution of (2.5). $\square$

It is interesting to note that in the above theorem we did not use asymptotic assumptions on g; the condition involving the gap of F is sufficient, precisely as in the case where g is absent as in [9]. However, in the absence of F, the argument of the above proof may be repeated (with slightly simpler calculations) to obtain the Lipschitz case of [10, Theorem 1.1]

Theorem 2.2. *Let e be a 2π-periodic continuous function, and g a bounded continuous function in $\mathbb{R}$ such that $g(\infty)$ and $g(-\infty)$ exist. By setting*

$$e_c = \int_0^{2\pi} \cos(x)e(x)\,dx, \quad e_s = \int_0^{2\pi} \sin(x)e(x)\,dx,$$

we have that $2(g(\infty)-g(-\infty)) > \sqrt{e_c^2+e_s^2}$ is a sufficient condition for the existence of a 2π-periodic solution of the differential equation $u''(t)+u(t)+g(u(t)) = e(t)$.

Remark 2.3. Korman and Li [6] studied 2π-periodic solutions to the equation that is analogous to (1.1) when resonance occurs at the nth eigenvalue, that is

$$u''(t) + n^2u(t) + \frac{d}{dt}F(u(t)) = e(t) \tag{2.6}$$

where $n \in \mathbb{N}$. Setting

$$e_{c,n} = \int_0^{2\pi} \cos(nx)\,e(x)\,dx, \quad e_{s,n} = \int_0^{2\pi} \sin(nx)\,e(x)\,dx. \tag{2.7}$$

they found the condition

$$2n(F(\infty)-F(-\infty)) > \sqrt{e_{c,n}^2+e_{s,n}^2} \tag{2.8}$$

for the existence of a solution. Following our procedure in the proof of Theorem 2.1 it may be seen that the condition (2.8) still implies existence when a bounded nonlinear term $g(u)$ is included in the equation (2.6). Without going to the details, let us mention that it suffices to use the decomposition

$$u(x) = A\cos(nx) + \frac{B}{n}\sin(nx) + W(x,A,B),$$

where $A = u(0)$, $B = u'(0)$, so that W satisfies

$$W(t, A, B) = \frac{1}{n} \int_0^t \sin\, n(t-x)(e(x) - F(u(x))' - g(u(x))\, dx \tag{2.9}$$

and in the final estimate we choose R and ϕ so that $A = R \sin\, \phi$, $B = nR \cos\, \phi$.

Remark 2.4. In [9] it is mentioned that in a previous paper [8] the case where the linear operator reduces to u'' (that is, when the zero eigenvalue is considered) and the dissipative term is of the form cu', $c > 0$, was discussed.

Let us then consider

$$u''(t) + g(u) + \frac{d}{dt} F(u(t)) = e(t) \tag{2.10}$$

where e is continuous and T-periodic. First, it is obvious that if (2.10) has a T-periodic solution and $m \le g \le M$ for some constants $m \le M$ and setting

$$\bar{e} = \frac{1}{T} \int_0^T e(t)\, dt$$

for the mean value of e, it turns out that $m \le \bar{e} \le M$. Moreover, letting $g_\pm = \lim_{s \to \pm\infty} g(s)$ the condition

$$g_- < \bar{e} < g_+ \tag{2.11}$$

is sufficient to the existence of a T-periodic solution of the differential equation (2.10).

This has been observed in [11] (even with less demanding inequalities). We note that in the special case where F and g are locally Lipschitz functions, the outline of our proof in section 2 may be followed to yield the partial converse; the situation here is simpler and the argument relies on the Poincaré-Miranda theorem.

In fact, we write any solution u of (2.10) as

$$u(t) = A + Bt + w(t)$$

where $A = u(0)$, $B = u'(0)$ and w satisfies the integral equation

$$w(t) = \int_0^t (t-s)[e(s) - g(A + Bs + w(s))] - \frac{d}{ds} F(A + Bs + w(s))]\, ds \tag{2.12}$$

Now there exists a solution $w(t, A, B)$ of (2.12), continuous in the set of its variables. Therefore we must show that there exist A, B so that $u(T) = A$, $u'(T) = B$, that is, $w(T) = -BT$ and $w'(T) = 0$. Since

$$w'(t) = \int_0^t [e(s) - g(A + Bs + w(s)) - \frac{d}{ds} F(A + Bs + w(s))]\, ds$$

we obtain the following system of equations for A and B,

$$\begin{aligned} &\int_0^T (T-s)[e(s) - g(A + Bs + w(s))]\, ds \\ &- \int_0^T F(A + Bs + w(s))\, ds + TF(A) + BT = 0, \end{aligned} \tag{2.13}$$

$$\int_0^T e(s)\, ds - \int_0^T g(A + Bs + w(s))\, ds = 0. \tag{2.14}$$

We define a vector field (X, Y) in the plane such that $X(A, B)$ (respectively $Y(A, B)$) is the left-hand side of (2.13) (respectively (2.14)). Invoking the boundedness of g

and F it is clear that there exists $b > 0$ such that $X(A, B) > 0$ (respectively < 0) if $B \geq b$ (respectively $B \leq -b$), $\forall A \in \mathbb{R}$.

On the other hand, using the boundedness of g again, together with (2.11),

$$\pm \lim_{A \to \pm\infty} Y(A, B) < 0$$

uniformly in $B \in [-b, b]$. We infer that we may choose $a > 0$ so that the field $(X(A, B), Y(A, B))$ satisfies the conditions of the Poincaré-Miranda theorem in the rectangle $[-a, a] \times [-b, b]$. It follows that the vector field (X, Y) has a zero in this rectangle and the proof is complete.

Remark 2.5. If $g \equiv 0$ in equation (2.10), we can be more specific. Namely, it is not difficult to see that $\bar{e} = 0$ is necessary and sufficient to the existence of a periodic solution of

$$u''(t) + \frac{d}{dt} F(u(t)) = e(t) \tag{2.15}$$

Moreover, the initial value $u(0)$ of the periodic solution can be arbitrarily prescribed.

In fact, let $E(t) = \int_0^t e(s)\, ds$. The T-periodic solution u of (2.15) with $A = u(0)$, $B = u'(0)$ solves the parametric first order initial value problem

$$u' + F(u) = E(t) + B + F(A), \quad u(0) = A \tag{2.16}$$

subject to the condition

$$\int_0^T E(s)\, ds + BT + F(A)T - \int_0^T F(u(s))\, ds = 0. \tag{2.17}$$

Clearly, (2.17) has a solution B for every $A \in \mathbb{R}$, but more can be said: if F is C^1 and increasing, (2.17) defines a C^1 function $B = B(A)$.

To see this, we check that the partial derivative of the left-hand side of (2.17) with respect to B does not vanish. Setting $v = \frac{\partial u}{\partial B}$ and $f = F'$, we have

$$v' + f(u)v = 1, \quad v(0) = 0 \tag{2.18}$$

so that

$$v(T) + \int_0^T f(u(s))v(s)\, ds = T. \tag{2.19}$$

Since $v(T) > 0$ our claim is proved. It follows that, under the stated conditions, in addition to the periodic solutions defined by (2.16), the problem (2.15) has a family of solutions bounded to the right, given by

$$u' + F(u) = E(t) + B(A) + F(A), \quad u(0) = C. \tag{2.20}$$

Moreover, if we denote by z_A the periodic solution of (2.15) given by (2.16)-(2.17), it is easily seen that for each solution u as in (2.20) the function $|u(t) - z_A(t)|$ is decreasing for $t \geq 0$; therefore there exists a constant K such that

$$\lim_{t \to +\infty} (u(t) - z_A(t)) = K.$$

Remark 2.6. We can discard the Lipschitz continuity of g in theorem 2.1 or in the setting of Remark 2.4. Let us briefly describe how to proceed with respect to g in the simpler case of Remark 2.4.

Let $L_\pm = \lim_{s \to \pm\infty} g(s)$. For each $n \in \mathbb{N}$ let $\pm\alpha_{n\pm} > 0$ be chosen so that $s \geq \alpha_{n+}$ (resp. $s \leq \alpha_{n-}$) implies $g(s) \geq L_+ - \frac{1}{n}$ (resp $g(s) \leq L_- + \frac{1}{n}$). Then construct a Lipschitz function g_n such that $g_n(s) = g(\alpha_{n+})$ (resp. $g_n(s) = g(\alpha_{n-})$) $\forall s \geq \alpha_{n+}$ (resp. $\forall s \leq \alpha_{n-}$) and $\max_{[\alpha_{n-}, \alpha_{n+}]} |g - g_n| < \frac{1}{n}$. Consider a sequence of approximation problems where g is replaced with g_n in equation (2.10). Each one of these problems has a T-periodic solution u_n whose sequence of initial conditions (A_n, B_n) is bounded, as the corresponding systems (2.13)-(2.14) show. Using the sequence of differential equations it is easy to obtain uniform C^1-estimates for (u_n). By the Ascoli-Arzelà's theorem, a subsequence of u_n converges uniformly to a function u which, clearly, is a periodic solution of (2.10).

3. Boundedness of solutions

Following [4, 9] a naturally related problem is the existence of bounded or almost periodic solutions when e is a continuous almost periodic function. For such functions the mean value exists. Let us consider the case where the following limit exists and is finite, uniformly in $a \in \mathbb{R}_+$:

$$P(e) = \lim_{T\to+\infty} \frac{1}{T}\Big(\Big(\int_a^{a+T} e(t)\cos t\,dt\Big)^2 + \Big(\int_a^{a+T} e(t)\sin t\,dt\Big)^2\Big)^{1/2}. \tag{3.1}$$

Lemma 3.1. *The condition*

$$F(+\infty) - F(-\infty) > \pi\, P(e)$$

is sufficient for the existence of a solution of (1.4) *which is bounded to the right.*

Proof. Following the argument in [4], let us set, for solutions such that u and $u' + F(u)$ do not vanish simultaneously,

$$u = r\cos\varphi, \quad u' + F(u) = r\sin\varphi$$

($r > 0$), and replace (2.1) with the system

$$r' = -\cos\varphi F(r,\cos\varphi) - g(r\cos\varphi)\sin\varphi + e\sin\varphi \tag{3.2}$$

$$\varphi' = -1 - \frac{\sin\varphi F(r,\cos\varphi)}{r} - \frac{\cos\varphi g(r\cos\varphi)}{r} + \frac{e\cos\varphi}{r} \tag{3.3}$$

A solution with $r(t_0) = R$ and $\varphi(t_0) = \theta$ is well defined in an arbitrary interval $[t_0, t_0+T]$ provided that R is sufficiently large, since r' is bounded. In fact $r(t) = R + \alpha(t)$ and $\varphi(t) = \theta - t + \beta(t)/R$ where α and β are bounded functions. Fix $\epsilon > 0$ so that

$$F(+\infty) - F(-\infty) > \pi(P(e) + 2\epsilon). \tag{3.4}$$

By definition of $P(e)$ there exists $\bar{T} > 0$ so that if $T > \bar{T}$ we have for every $t_0 \ge 0$ and every α,

$$\int_{t_0}^{t_0+T} \sin(\alpha - s)e(s)\,ds < T(P(e) + \epsilon). \tag{3.5}$$

Next we choose $\bar{T} > 1$ so that if T is a multiple of 2π such that $\bar{T} \le T \le \bar{T} + 2\pi$, and then taking R sufficiently large, we have

$$\int_{t_0}^{t_0+T} \cos\varphi(s)F(r(s)\cos\varphi(s))\,ds > \frac{T}{\pi}(F(+\infty) - F(-\infty))$$

as well as (by Lemma 1.2)

$$\Big|\int_{t_0}^{t_0+T} g(r\cos\varphi(t))\sin\varphi(t)\Big| < \epsilon.$$

Using the differential equation (3.2), the two preceding inequalities and the assumption of the Lemma we conclude that $r(t_0 + T) < R$. Since r' is bounded and t_0 is arbitrary, the proof is complete. $\square$

On the basis of this property, almost periodicity and compactness arguments (see [2, 4]) allow to obtain a bounded solution of (2.1) in the whole real line, provided the limit (3.1) is uniform in $a \in \mathbb{R}$ and $e(t)$ is almost periodic.

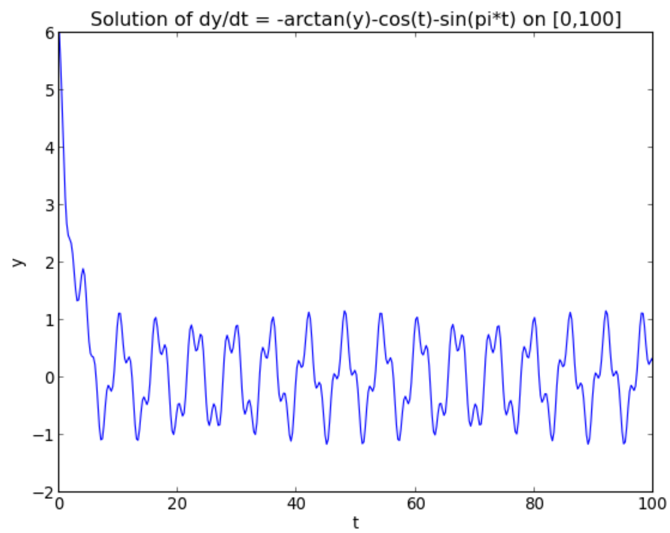

FIGURE 1. Solution of (3.6) with $F(u) = \arctan u$ and $E(t) = -\cos t - \sin(\pi t)$.

Remark 3.2. Perhaps the simplest context where the above argument may be displayed consists in dealing with a first order equation of the type (2.16), say

$$u' + F(u) = E(t) \tag{3.6}$$

where E has a mean value, at least in the sense that

$$\bar{E} := \lim_{T\to+\infty} \frac{1}{T} \int_a^{a+T} E(s)\, ds$$

exists uniformly in $a \geq 0$. Then it is easy to see that the condition

$$F(-\infty) < \bar{E} < F(+\infty)$$

implies that all the solutions of (3.6) are bounded to the right. In fact, fix $\varepsilon > 0$ so that

$$F(-\infty) + 2\varepsilon < \bar{E} < F(+\infty) - 2\varepsilon,$$

and take $T > 0$ so that for all $a \geq 0$,

$$\Big|\frac{1}{T} \int_a^{a+T} E(s)\, ds - \bar{E}\Big| < \varepsilon,$$

and $R > 0$ such that $x > R$ (resp. $x < -R$) implies $F(x) > F(+\infty) - \varepsilon$ (resp. $F(x) < F(-\infty) + \varepsilon$). If for some solution $u(t)$ of (3.6) and $t_0 \geq 0$ we have $u(t_0) \geq R$

(resp. $u(t_0) \le -R$), then for some $s \in [t_0, t_0+T]$ the inequality $u(s) \le R$ (resp. $u(s) \ge -R$) must hold. Since u' is bounded the claim follows.

A little strengthening of the assumptions would lead to the existence of an almost periodic solution which, in case F is increasing, must be unique.

Figure 1 shows a graphic simulation for a solution of (3.6) with $F(u) = \arctan u$ and $E(t) = -\cos t - \sin(\pi t)$. The figure was drawn using the open source software available at the web address [15].

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